

## STRICTLY CONVEX SIMPLEXWISE LINEAR EMBEDDINGS OF A 2-DISK

BY

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ABSTRACT. Let  $K \subset \mathbf{R}^2$  be a finitely triangulated 2-disk; a map  $f: K \rightarrow \mathbf{R}^2$  is called *simplexwise linear (SL)* if  $f|_\sigma$  is affine linear for each (closed) 2-simplex  $\sigma$  of  $K$ . Let  $E(K) = \{\text{orientation preserving SL embeddings } K \rightarrow \mathbf{R}^2\}$ ,  $E_{\text{sc}}(K) = \{f \in E(K) | f(K) \text{ is strictly convex}\}$ , and let  $\overline{E(K)}$  and  $\overline{E_{\text{sc}}(K)}$  denote their closures in the space of all SL maps  $K \rightarrow \mathbf{R}^2$ . A characterization of certain elements of  $\overline{E(K)}$  is used to prove that  $E_{\text{sc}}(K)$  has the homotopy type of  $S^1$  and to characterize those elements of  $\overline{E(K)}$  which are in  $\overline{E_{\text{sc}}(K)}$ , as well as to relate such maps to SL embeddings into the nonstandard plane.

**1. Introduction.** In this paper we apply the methods of [B] to the study of simplexwise linear embeddings of a 2-disk in  $\mathbf{R}^2$  with strictly convex image, and simplexwise linear maps which are the limits of such embeddings. (A map from a triangulated 2-disk into  $\mathbf{R}^2$  is called *simplexwise linear (SL)* if it is affine linear on each (closed) 2-simplex.) Our results are the analogs in the strictly convex case of [BCH, Theorem 5.1] and [B, Theorem 1.2]. The interest in strictly convex SL embeddings is threefold. First, knowing the homotopy type of the space of strictly convex embeddings (Theorem 1.1) could help calculate the homotopy type of the space of all SL embeddings of a 2-disk (which in turn would be useful in the study of SL homeomorphisms of surfaces); second, characterizing SL maps which are the limits of strictly convex embeddings (Theorem 1.2) answers a problem arising in the attempt to extend the foundations of algebraic topology to the nonstandard case, as suggested by C.-H. Sah (private communication); third, if  $C(p_1, \dots, p_n)$  denotes the configuration space of  $n$  points in the plane having the same order type as  $p_1, \dots, p_n \in \mathbf{R}^2$  (see [GP]), then it can be seen that

$$C(p_1, \dots, p_n) = E_{\text{sc}}(K_1) \cap \dots \cap E_{\text{sc}}(K_r)$$

for some triangulated 2-disk  $K_1, \dots, K_r$ ; hence Theorem 1.1 should help determine the homotopy type of  $C(p_1, \dots, p_n)$ .

We will use the definitions, notation and results of [B] without restating them. For background on SL maps, see the introductions to [B and BCH], as well as [BS and CHHS].

**DEFINITION.** A polyhedral 2-disk  $D$  in  $\mathbf{R}^2$  is called *strictly convex* if for every boundary vertex  $v$  of  $D$ , there is a line  $l$  in  $\mathbf{R}^2$  such that  $l \cap D = \{v\}$ . If a polygonal circle  $C$  in  $\mathbf{R}^2$  bounds a strictly convex disk, we also say  $C$  is strictly convex.

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Let  $K$  be a (finitely triangulated, rectilinear) 2-disk in  $\mathbf{R}^2$ . We regard simplices as closed, and will write  $K$  when we mean the topological space  $|K|$  underlying  $K$ . Let  $K^i$  denote the set of (closed)  $i$ -simplices of  $K$  and let  $(\text{int } K)^0$  and  $(\partial K)^0$  denote the interior and boundary vertices of  $K$ , respectively.

DEFINITION.  $E(K) = \{\text{orientation preserving SL embeddings } K \rightarrow \mathbf{R}^2\}$  and  $E_{\text{sc}}(K) = \{f \in E(K) \mid f(K) \text{ is strictly convex}\}$ .

As in [B, §1],  $E(K)$  and  $E_{\text{sc}}(K)$  are identified with open subsets of Euclidean spaces, and hence their closures  $\overline{E(K)}$  and  $\overline{E_{\text{sc}}(K)}$  are well defined. Let  $E(K, (*\mathbf{R})^2)$  and the “standard part” map  $^\circ$  be as in [B, §1]. The following definition gives the simplest way of extending the notion of strict convexity to the infinitesimal case.

DEFINITION.  $g \in E(K, (*\mathbf{R})^2)$  is *strictly boundary-convex* if

$$\det \begin{pmatrix} 1 & g(w_{i+2}) \\ 1 & g(w_{i+1}) \\ 1 & g(w_i) \end{pmatrix} > 0 \quad (\text{in } *\mathbf{R})$$

for all  $0 \leq i \leq s$ , where  $\{w_0, \dots, w_s\}$  are the boundary vertices of  $K$  in clockwise order.

Note. For a “standard” map  $g \in E(K) \subset E(K, (*\mathbf{R})^2)$ ,  $g$  is strictly boundary-convex iff  $g(K)$  is strictly convex (i.e.  $g \in E_{\text{sc}}(K)$ ).

In §§4 and 5, respectively, we prove the following results.

THEOREM 1.1. *For any  $K$ ,  $E_{\text{sc}}(K)$  is homeomorphic to  $S^1 \times C$  for some contractible  $C$ . In particular,  $E_{\text{sc}}(K)$  has the homotopy type of  $S^1$ .*

THEOREM 1.2. *For any  $K$  and any  $f \in \overline{E(K)}$ , the following are equivalent:*

- (1)  $f \in \overline{E_{\text{sc}}(K)}$ ,
- (2)  $f = ^\circ g$ , for some strictly boundary-convex  $g \in E(K, (*\mathbf{R})^2)$ , and
- (3) for any line  $l$  in  $\mathbf{R}^2$ ,  $f^{-1}(l) \cap \partial K$  has at most two components.

REMARKS. (1) Condition (3) in Theorem 1.2 is a natural generalization of convexity. If  $f \in \overline{E(K)}$  has image a 2-disk, then the condition can be stated more simply:  $f(K)$  is convex and for any two distinct noncollapsed boundary 1-simplices  $A$  and  $B$ ,  $\text{int } f(A) \cap \text{int } f(B) = \emptyset$  (where “int” denotes relative interior, i.e. interior in the affine span). This last condition on  $f|\partial K$  cannot be eliminated. The main case of implication (3) $\Rightarrow$ (1), which requires the bulk of the proof, can then be stated as follows: Suppose  $f: K \rightarrow \mathbf{R}^2$  is the limit of maps in  $E(K)$  (none of which need have convex image); if  $f(K)$  is a convex 2-disk and  $f|\partial K$  behaves nicely, then  $f$  is actually the limit of maps in  $E_{\text{sc}}(K)$ .

(2) Let  $L(K) = \{\text{SL homeomorphisms } K \rightarrow K \text{ fixing } \partial K \text{ pointwise}\}$ , as in [BCH and B]. The methods used to prove Theorem 1.1 can also be used to give an alternate proof that  $L(K)$  is contractible if  $K$  is strictly convex, which is a version of Theorem 5.1 of [BCH].

**2. Preliminary lemmas.**

DEFINITION. Let  $\overline{E_*}(K) = \{f \in \overline{E(K)} \mid f|\partial K \text{ is injective}\}$ .

REMARK. If  $K$  has  $p$  vertices, then  $\overline{E_*}(K) \subset \overline{E(K)} \subset \mathbf{R}^{2p}$ , as mentioned in [B, §1].

DEFINITION. Let  $\pi_x: \mathbf{R}^2 \rightarrow \mathbf{R}^1$  be the projection onto the  $x$ -coordinate and let

$$\Pi_x: \mathbf{R}^{2p} \rightarrow \mathbf{R}^p = \mathbf{R}^1 \times \{0\} \times \mathbf{R}^1 \times \{0\} \times \dots \subset \mathbf{R}^{2p}$$

be the projection map given by

$$(x_1, y_1, x_2, y_2, \dots, x_p, y_p) \mapsto (x_1, x_2, \dots, x_p),$$

so that if  $f: K \rightarrow \mathbf{R}^2$  is SL, then  $\Pi_x(f) = (\pi_x f(v_1), \dots, \pi_x f(v_p))$ .

REMARK. By rotation, any direction in  $\mathbf{R}^2$  could be used as the “ $x$ -direction”. We can think of the “ $x$ -directions” as points in  $S^1$ .

LEMMA 2.1.  $[\overline{E_*(K)} - E(K)] \cap [\bigcup_{x \in S^1} \Pi_x^{-1} \Pi_x E(K)]$  is collared in  $\overline{E_*(K)}$ , where the union is over all possible choices of the “ $x$ -direction” in  $\mathbf{R}^2$ .

PROOF. First, for a fixed  $x \in S^1$ , one can deduce the existence of local collarings using the convex disk decomposition method of [BCH, §4]. The existence of the global collaring follows from the local one by standard arguments (as in [Br]).  $\square$

DEFINITION. Let

$$M(K) = \{f: K \rightarrow \mathbf{R}^2 \mid f \text{ is SL and } f|\partial K \text{ is an orientation preserving embedding}\}.$$

DEFINITION. For  $f, g \in M(K)$ ,  $f$  has the same collapsing as  $g$  if, for any  $\delta \in K^2$ ,  $f(\delta)$  is a 2-simplex, line segment, or a point iff  $g(\delta)$  is, respectively, a 2-simplex, line segment, or a point and, for any  $A \in K^1$ ,  $f(A)$  is a 1-simplex, or a point iff  $g(A)$  is, respectively, a 1-simplex, or a point.

REMARK. If  $f, g \in M(K)$  have the same collapsing, then  $\delta \in K^2$  is of type PC, EC, or SC (as in [B, §2]) with respect to  $f$  iff it is of the same type with respect to  $g$ .

Let  $R(K)$  be as in [B, §1].

LEMMA 2.2. If  $f_t: [0, 1] \rightarrow M(K)$  is a continuous map such that  $f_0 \in R(K)$  and  $f_1 \notin R(K)$ , then, for some  $t_0 \in (0, 1)$ ,  $f_{t_0}$  has different collapsing then  $f_0$ .

PROOF. First, note that for fixed  $\delta \in K^2$ , the map  $M(K) \rightarrow \mathbf{R}$ , given by  $f \mapsto \det(f|\delta)$ , is continuous (see [B, §1] for definitions). The lemma can then be deduced easily, using [B, Lemma 1.1] and the Intermediate Value Theorem.  $\square$

LEMMA 2.3. Let  $f_t: [0, 1] \rightarrow M(K)$  be a continuous map such that  $f_0 \in \overline{E_*(K)}$ , and let  $f_t$  have the same collapsing as  $f_0$  for all  $t \in [0, 1)$ . Then  $f_t \in \overline{E_*(K)}$  for all  $t \in [0, 1]$ .

PROOF.  $\overline{E_*(K)}$  is closed in  $M(K)$ , so it suffices to show  $f_t \in \overline{E_*(K)}$  for all  $t \in [0, 1)$ . Since  $f_0 \in \overline{E_*(K)}$ ,  $f_0 \in R(K)$ , and it follows from Lemma 2.2 that  $f_t \in R(K)$  for all  $t \in [0, 1)$ . Let

$$r = \sup\{t_0 \in [0, 1) \mid f_t \in \overline{E_*(K)} \text{ for } t \in [0, t_0]\},$$

which is well defined. If  $r = 1$  the lemma is proved, so assume  $0 < r < 1$ .  $f_t \in \overline{E_*(K)}$  for  $t < r$  by definition of  $r$ , so  $f_r \in \overline{E_*(K)}$  also; in other words, there is a map in  $E(K)$  as close as desired to  $f_r$ .  $f_t$  has the same collapsing as  $f_r$  for  $t \in [0, 1)$ , so  $\varepsilon(f_t)$  (as in [B, Theorem 1.2]) varies continuously with  $t$  (although  $\varepsilon$  is not a continuous function in general). It now follows by standard arguments that there is some  $\eta > 0$  such that  $[r, r + \eta) \subset [0, 1)$ , and for each  $t \in [r, r + \eta)$  there is a map in  $E(K)$  within  $\varepsilon(f_t)$  of  $f_t$ . [B, Theorem 1.2] now implies that  $f_t \in \overline{E(K)}$  for all  $t \in [r, r + \eta)$ ; since  $f_t \in M(K)$  for all  $t$ , it follows that  $f_t \in \overline{E_*(K)}$  for  $t \in [r, r + \eta)$ , a contradiction; so  $r = 1$ .  $\square$

DEFINITION. Let  $A_1 = \langle a_1, b_1 \rangle$ ,  $A_2 = \langle a_2, b_2 \rangle$  be line segments in  $\mathbf{R}^2$ .  $A_1$  *immediately dominates*  $A_2$  if:

- (1)  $A_1 \cap A_2 = \{a_2\} \in \text{int } A_1$ ; and
- (2)  $A_1$  is not parallel to the  $y$ -axis; and
- (3)  $A_2$  is not parallel to the  $y$ -axis and
- (4) for any  $p \in \pi_x(A_1) \cap \pi_x(A_2)$  such that  $p \neq \pi_x(a_2)$ ,

$$(\pi_x|_{A_1})^{-1}(p) > (\pi_x|_{A_2})^{-1}(p),$$

or

- (3')  $A_2$  is parallel to the  $y$ -axis and
- (4')  $a_2$  has larger  $y$ -coordinate than  $b_2$ . See Figure 2.1(i).

Note that the inequality in (4) holds independently of the choice of  $p$ , so the above definition makes sense.

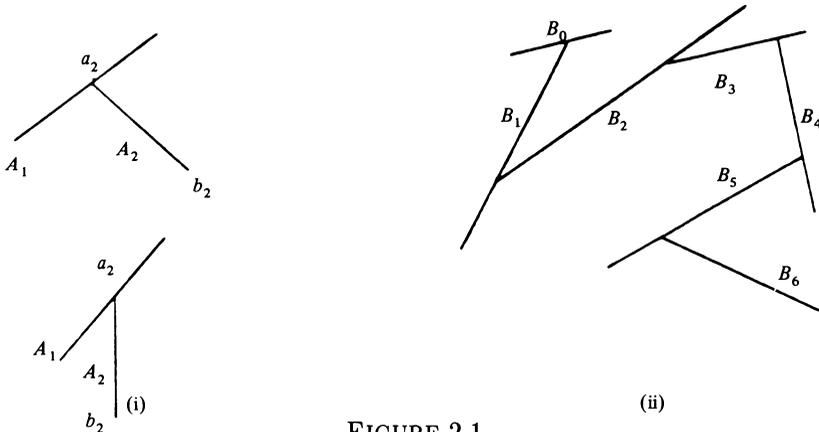


FIGURE 2.1

DEFINITION. Let  $A_1, A_2$  be the line segments in  $\mathbf{R}^2$ .  $A_1$  *dominates*  $A_2$  if there is a chain  $\mathbf{C}$  of line segments  $A_1 = B_0, B_1, \dots, B_m = A_2$ , called a *domination chain*, such that  $B_i$  immediately dominates  $B_{i+1}$  for  $0 \leq i \leq m - 1$ . We write  $D(A_1, A_2; \mathbf{C}) = m$ ; let  $D(A, A; \{A\}) = 0$ . See Figure 2.1(ii).

The following lemma can be proved by elementary methods.

LEMMA 2.4. Let  $A_0, \dots, A_n$  be a domination chain, and suppose

$$A_i \cap A_j = \begin{cases} A_i = A_j, \text{ or} \\ \emptyset, \text{ or} \\ \text{single point, which is an} \\ \text{endpoint of at least one of } A_i \text{ and } A_j. \end{cases}$$

Then  $A_0 \neq A_n$ .  $\square$

**3. The basic homotopy.** Let  $\{w_0, \dots, w_s\}$  be some labeling of the boundary vertices of  $K$  in clockwise order and let  $\mathcal{A} \subset E_{sc}(K)$  be a compact subset. For some boundary vertex, say  $w_0$ , suppose we are given a continuous function  $\phi: \mathcal{A} \rightarrow \mathbf{R}^2$  such that, for each  $g \in \mathcal{A}$ ,  $\phi(g)$  is outside  $g(K)$  and the circle  $(\phi(g), g(w_1), \dots, g(w_s))$  is strictly convex. We will then construct an SL isotopy  $F_t: \mathcal{A} \times [0, 1] \rightarrow E_{sc}(K)$  such that, for all  $g \in \mathcal{A}$ ,  $F_0(g) = g$  and  $F_1(g)(w_0) = \hat{\phi}(g)$ , where  $\hat{\phi}: \mathcal{A} \rightarrow \mathbf{R}^2$  is some (unspecified) continuous function as close as desired to

$\phi$ . To construct  $F_t$ , we will construct a homotopy

$$H_t: \mathcal{A} \times [0, 1] \rightarrow \overline{E_*(K)} \cap \left[ \bigcup_{x \in S^1} \Pi_x^{-1} \Pi_x E(K) \right]$$

such that, for all  $g \in \mathcal{A}$ ,  $H_0(g) = g$ ,  $H_1(g)(w_0) = \phi(g)$ , and  $H_t(g)(K)$  is strictly convex for all  $t \in [0, 1]$ ;  $F_t$  is then obtained from  $H_t$  by using Lemma 2.1 and the compactness of  $\mathcal{A}$ .

To define  $H$ , we fix  $g \in \mathcal{A}$  and define  $H_t(g)$  for all  $t \in [0, 1]$ . Define  $H_0(g) = g$ .  $H_t(g)$  will be defined by specifying  $H_t(g)(v)$  for all  $v \in K^0$ . Without loss of generality we assume that  $\phi(g)$  is the origin in  $\mathbf{R}^2$  and that  $g(w_0)$  is on the positive  $y$ -axis, noting that  $g(w_0)$  can never equal  $\phi(g)$ . The idea for defining  $H_t(g)$  is very simple: the image of  $w_0$  will move uniformly toward the origin, reaching it at  $t = 1$ ; the images of other vertices will move on lines parallel to the  $y$ -axis if they get “caught” on moving 1-simplices. (This is very similar to the proof of Lemma 4.2 in [BS], except that they used radial “tracks” whereas we use parallel “tracks”.) Because of the restrictions on  $\phi$ , and the parallel track method, it is easy to check that all maps are injective on  $\partial K$ , have strictly convex images, and are in  $\bigcup_{x \in S^1} \Pi_x^{-1} \Pi_x E(K)$ , so we will not mention these properties further. (Note that no boundary vertex besides  $w_0$  will have its image moved, by strict convexity.)

More precisely, suppose  $g(w_0) = (0, w_{0,2})$ . Then define

$$H_t(g)(w_0) = (0, (1 - t)w_{0,2}).$$

Define  $H_t^0(g): [0, 1] \rightarrow M(K)$  by  $H_t^0(g)(w_0) = H_t(g)(w_0)$  and  $H_t^0(g)(v) = v$  for all  $v \in K^0$ ,  $v \neq w_0$ . By hypothesis,  $H_0^0(g) = g \in \mathcal{A} \subset E(K)$ . Let

$$\eta_1 = \sup\{t_0 \in [0, 1] \mid H_t^0(g) \in E(K) \text{ for all } t \in [0, t_0]\}.$$

By the continuity of  $H_t^0(g)$  as a function of  $t$  and the fact that  $E(K)$  is open,  $H_t^0(g) \in E(K)$  for all  $t$  close enough to 0, and thus  $\eta_1$  is well defined,  $0 < \eta_1$ .  $H_t^0(g) \in E(K)$  for all  $t \in [0, \eta_1)$ , so the continuity of  $H_t^0(g)$  implies that  $H_{\eta_1}^0(g) \in \overline{E_*(K)}$ ; once again the definition of  $\eta_1$ ,  $E(K)$  being open, and the continuity of  $H_t^0(g)$  imply that, in fact,  $H_{\eta_1}^0(g) \in \overline{E_*(K)} - E(K)$ . Now, define  $H_t(g) = H_t^0(g)$  for all  $t \in [0, \eta_1]$ . If  $\eta_1 = 1$  we have defined  $H_t(g)$  all  $t$ ; if not we proceed as follows.

By [B, Lemma 1.1],  $H_{\eta_1}(g)$  collapses some 2-simplices of  $K$  (as in [B, §2]). Since only the image of  $w_0$  moved for  $t \in [0, \eta_1]$ , any 2-simplex collapses by  $H_{\eta_1}(g)$  must contain  $w_0$  as a vertex. The image of  $w_0$  does not hit the images of any other vertices for  $t \in [0, \eta_1]$ , so all 2-simplices collapsed by  $H_{\eta_1}(g)$  are of type SC (as in [B, §2]). Since only 1-simplices containing  $w_0$  (as an endpoint) have moving images and since the image of  $w_0$  is moved in the negative  $y$ -axis direction, the only possible  $H_{\eta_1}(g)$ -segment complexes (as in [B, §3]) are as seen in Figure 3.1. (Of course, there may be a number of such complexes.)

Let  $\{v_1, \dots, v_n\}$  be the vertices of  $M$  such that  $H_{\eta_1}(g)(v_i) \in \text{int } H_{\eta_1}(g)(A_i)$  for some 1-simplices  $A_i$  whose images are moved for  $t \in [0, \eta_1]$  ( $w_0$  is an endpoint of

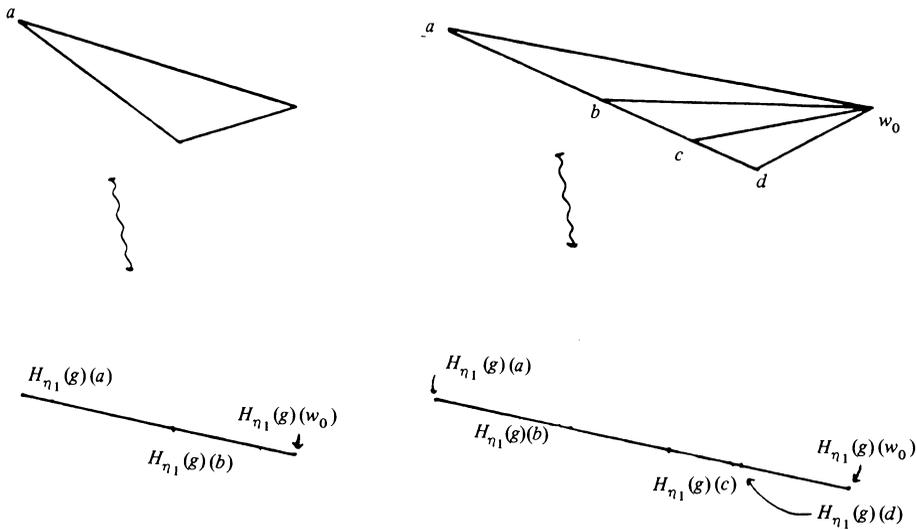


FIGURE 3.1

each  $A_i$ ); there may be more than one choice for each  $A_i$ , but we can always pick it uniquely so that the endpoint of  $A_i$  other than  $w_0$  is not one of the  $v_i$ ,  $1 \leq i \leq m$ . Define  $H_t^1(g) : [\eta_1, 1] \rightarrow M(K)$  as follows:

$$H_{\eta_1}^1(g) = H_{\eta_1}(g), \quad H_t^1(g)(w_0) = H_t(g)(w_0) \quad \text{for all } t,$$

$$H_t^1(g)(v) = v \quad \text{for } v \neq v_i, \quad 1 \leq i \leq m,$$

and  $H_t^1(g)(v_i)$  is the intersection of the lines  $H_t^1(g)(A_i)$  and  $\pi_x^{-1}\pi_x(v_i)$ .  $H_t^1(g)$  is well defined since the image of  $K$  is always strictly convex. It is also evident that for all  $t$  close enough to  $\eta_1$ ,  $H_t^1(g)$  has the same collapsing as  $H_{\eta_1}^1(g)$ .

Let

$$\eta_2 = \sup\{t_0 \in [\eta_1, 1] \mid H_t^1(g) \text{ has the same collapsing as } H_{\eta_1}^1(g) \text{ for all } t \in [\eta_1, t_0]\}.$$

By the above remarks,  $\eta_2$  is well defined and  $\eta_1 < \eta_2$ .  $H_t^1(g)$  is continuous as a function of  $t$ ,  $H_{\eta_1}^1(g) = H_{\eta_1}(g) \in \overline{E_*(K)}$ , and  $H_t^1(g)$  has the same collapsing as  $H_{\eta_1}^1(g)$  for all  $t \in [\eta_1, \eta_2)$ , so Lemma 2.3 implies that  $H_t^1(g) \in \overline{E_*(K)}$  for  $t \in [\eta_1, \eta_2]$ . Define  $H_t(g) = H_t^1(g)$  for  $t \in [\eta_1, \eta_2]$ , which makes  $H_t(g)$  continuous on  $[0, \eta_2]$ , since  $H_{\eta_1}^0(g) = H_{\eta_1}^1(g)$ . If  $\eta_2 = 1$ , we have defined  $H_t(g)$  as desired on  $[0, 1]$ ; otherwise, the above process continues. Assuming  $\eta_2 < 1$ , it is easy to see that  $H_{\eta_2}^1(g)$  collapses some 2-simplices not collapsed by  $H_t^1(g)$  for  $t \in [\eta_1, \eta_2)$ , although everything collapsed for those  $t$  is collapsed in the same way by  $H_{\eta_2}^1(g)$ .

We will now define  $H_t^2(g) : [\eta_2, 1] \rightarrow M(K)$ , with  $H_{\eta_2}^2(g) = H_{\eta_2}(g)$ ,  $H_t^2(g)(w_0) = H_t(g)(w_0)$  for all  $t$ , and find  $\eta_3 > \eta_2$  so that  $H_t^2(g) \in \overline{E_*(K)}$  for  $t \in [\eta_2, \eta_3]$ . We then define  $H_t(g) = H_t^2(g)$  for  $t \in [\eta_2, \eta_3]$ . Also, it will be seen that if  $\eta_3 < 1$  (we are done if  $\eta_3 = 1$ ), then  $H_{\eta_3}^2(g)$  collapses some 2-simplices not collapsed by  $H_t^2(g)$  for  $t \in [\eta_2, \eta_3)$ . We then continue this process, finding  $\eta_4, \eta_5, \dots$  until some  $\eta_k = 1$ , which must be the case, since at each successive  $\eta_i$  more 2-simplices are collapsed and  $K$  is finitely triangulated.  $H_t(g)$  will be defined to equal  $H_t^i(g)$  on

$[\eta_i, \eta_{i+1}]$  for  $i = 0, \dots, k - 1$  (setting  $\eta_0 = 0$ ).  $H_t(g)$  will then be a continuous map  $[0, 1] \rightarrow \overline{E_*(K)}$  with  $H_1(g)(w_0) = \phi(g)$  as desired. Hence it suffices to show how to define  $H_t^2(g)$  satisfying the appropriate conditions.

To define  $H_t^2(g)$  we need the following claim, which can be checked from the definition of  $H_t(g)|_{[0, \eta_2]}$ . Let  $\{\Lambda_1, \dots, \Lambda_p\}$  be the set of  $H_{\eta_2}(g)$ -segment complexes and let  $I_j = H_{\eta_2}(g)(\Lambda_j)$  for  $1 \leq j \leq p$ ;  $I_j$  is a line segment in  $\mathbf{R}^2$ . Since  $H_{\eta_2}(g) \in \overline{E_*(K)}$ , all  $\Lambda_j$  are simple (as in [B, §3]).

CLAIM. (i) Each  $I_j$ , neither end of which contains  $w_0$ , has at least one end with image in the interior of some other (unique)  $I_k$ ;

(ii) no 2-simplex is collapsed to a point by  $H_t(g)$ ,  $t \in [0, \eta_2]$ ;

(iii)  $I_j$  is not parallel to the  $y$ -axis for all  $j$ ;

(iv) the “top” side of  $\Lambda_j$  (which is defined by (iii)) is a single 1-simplex;

(v)  $H_{\eta_2}(g)(w_0) \notin \text{int } I_j$  for any  $j$ .

We now define  $H_t^2(g): [\eta_2, 1] \rightarrow M(K)$  as follows: Let  $H_{\eta_2}^2(g) = H_{\eta_2}(g)$  and let  $H_t^2(g)(w_0) = H_t(g)(w_0)$  for  $t \in [\eta_2, 1]$ . If  $v \neq w_0$  and  $H_{\eta_2}^2(g)(v) \notin \text{int } I_j$  for any  $j$ , let  $H_t^2(g)(v) = v$  for all  $t$ . Because of the claim, we can apply the notions of immediate domination and domination to the  $I_j$ , in order to define  $H_t^2(g)$  on the rest of the vertices of  $K$ . Clearly any two  $I_j$  intersect in at most one point, which is the endpoint of at least one. Lemma 2.4 shows that the following definition makes sense.

DEFINITION. For  $I_j$  as above, let

$$n(I_j) = \sup\{D(I_k, I_j; \mathbf{C}) \mid \mathbf{C} \text{ is a domination chain from } I_k \text{ to } I_j\}.$$

For a vertex  $v$  such that  $H_{\eta_2}(g)(v) \in \text{int } I_j$  (note that  $I_j$  is unique), let  $n(v) = n(I_j) + 1$ . Also, let  $n(w_0) = 0$ , and  $n(v) = -1$  for  $v \neq w_0$  such that  $H_{\eta_2}(g)(v) \notin \text{int } I_j$  for any  $j$ .  $n(I_j)$  and  $n(v)$  are called the *order* of  $I_j$  (resp.  $v$ ).

REMARK.  $n(I_j) = \max\{n(v) \mid v \text{ is a vertex in an end of } \Lambda_j\}$ , where “end” is defined in [B, §3].

So far, we have defined  $H_t^2(g)(v)$  for vertices of order  $-1$  and  $0$ . Next, let  $n(v) = 1$ , so  $H_{\eta_2}^2(g)(v) \in \text{int } I_j$  for some  $I_j$  with  $n(I_j) = 0$ . Define  $H_t^2(g)(v)$  to be the intersection of  $\langle H_t^2(g)(w), H_t^2(g)(w_0) \rangle$  and the line parallel to the  $y$ -axis containing  $H_{\eta_2}^2(g)(v)$ , where  $w$  is any vertex contained in the end of  $\Lambda_j$  not containing  $w_0$  (noting that  $w_0$  is the single vertex in one end of  $\Lambda_j$ , since  $n(I_j) = 0$ ). Any such  $w$  has order  $-1$ , so the choice of  $w$  does not effect  $H_t^2(g)(v)$ . We can continue in this fashion, defining  $H_t^2(g)$  on the vertices of successively higher order, until  $H_t^2(g)$  is defined on all vertices. It is seen that vertices with the same  $H_{\eta_2}^2(g)$ -images have the same  $H_t^2(g)$ -images for all  $t \in [\eta_2, 1]$ , that the  $\Lambda_j$  are the  $H_t^2(g)$ -segment complexes for  $t$  near enough to  $\eta_2$ , and that indeed  $H_t^2(g)$  has the same collapsing as  $H_{\eta_2}^2(g)$  for  $t$  near enough to  $\eta_2$ . As before, there exists maximal  $\eta_3 > \eta_2$  such that  $H_t^2(g) \in \overline{E_*(K)}$  for  $t \in [\eta_2, \eta_3]$ . This completes the definition of  $H_t$ . It only remains to be noted that  $H_t(g)$  is continuous as a function of both  $t$  and  $g$ ; the following claim, which can be proved by induction on  $n(v)$  (with respect to  $t$  and  $g$ ), suffices.

CLAIM. For fixed  $(g, t) \in \mathcal{A} \times [0, 1]$  and for any  $v \in K^0$ ,  $H_s(h)(v)$  is as close as desired to  $H_t(g)(v)$  for all  $(h, s) \in \mathcal{A} \times [0, 1]$  sufficiently close to  $(g, t)$ .

**4. Strictly convex embeddings.**

PROOF OF THEOREM 1.1. For  $f \in E_{sc}(K)$ , one can define the center of gravity of  $f(K)$ , denoted  $\text{cog}(f)$ , in the usual way; the map  $E_{sc}(K) \rightarrow \mathbf{R}^2$  given by  $f \mapsto \text{cog}(f)$  is continuous. Note that  $\text{cog}(f) \in \text{int } f(K)$ . Let  $\{w_0, \dots, w_s\}$  be the boundary vertices of  $K$  in clockwise order.

It is clear that

$$\begin{aligned} E_{sc}(K) &\approx \mathbf{R}^2 \times \{f \in E_{sc}(K) \mid \text{cog}(f) = (0, 0) \in \mathbf{R}^2\} \\ &\approx \mathbf{R}^2 \times S^1 \times D, \end{aligned}$$

where

$$D = \{f \in E_{sc}(K) \mid \text{cog}(f) = (0, 0), f(w_0) \in (0, \infty) \times \{0\}\}.$$

The theorem will be proved if we show  $D$  is contractible.  $D$  is a CW-complex, being dominated by  $E_{sc}(K)$  (which is an open subset of some Euclidean space, and hence a CW-complex itself); see [M, p. 272]. A theorem of Whitehead implies that  $D$  is contractible if we can show all its homotopy groups are trivial, and hence it suffices to show that any compact subset of  $D$  contracts in  $D$  to a point. Let  $C \subset D$  be compact.

Let  $B^2$  be the unit ball in  $\mathbf{R}^2$  and  $S^1$  the unit circle. We then deform  $C$  into a compact set  $\hat{C}$  such that, for  $f \in \hat{C}$ ,  $f(K) \subset \text{int } B^2$ ; this is done by radially shrinking (about  $(0,0)$ ) the images of maps in  $C$  (by which it is meant that the images of vertices are moved radially, thus specifying SL maps, in order to avoid the “standard mistake”). Such a deformation takes place entirely in  $D$  and it will suffice to prove  $\hat{C}$  contracts in  $D$ . To do so, we will construct a homotopy  $F_t: \hat{C} \times [0, 1] \rightarrow G$ , where

$$G = \{f \in E_{sc}(K) \mid f(w_0) \in (0, \infty) \times \{0\}, \text{cog}(f) \in (f(w_0), \infty) \times \{0\}\},$$

such that  $F_0$  is the inclusion of  $\hat{C}$  in  $G$ , and  $F_1(\hat{C})$  is a single point. For any such  $F$ , we construct a map  $H: F(\hat{C} \times [0, 1]) \rightarrow D$  such that  $H$  fixes  $F(\hat{C} \times [0, 1]) \subset D$ ; the homotopy  $H \circ F_t: \hat{C} \times [0, 1] \rightarrow D$  is then the desired contraction of  $\hat{C}$  in  $D$ , so that constructing  $F_t$  will suffice to prove the theorem, once we see that  $H$  exists.  $H$  is constructed as follows: for  $g \in F(\hat{C} \times [0, 1])$ , let  $H(g)$  be the map obtained by first pivoting the image of  $g$  about  $g(w_0)$  (which lies in  $(0, \infty) \times \{0\}$ ) until the center of gravity is in  $(-\infty, g(w_0)) \times \{0\}$  (where this rotation is such that the center of gravity misses  $(g(w_0), \infty) \times \{0\}$ ), and then radially expanding or shrinking about  $g(w_0)$  (as before) the image of the pivoted map until the center of gravity is at  $(0, 0)$ .  $F(\hat{C} \times [0, 1])$  is compact and this definition of  $H$  is seen to be continuous.

Let  $B^\circ(x, r)$  denote the open ball in  $\mathbf{R}^2$  of radius  $r$  centered at  $x$ . Let  $\{p_0, \dots, p_s\}$  be the  $s+1$  roots of unity, lying in clockwise order in  $S^1$  with  $p_0 = (1, 0)$ . Choose  $\varepsilon > 0$  so small so that the  $B^\circ(p_i, 2\varepsilon)$  are disjoint and any polygonal circle  $(z_0, z_1, \dots, z_s)$ , with  $z_i \in B^\circ(p_i, 2\varepsilon)$ , is embedded and bounds a strictly convex 2-disk.

We will construct  $F_t$  in four steps, using the method of §3 repeatedly. The first step,  $F_t|_{[0, \frac{1}{4}]}$ , moves the images of the vertices of  $\partial K$  radially outward toward  $S^1$  (recall that, for  $f \in \hat{C}$ ,  $f(K) \subset \text{int } B^2$ ). We will move one boundary vertex at a time, using the isotopy constructed in §3 to do so. In order to stay in  $E_{sc}(K)$  we we pick some large integer  $n$ , using the compactness of  $\hat{C}$ , so that if we move the images of the vertices of  $\partial K$  radially outward one at a time (going around  $\partial K$   $n$  times) by amounts equal to  $1/n$  of their initial distances from  $S^1$ , then at each

stage the image of  $\partial K$  is strictly convex. It is easy to see what the appropriate  $\phi$  is for each of the isotopies. Because only one boundary vertex moves during each isotopy and the image of that vertex ends up as close as desired to  $\phi(g)$ , we can insist that at the end of  $F_t|[0, \frac{1}{4}]$  the images of the boundary vertices are no more than  $\varepsilon/4$  from  $S^1$ . This deformation clearly stays in  $G$  and we have thus defined  $F_t|[0, \frac{1}{4}]$ .

By a similar process,  $F_t|[\frac{1}{4}, \frac{1}{2}]$  is constructed so that its approximates moving the images of the  $w_i, i \neq 0$ , by a rotation about the origin, so that the final image of each  $w_i$  is close to  $p_i$ . This isotopy moves the images of the  $w_i, i \neq 0$ , one at a time in short straight lines approximating a rotation. We can construct the isotopy so that the image of  $w_0$  is fixed throughout, and  $F_{1/2}(g)(w_i) \in B(p_i, \varepsilon)$  for all  $g \in \hat{C}$  and all  $i$ .

$F_t|[\frac{1}{2}, \frac{3}{4}]$  is constructed as before, this time moving the images of each  $w_i, i \neq 0$ , only once in a straight line toward  $r_i$ , which is the point of  $B(p_i, \varepsilon)$  farthest from  $(0,0)$ . Hence, the final images of the  $w_i$  lie arbitrarily closely to  $r_i$  (except  $w_0$ , whose image is fixed), so that we can insist that they are in  $B(p_i, 2\varepsilon)$ ; also, for each  $0 \leq i \leq s, r'_i \equiv F_{3/4}(g)(w_i)$  is constant for all  $g \in \hat{C}$ . Hence,  $F_{3/4}(g)(\partial K)$  is the strictly convex polygonal circle  $(r'_0, r'_1, \dots, r'_s)$  for all  $g$ . Let  $K_1 = F_{3/4}(g)(K)$  for some choice of  $g \in \hat{C}$ ; it follows that  $F_{3/4}(\hat{C}) \subset L(K_1)$  (see [B, §1] for the definition of  $L(K_1)$ ). Note that since all choices of  $g$  yield SL homeomorphic triangulations of the 2-disk bounded by  $(r'_0, \dots, r'_s)$ , any choice of  $g$  could be used.

Finally, let  $F_t|[\frac{3}{4}, 1]$  be a contraction in  $L(K_1)$  of  $F_{3/4}(\hat{C})$  to a point, using [BCH, Theorem 5.1]. Putting  $F_t|[0, \frac{1}{4}], \dots, F_t|[\frac{3}{4}, 1]$  together, we obtain a contraction of  $\hat{C}$  to a point, and it is easy to check that the contraction remains in  $G$ .  $\square$

**5. Convex near-embeddings.**

LEMMA 5.1. For  $f \in \overline{E(K)}$ , the following two conditions are equivalent:

- (i) for any line  $l$  in  $\mathbf{R}^2, f^{-1}(l) \cap \partial K$  has at most two components;
- (ii)  $f(K)$  is either:
  - (a) a point,
  - (b) a convex 2-disk with  $\partial K$  the union of subcomplexes  $A_1, \dots, A_r, B_1, \dots, B_m$ , where the  $A_i$  are the noncollapsed 1-simplices of  $\partial K$  in clockwise order, each  $B_j$  is a maximal connected (nontrivial) subarc of  $\partial K$  which is mapped to a point,  $f(A_i) \cap f(A_{i+1}) = \{\text{point}\}$  for all  $i \pmod r$ , and  $f(A_i) \cap f(A_j) = \emptyset$  for all  $i$  and all  $j \neq i \pm 1$ , or
  - (c) a line segment with  $\partial K = E_1 \cup S_1 \cup E_2 \cup S_2$ , where the  $E_i$  and  $S_i$  are as in the definition of a simple  $f$ -segment complex (following [B, Lemma 3.3]) and each  $S_i$  is decomposed as is all of  $\partial K$  in case (ii).

PROOF. (i) $\Rightarrow$ (ii). First, let us examine  $f(\partial K)$ . Suppose there exist two distinct points  $x, y \in \partial K$  such that  $f(x) = f(y)$ , yet neither subarc of  $\partial K$  joining  $x$  to  $y$  is mapped to a point. By the hypothesis on  $f$ , if any line  $l$  in  $\mathbf{R}^2$  contains  $f(x) = f(y)$  and intersects  $f(\partial K)$  in some other point (say  $f(z)$ ), then the image of one of the subarcs of  $\partial K$  from  $z$  to  $x$  or  $y$  must lie entirely in  $l$ . Because this holds for any  $f(z)$  in any such  $l$ ,  $f(\partial K)$  must be a figure "V" with vertex at  $f(x) = f(y)$ . In that case, however, we could pick some other  $x', y'$  satisfying the same conditions as  $x, y$ , but with  $f(x') = f(y') \neq f(x) = f(y)$ . It would then follow that  $f(\partial K)$

is a figure “ $V$ ” with vertex at  $f(x') = f(y')$ , and this could only happen if  $f(\partial K)$  were actually a line segment. Assume  $f(\partial K)$  is not a point, or there is nothing to prove. As in [B, Lemma 3.4], it is seen that  $\partial K = E_1 \cup S_1 \cup E_2 \cup S_2$  as desired; it is also seen that for any distinct  $u, v \in S_i$  with  $f(u) = f(v)$ , the arc in  $S_i$  joining  $u$  and  $v$  must be mapped to a point. It then follows that each  $S_i$  can be decomposed as in case (c). If no  $x$  and  $y$  as above exist, then it follows similarly that  $\partial K$  can be decomposed as in (b); if  $\partial K = B_1$  then  $f(\partial K)$  is a point, and otherwise it is a polygonal circle. Finally,  $f \in \overline{E(K)}$ , so an argument like one given in the proof of [B, Lemma 2.4] shows that  $f(K)$  is contained in the region bounded by  $f(\partial k)$  and (i) $\Rightarrow$ (ii) follows easily.

(ii) $\Rightarrow$ (i). This is straightforward.  $\square$

PROOF OF THEOREM 1.2. (3) $\Rightarrow$ (1). We prove  $f \in \overline{E_{sc}(K)}$  in three cases, corresponding to the cases in (ii) in Lemma 5.1.

*Case 1.  $f(K)$  is a point.* Even if  $K$  is not convex, there is always some embedding  $g \in E_{sc}(K)$  by [BS, Theorem 2.2, p. 207]. We may assume that  $f(K)$  is in the interior of  $g(K)$ , and then  $f$  is the limit of a sequence of embeddings in  $E_{sc}(K)$  obtained by radially shrinking the image of  $g$  toward  $f(K)$  (more precisely, radially moving the images of vertices toward  $f(K)$ , to avoid the Standard Mistake).

*Case 2.  $f(K)$  is as in case (ii)(b) of Lemma 5.1.* Because  $f(K)$  is convex,  $\sphericalangle(f(A_i), f(A_{i+1})) \leq \pi$  for all  $i = 1, \dots, r \pmod{r}$ , where  $\sphericalangle(\ , \ )$  denotes the interior angle. In fact, we may assume without loss of generality that if  $A_i$  and  $A_{i+1}$  are not in the same  $f$ -segment complex, then  $\sphericalangle(f(A_i), f(A_{i+1})) < \pi$ ; this condition need not hold in general, but we can always find a map in  $\overline{E(K)}$  arbitrarily close to  $f$  satisfying this condition by moving the images of all the vertices mapped to  $f(A_i) \cap f(A_{i+1})$  by a small amount if  $\sphericalangle(f(A_i), f(A_{i+1})) = \pi$ . Call this new map  $f$ . In other words, we may assume that every natural edge of  $f(K)$  (that is, a line segment which is the intersection of  $\partial f(K)$  with a line in  $\mathbf{R}^2$ ) is either the image of a single 1-simplex or the image of a single  $f$ -segment complex; this condition implies that all vertices  $v \in \partial K$  for which the interior angle of  $\partial f(K)$  at  $f(v)$  is  $\pi$  are, in fact, side vertices of  $f$ -segment complexes. (Note that this argument uses 2-dimensionality in a crucial way; R. Connelly and D. W. Henderson show in [CH] that the analogous “budging” cannot always happen in 3-dimensions.)

We now proceed to “pull  $f(K)$  apart” in such a way that the final image is strictly convex, using the method of proof of [B, Proposition 7.1]. If  $S(f) = 0$ , i.e. no 2-simplices are collapsed by  $f$ , then  $f$  is actually an embedding and the  $A_i$  are all of  $\partial K$ . There are no  $f$ -segment complexes, so by our assumption in the above paragraph  $\sphericalangle(f(A_i), f(A_{i+1})) < \pi$  for all  $i$ , and therefore  $f \in E_{sc}(K)$ . Since  $f$  is arbitrarily close to the original map, it is seen that the original map is in  $\overline{E_{sc}(K)}$ . Assume from now on that  $S(f) > 0$ .

Since  $f \in \overline{E(K)}$ , choose  $g \in E(K)$  very close to  $f$ .  $f(B_i)$  is a point for all  $i$ , so  $g(B_i)$  is contained in a small open disk about  $f(B_i)$ . For each  $B_i$ , we now add some 2-simplices to  $g(\partial K)$ , as follows. First, for each  $B_i$ , we adjoin a convex polygonal arc  $s_i$  to  $g(\partial K)$  where  $s_i \cap g(K) = \{\text{endpoints of } s_i\} = \{\text{endpoints of } g(B_i)\}$ , and the  $s_i$  are disjoint, as in Figure 5.1(i). Next, we add a triangulated collar  $C_i$  to the outside of  $s_i$  such that  $(C_i - s_i) \cap g(K) = \emptyset$ , and the (finite) triangulation of  $C_i$  (which contains  $s_i$  without subdivision) is such that (1) every 2-simplex of  $C_i$  has either one vertex in  $s_i$  and two in  $\partial C_i - s_i$ , or vice-versa, and (2) no vertex

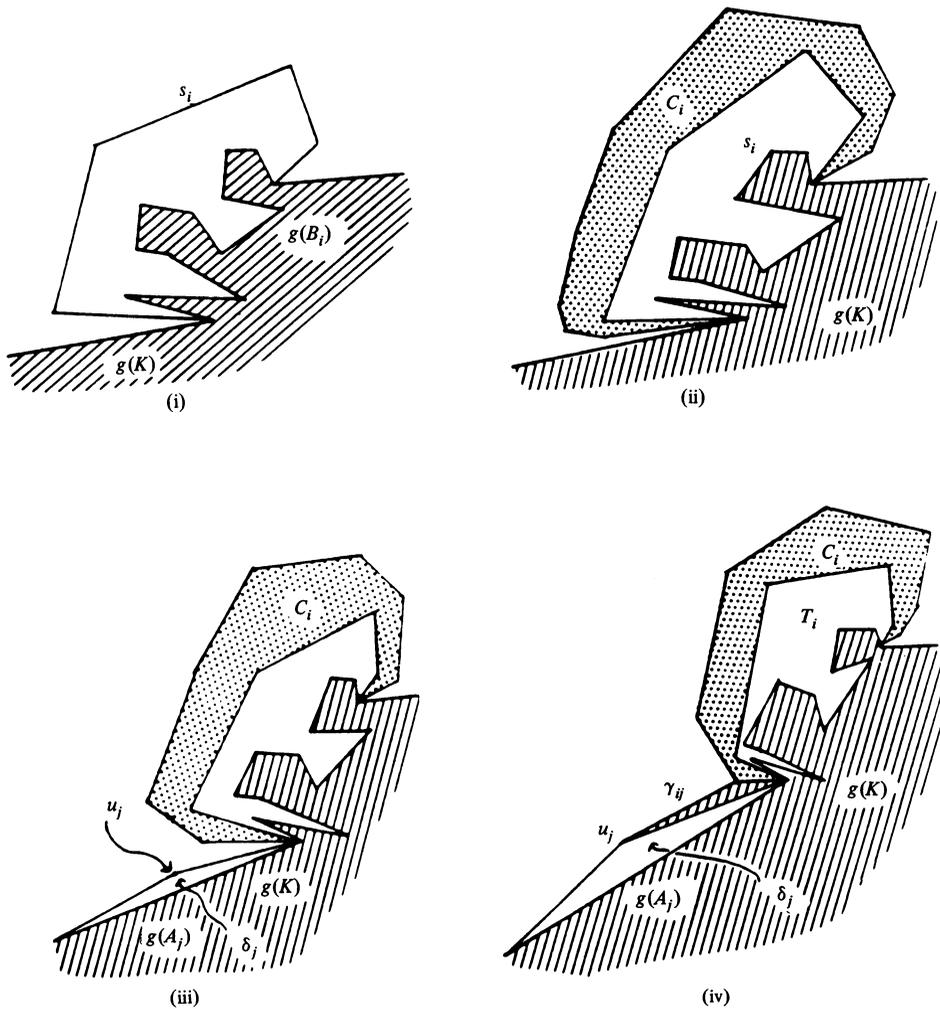


FIGURE 5.1

of  $\partial C_i - s_i$  is contained in more than one 2-simplex which has two vertices in  $s_i$ ; by making  $C_i$  thin enough, we can insist that the  $C_i$  are disjoint. (See Figure 5.1(ii).) For each  $A_j$  which intersects some  $B_i$ , add a very thin 2-simplex  $\delta_j$  to  $G(A_j)$  so that  $\delta_j \cap [g(K) \cup C_1 \cup \dots \cup C_m] = g(A_j)$ , as in Figure 5.1(iii). Let  $u_j$  be the vertex of  $\delta_j$  not in  $\partial g(K)$ . For each  $C_i$  and  $\delta_j$  which intersect (at most in a single point, which is a vertex, and an endpoint of  $s_i, g(B_i)$  and  $g(A_j)$ ), we would like to add the 2-simplex  $\gamma_{ij}$  containing  $u_j$ , the point of intersection of  $C_i$  and  $\delta_j$ , and the vertex of  $\partial C_i - s_i$  which is joinable to  $C_i \cap \delta_j$ , as in Figure 5.1(iv); this 2-simplex might overlap with  $C_i$ , but we could have chosen  $s_i, C_i, \delta_i$  so that  $\gamma_{ij}$  only intersects  $C_i$  and  $\delta_j$  in one boundary 1-simplex each, as in Figure 5.1(iv). Finally, the region  $T_i$  bounded by  $s_i \cup g(B_i)$  is a polyhedral 2-disk, and we extend the triangulation of its boundary to a triangulation of the whole disk (see Figure 5.1(iv)). We now let

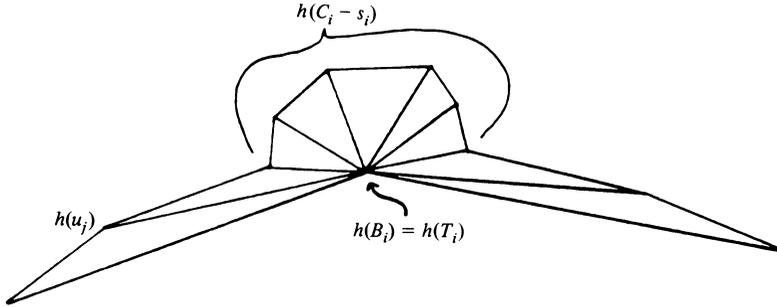


FIGURE 5.2

$K_1$  be the union of  $g(K)$  with all the  $C_i, T_i, \delta_j, \gamma_{ij}$ . It is easy to see that  $K_1$  is a finitely triangulated 2-disk.

The map  $f \circ g^{-1}: g(K) \rightarrow \mathbf{R}^2$  is in  $\overline{E(g(K))}$  and we extend it to an SL map  $h: K_1 \rightarrow \mathbf{R}^2$  by defining  $h$  on the vertices of  $K_1 - g(K)$ , as follows. First, let  $h(u_j)$  be outside of  $f(K)$  very close to the midpoint of  $f(A_j)$ , so that  $\det(h|\delta_j) > 0$  for all  $j$ . Next, map the vertices of  $\partial C_i - s_i$  onto a small circle about  $f(B_i)$  such that the images of these vertices are distinct, in order, and lie outside of  $h(g(K) \cup \delta_1 \cup \delta_2 \cup \dots)$ , as in Figure 5.2. Finally, for  $v \in T_i^0$ , let  $h(v) = f(B_i)$ .

It is easy to see that  $h: K_1 \rightarrow \mathbf{R}^2$  is boundary-nice and in  $R(K_1)$  (as in [B, §§1 and 2]). Also, the edge-point-inverses of  $h$  [B, §2] are either from  $f$  or unions of spanning line segments of 2-simplices in the  $C_i$ , and in either case it is easy to check that these edge-point-inverses are arcs; hence  $h$  is ordered (as in [B, §2]).  $h$  thus satisfies the hypotheses of [B, Proposition 7.1], and we proceed to construct the homotopy  $h_t: [0, 1] \rightarrow \text{OBR}(K_1)$ , with  $h_0 = h$  and  $h_1 \in E(K_1)$ , given by the proposition, with  $\rho$  some arbitrarily small number. By examining the proof of the proposition, it is seen that there is some freedom in how  $h_t$  is constructed, and we will now specify a particular choice of  $h_t$  (which is needed for the proof of this implication).  $h_t$  is constructed as a finite sequence of homotopies, each of which results from “pulling apart” an  $h$ -segment complex or  $h$ -vertex inverse (as in [B, §§3 and 5]). In each pulling-apart, the exact direction and distance in which the set  $V(h_t)$  is moved is not specified in the proof of [B, Proposition 7.1]; we will be more specific now.

As mentioned previously, we may assume that every natural edge of  $h(g(K)) = f(K)$  in either the image of a single 1-simplex of  $\partial g(K)$ , or the image of a single  $h$ -segment complex (noting that by construction the  $h$ -segment complexes with images in  $h(g(K))$  are precisely the  $f$ -segment complexes). Let  $C(t)$  be the following condition:

$$C(t): \text{ For every } E, F \in (\partial g(K))^1 \text{ such that } h_t(E), h_t(F) \text{ are not points and } h_t(E) \cap h_t(F) \neq \emptyset, \text{ then } \sphericalangle(h_t(E), h_t(F)) < \pi.$$

(Note that  $h_t(E)$  and  $h_t(F)$  intersect in precisely one point, by Lemma 5.1, so the condition makes sense.) If every natural edge of  $h(g(K))$  is the image of a single 1-simplex, then  $C(0)$  holds, and set  $t_0 = 0$ ; if not, we will start constructing  $h_t$  in

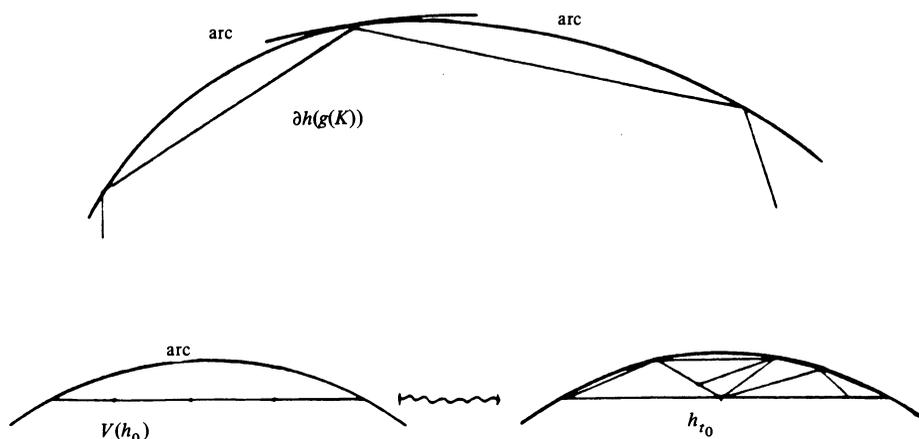


FIGURE 5.3

such a way that  $C(t_0)$  holds for some  $t_0 \in (0, 1)$ .  $h_t|_{[0, t_0]}$  is constructed as follows. The natural edges of  $h(g(K))$  which are not the images of single 1-simplices must be the images of single  $h$ -segment complexes which may have side vertices on their “outside” sides. By the proof of [B, Proposition 7.1], we can start our “pulling-apart” by moving the  $V(h_0)$  corresponding to these side vertices one at a time. Each  $V(h_0) \cap \partial g(K)$  is precisely a single  $B_j$  and, when we move the  $V(h_0)$ , we will not be uncollapsing any of the 1-simplices in the  $B_j$ . More specifically, for each natural edge with these side vertices, construct an arc of a circle intersecting its endpoints, with the arc belonging to a circle of such large radius that (1)  $\text{int } h(g(K))$  lies inside the circle, (2) the region bounded by the union of  $\partial h(g(K))$  and the arc contains no vertices of  $K_1$  not in  $h(g(K))$ , and (3) if any two such arcs intersect (in a common endpoint) the interior angle between their tangents at the point of intersection is strictly less than  $\pi$ . See Figure 5.3. By properly choosing these arcs we can define  $h_t|_{[0, t_0]}$  to be the result of “pulling apart” the  $V(h_0)$ , where each pulling-apart of the  $V(h_0)$  moves images of vertices perpendicularly to the corresponding edges until they reach the appropriate arcs.  $h_{t_0}$  has the images of all the  $V(h_0)$  on the arcs, and clearly  $C(t_0)$  holds. See Figure 5.3.

We now proceed to construct  $h_t|_{[t_0, 1]}$ ;  $C(t_0)$  holds, and we will construct  $h_t|_{[t_0, 1]}$  so that  $C(t)$  holds for all  $t \in [t_0, 1]$ . In particular  $C(1)$  will hold and, since  $h_1$  will be in  $E(K_1)$ , it will follow that  $[h_1|g(K)] \circ g: K \rightarrow \mathbf{R}^2$  will be in  $E_{\text{sc}}(K)$ , thus completing the proof of this case. (Note that  $h_1$  could have been made as close as desired to  $h = h_0$ , so that  $[h_1|g(K)] \circ g$  is as close as desired to  $f$ .)

$h_t|_{[t_0, 1]}$  is constructed by a finite sequence of “pullings-apart”, on which we proceed inductively; assume that after a certain pulling-apart condition  $C(t)$  is still satisfied. In the next pulling-apart, if none of the collapsed 1-simplices of  $\partial g(K)$  are uncollapsed, then, by simply insisting that the vertices of  $K_1$  are moved by very small amounts, it is evident that condition  $C(t)$  can be preserved throughout this pulling-apart. Now assume that in some pulling-apart, some collapsed 1-simplices in  $\partial g(K)$  are pulled apart. These 1-simplices are contained in an  $h_t$ -vertex inverse

which is being pulled apart; since we have freedom in choosing the pulling path used (as in [B, §5]), this path could have been chosen so that it only intersected one 1-simplex of  $\partial g(K)$  (in particular, one end of the path is in  $h_t(g(K))$ , and the other is in  $h_t(K_1 - g(K))$ ), so that we may assume only one 1-simplex of  $\partial g(K)$  is being pulled apart. Let  $E, F \in (\partial g(K))^1$  be the two distinct, noncollapsed 1-simplices of  $\partial g(K)$  which intersect the  $h_t$ -vertex inverse being pulled apart. By assumption, condition  $C(t)$  holds before we start “pulling apart”, so the interior angle between  $h_t(E)$  and  $h_t(F)$  is strictly less than  $\pi$ . Hence, both exterior angles between the lines containing  $h_t(E)$  and  $h_t(F)$  lie outside of  $h_t(g(K))$ . It is easy to see that  $V(h_t)$  may be moved by a small enough amount into the interior of one of these exterior angles, and that by doing so the “uncollapsed” 1-simplex of  $\partial g(K)$  will be pulled apart in such a way that condition  $C(t)$  is satisfied throughout the pulling-apart. This completes the proof of Case 2.

*Case 3.  $F(K)$  is as in Case (ii)(c) of Lemma 5.1.* This case will be reduced to the previous one.  $K$  must be either a single  $f$ -segment complex, or the union of a finite number of  $f$ -segment complexes together with their  $f$ -vertex inverses. In the latter case the images of the  $f$ -segment complexes intersect in at most the images of their ends; if  $E$  is such an end and  $e \in E^0$ , then [B, Lemma 5.1(ii)] implies that  $\Gamma(e)$  (as in [B, §5]) is a 1-connected subcomplex of  $K$ . In either case (one or many  $f$ -segment complexes), we construct  $h: K_1 \rightarrow \mathbf{R}^2$  as in Case 2, and  $g(K) \subset K_1$  is then the union of  $h$ -segment complexes and  $h$ -vertex inverses (which are 1-connected subcomplexes) corresponding to the  $f$ -segment complexes and  $f$ -vertex inverses mentioned above. The set of boundary vertices of  $g(K)$  which are mapped into the interior of  $h(g(K)) = f(K)$  can be divided into two subsets, corresponding to the two sides of  $h(g(K))$ . We can find an arc of a circle, intersecting  $h(g(K))$  in its endpoints, very close to  $h(g(K))$ , as was done for each natural edge in the previous case. We would like to start pulling apart  $h(g(K))$  by moving those boundary vertices of  $g(K)$  corresponding to the side of  $h(g(K))$  on which the arc lies until they lie on the arc. If  $g(K)$  were exactly one  $h$ -segment complex, this could be done as in the previous case; if not, there would still be no problem, since all the  $h$ -vertex inverses in  $g(K)$  are 1-connected subcomplexes, so we can pull them apart at will, rather than waiting until no  $h$ -segment complexes have side vertices as in the proof of [B, Proposition 7.1] (since this last condition was only required in order to insure that no image of an  $h_t$ -vertex inverse is in the interior of the image of an  $h_t$ -segment complex, which holds automatically here). By partially pulling apart  $h(g(K))$  in this fashion while moving vertices less than any given amount, we thus obtain, for some  $t \in [0, 1]$ ,  $[h_t|g(K)] \circ g: K \rightarrow \mathbf{R}^2$  which is as in Case 2, and the present case is proved.

(1) $\Rightarrow$ (3). If  $f \in \overline{E_{sc}(K)}$ , then  $f = \lim f_i$ , for some  $f_i \in E_{sc}(K)$ ,  $i = 1, 2, 3, \dots$ . For any line  $l$  in  $\mathbf{R}^2$ ,  $f_i^{-1}(l) \cap \partial K$  is either empty, or one or two points; it follows that  $f^{-1}(l) \cap \partial K$  has at most two components.

(2) $\Rightarrow$ (3). If (3) does not hold, then it follows from Lemma 5.1 that there must be some line  $l$  in  $\mathbf{R}^2$  such that  $l$  intersects  $f(\partial K)$  in the interiors of the images of at least three distinct, noncollapsed 1-simplices, say  $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle, \langle a_3, b_3 \rangle$ . It then follows that there is a line  $*l$  in  $(*\mathbf{R})^2$  which intersects the interiors of the  $\langle g(a_i), g(b_i) \rangle$ ,  $i = 1, 2, 3$ , where  $\langle \ , \ \rangle$  is now convex hull in  $(*\mathbf{R})^2$ . However, the existence of such a line contradicts the fact that  $g$  is strictly boundary convex (using

the analogous result in the standard case and the Transfer Principle, as in [D, p. 28]), so (3) must hold.

(3) $\Rightarrow$ (2). Proceed as in the proof of (3) $\Rightarrow$ (1), but pull apart by infinitesimally small amounts.  $\square$

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