VARIETIES OF AUTOMORPHISM GROUPS OF ORDERS

BY

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ABSTRACT. The group $A(\Omega)$ of automorphisms of a totally ordered set $\Omega$ must generate either the variety of all groups or the solvable variety of class $n$. In the former case, $A(\Omega)$ contains a free group of rank $2^{\aleph_0}$; in the latter case, $A(\Omega)$ contains a free solvable group of class $n-1$ and rank $2^{\aleph_0}$.

1. Introduction. This work was prompted by the following question of A. Ehrenfeucht (see the footnote on p. 47 of [8]); I thank J. Mycielski for calling it to my attention:

Must the group of automorphisms of an ordered set have a free subgroup of rank $2^{\aleph_0}$ whenever it has a free subgroup of rank 2?

The answer is yes. In fact, if the group $A(\Omega)$ of automorphisms of the (totally) ordered set $\Omega$ fails to satisfy every nontrivial equational group law, then $A(\Omega)$ must contain a free subgroup of rank $2^{\aleph_0}$ (Theorem 5.6).

Phrased in the last way, the question suggests an analogous question for groups $A(\Omega)$ which do satisfy some equational laws. In the first place, one should ask what varieties (equationally defined classes) can be generated by the groups $A(\Omega)$. We will show that these are just the varieties $\mathfrak{S}_n$ of $n$-solvable groups, together with the variety $\mathfrak{S}$ of all groups (Theorem 2.10). Then we show that if $A(\Omega)$ generates $\mathfrak{S}_n$, $A(\Omega)$ contains a free-$\mathfrak{S}_{n-1}$ subgroup of rank $2^{\aleph_0}$ (Theorem 5.3). As a byproduct of this investigation, we can describe precisely the structure of a group $A(\Omega)$ if $A(\Omega)$ generates $\mathfrak{S}_n$ (Corollaries 4.3 and 2.6), thus generalizing a theorem of Chang and Ehrenfeucht [1] from the case $\mathfrak{S}_1 = \text{abelian}$. In the special cases that $A(\Omega)$ is transitive on $\Omega$, we characterize not only the group $A(\Omega)$ but the set $\Omega$ if $A(\Omega)$ generates $\mathfrak{S}_n$ (Theorem 3.2 and Corollary 2.6), thus generalizing a theorem of Ohkuma [10], again from the case $\mathfrak{S}_1$.

In many ways the most natural context for the study of $A(\Omega)$ is as a lattice ordered group. The lattice operations are defined pointwise: for $f, g \in A(\Omega)$ and $\alpha \in \Omega$, $\alpha(f \lor g) = (\alpha f) \lor (\alpha g)$, and dually. The questions posed above have analogues in the language of lattice ordered groups ($l$-groups), where the operations $\land, \lor$ together with the group operations can be used in equational laws for $A(\Omega)$. In Corollary 2.6 we show that the group $A(\Omega)$ generates $\mathfrak{S}_n$ iff the $l$-group $A(\Omega)$ generates $\mathfrak{A}^n$ (the $l$-group analogue of $\mathfrak{S}_n$), and in this case $A(\Omega)$ contains a free-$\mathfrak{A}^{n-1}$ $l$-subgroup of rank $2^{\aleph_0}$ (Theorem 5.2). Finally, the group $A(\Omega)$ generates $\mathfrak{S}$ iff the $l$-group $A(\Omega)$ generates either $\mathcal{N}$ or $\mathcal{L}$, where $\mathcal{N}$ is the join of all $\mathfrak{A}^n$
and \( \mathcal{L} \) is the variety of all \( l \)-groups (Corollary 2.8). In each case \( A(\Omega) \) contains an appropriate free \( l \)-subgroup of rank \( 2^{n_0} \) (Theorems 5.4 and 5.5).

2. The varieties of \( A(\Omega) \). For general background in the theory of ordered permutation groups and \( l \)-groups, the reader is referred to Glass [2]. We deal first with varieties of \( l \)-groups. Let \( \mathcal{A} \) denote the variety of abelian \( l \)-groups, \( \mathcal{A}^0 \) the variety of one-element \( l \)-groups, and, for each positive integer \( n \), \( \mathcal{A}^n \) the variety of those \( l \)-groups \( G \) which have an \( l \)-ideal \( H \) (convex normal sublattice subgroup) such that \( H \in \mathcal{A}^{n-1} \) and \( G/H \in \mathcal{A} \) (see Martinez [6]). Let \( \mathcal{N} \) be the join of all \( \mathcal{A}^n \). Then \( \mathcal{N} \) is covered by the variety \( \mathcal{L} \) of all \( l \)-groups [4, 5]. We show that the variety of \( l \)-groups generated by \( A(\Omega) \), \( l \)-var \( A(\Omega) \), is either \( \mathcal{L}, \mathcal{N} \), or \( \mathcal{A}^n \) for some \( n \).

We make use of the (orbital) wreath product, defined as follows. Let \((K, \Lambda)\) be an ordered permutation group, that is, \( K \) is a subgroup of \( \Lambda(\Lambda) \). For each \( \lambda \in \Lambda \) let \( H_{\lambda} \) be a group with \( H_{\lambda} = H_{\mu} \) when \( \lambda \) and \( \mu \) lie in the same \( K \)-orbit. Let \( H = \prod_{\lambda \in \Lambda} H_{\lambda} \). The wreath product \( \{H_{\lambda}\} \text{ Wr } (K, \Lambda) \) is the splitting extension of \( H \) by \( K \) where conjugation by an element of \( K \) permutes the indices of \( H \) in the obvious way. If \( K \) is a sublattice of \( A(\Lambda) \) and each \( H_{\lambda} \) is an \( l \)-group, then \( G \) becomes an \( l \)-group with the order \( e \leq hk \) (for \( e \) the identity element, \( h \in H, k \in K \)) iff for each \( \lambda \in \Lambda \), \( \lambda < \lambda k \) or both \( \lambda = \lambda k \) and \( h_{\lambda} \geq e \). If each \( H_{\lambda} \) is given as an \( l \)-subgroup of \( A(\theta_{\lambda}) \) for some totally ordered set \( \theta_{\lambda} \), the wreath product can be described another way. Let \( \Omega = \{(\sigma, \lambda) \in (\bigcup_{\lambda \in \Lambda} \theta_{\lambda}) \times \Lambda : \sigma \in \theta_{\lambda} \} \), and order \( \Omega \) lexicographically from the right. Then \( \{H_{\lambda}\} \text{ Wr } (K, \Lambda) \) consists of all \( g \in A(\Omega) \) having the form \((\sigma, \lambda)g = (\sigma g_{\lambda}, \lambda g)\), where \( g \in K \) and \( g_{\lambda} \in H_{\lambda} \) for each \( \lambda \).

Of special interest are the iterated wreath products of the \( l \)-group \( Z \) of integers (permuting itself), \( \text{Wr}^0 Z = \{e\} \), \( \text{Wr}^1 Z = Z \), \( \text{Wr}^2 Z = \{Z\} \text{ Wr } (Z, Z) \), and, generally, \( \text{Wr}^n Z = \{\text{Wr}^{n-1} Z\} \text{ Wr } (Z, Z) \).

An orbital of an ordered permutation group \( A(\Omega) \) is the convexification of an orbit \( \alpha A(\Omega) \), \( \alpha \in \Omega \). Note that if \( \Omega_i \) is an orbital of \( A(\Omega) \), then the restriction of \( A(\Omega) \) to \( \Omega_i \) is \( A(\Omega_i) \), and \( A(\Omega) \) is the full direct product of its restrictions to orbitals. A natural congruence on an orbital \( \Omega_i \) is an equivalence relation on \( \Omega_i \) whose classes are convex, which is respected by the action of \( A(\Omega_i) \), and which satisfies certain technical conditions (see McCleary [7] or Glass [2] for details). For our purpose here, the important properties are that the natural congruences on any orbital form a complete tower under containment, and if \( \mathcal{C}_i \subseteq \mathcal{C} \) are natural congruences such that \( \mathcal{C} \) covers \( \mathcal{C}_i \) (a covering pair) and if \( C \) is a \( \mathcal{C} \) class, then \( A(\Omega) \) induces a primitive component corresponding to \( C \) and \( \mathcal{C}_i \) by restriction to \( C \) followed by the natural homomorphism arising from the congruence \( \mathcal{C}_i \). The primitive component \( K = K(C, \mathcal{C}_i) \) is again an ordered permutation group, and either (i) \( K \) is regular, that is, \( K \) is a subgroup of the real numbers \( R \), and the set permuted by \( K \) is a union of cosets of \( K \) in \( R \), or (ii) \( K \) is 0-2-transitive on one of its orbits, which means that if \( \alpha < \beta \) and \( \gamma < \delta \) are points of that orbit, then there exists \( k \in K \) such that \( \alpha k = \gamma \) and \( \beta k = \delta \) (see McCleary [7] or Glass [2, especially Corollary 4.4.1]). It is also useful to note that if \( C \) is a congruence, then \( A(\Omega_i) \approx \{A(C_{\lambda})\} \text{ Wr } (G, \Omega_i/C) \), where \( \{C_{\lambda} : \lambda \in \Omega_i/C\} \) are the \( C \) classes and the induced group \( G \approx A(\Omega_i/C) \) in case \( A(\Omega_i) \) is transitive (though not in general).

We call an element \( e < f \in A(\Omega) \) join irreducible if \( f = a \lor b \) and \( a \land b = e \) imply \( a = e \) or \( b = e \).
LEMMA 2.1. If on one orbital of \( A(\Omega) \) there are covering pairs \((C_i, C_i')\), \( i = 1, 2, \ldots, n \), of natural congruences with \( C_i \subseteq C_{i+1} \), and there are \( C_i \) classes \( C_i' \) with \( C_i' \subseteq C_{i+1}' \), and induced primitive components \( K_i = K_i(C_i, C_i') \) all nontrivial, then \( A(\Omega) \) contains an \( l \)-subgroup isomorphic to \( \text{Wr}^n \mathbb{Z} \); moreover, we can assume the generator of the upper copy of \( \mathbb{Z} \) is join irreducible.

PROOF. We prove by induction that \( A(C_i') \) contains a copy of \( \text{Wr}^i \mathbb{Z} \). Since \( K_1 \) is nontrivial, there exists a join irreducible \( e < g \in A(C_1') \). Then \( g \) generates an infinite cyclic totally ordered subgroup, so \( A(C_1') \) contains an \( l \)-subgroup isomorphic to \( Z = \text{Wr}^1 \mathbb{Z} \). Assume \( A(C_i') \) contains an \( l \)-subgroup \( H_0 \) isomorphic to \( \text{Wr}^i \mathbb{Z} \). We may identify \( A(C_{i+1}) \) with a subgroup of \( A(\Omega) \) fixing each point not in \( C_i \). Let \( C \) be the \( C_{i+1} \) class containing \( C_i \). Since \( K_{i+1} \) is not trivial, there exists a join irreducible \( e < g \in A(C_{i+1}) \) such that \( Cg \neq C \) (this follows from the technical definition of natural congruence; see \([7 \text{ or } 2]\)). Then the set \( \{Cg^j\} \) is pairwise disjoint. Let \( H_j = g^{-j}H_0g^j \). Clearly \( A(C_{i+1}) \) contains the full direct product \( \prod H_j \), and conjugation by \( g \) permutes the indices of this product. Thus \( A(C_{i+1}) \) contains an \( l \)-subgroup isomorphic to \( H_0 \text{Wr}^i \mathbb{Z} \approx \text{Wr}^{i+1} \mathbb{Z} \).

If \( f \in A(\Omega) \), \( \text{supp} f = \{ \alpha \in \Omega : \alpha f \neq \alpha \} \) and, for any subgroup \( G \subseteq A(\Omega) \), \( \text{supp} G = \{ \alpha \in \Omega : \exists f \in G, \alpha f \neq \alpha \} \).

LEMMA 2.2. If some primitive component of \( A(\Omega) \) is not regular then \( A(\Omega) \) has \( l \)-subgroups \( G_n \approx \text{Wr}^n \mathbb{Z} \), \( n = 1, 2, \ldots, \), such that if \( n \neq m \), \( \text{supp} G_n \cap \text{supp} G_m = \emptyset \).

PROOF. If \( K \) is a nonregular primitive component, \( K \) is 0-2-transitive on one of its orbits. Let \( \alpha < \beta \) be any two points of this orbit. We construct \( G_n \approx \text{Wr}^n \mathbb{Z} \) with \( \text{supp} G_n \subseteq (\alpha, \beta) \). There exists \( k \in K \) such that \( e < k \) and \( \text{supp} k \subseteq (\alpha, \beta) \). Hence \( K \) contains \( \langle k \rangle \approx Z \). If \( \gamma k \neq \gamma \), the intervals \( (\gamma k^i, \gamma k^{i+1}) \) are pairwise disjoint, and \( (\gamma, \gamma k) \) contains \( \text{supp} k' \) for some \( e < k' \in K \). Then \( \text{supp} k^{-i}k'k \subseteq (\gamma k^i, \gamma k^{i+1}) \) and \( K \) contains the full direct product of the groups \( k^{-i}(k')k^i \). Hence \( K \) contains an \( l \)-subgroup isomorphic to \( Z \text{Wr}^i \mathbb{Z} = \text{Wr}^{i+1} \mathbb{Z} \). The construction is completed by induction. We can then choose disjoint intervals \( (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots \) and find \( G_n \approx \text{Wr}^n \mathbb{Z} \) with \( \text{supp} G_n \subseteq (\alpha_n, \beta_n) \).

We let \( l\text{-var} G \) denote the variety of \( l \)-groups generated by \( G \), and \( g\text{-var} G \) the variety of groups.

LEMMA 2.3. \( l\text{-var} \text{Wr}^n \mathbb{Z} = A^n [4, \text{Theorem } 4.6] \) and \( g\text{-var} \text{Wr}^n \mathbb{Z} = \mathfrak{S}_n \) (see Neumann \([9, 22.24 \text{ plus } 17.6]\)).

LEMMA 2.4. If \( A(\Omega) \) has just one orbital and there are only a finite number of natural congruences, then \( A(\Omega) \) can be embedded (as an \( l \)-group) in the iterated wreath product of its primitive components (see Glass \([2]\)).

We also note the following connection between wreath products and varieties. If \( H \in \mathcal{A} \) and each \( G_\lambda \in \mathcal{A}_1 \) then \( \{G_\lambda \} \text{Wr} (H, \Lambda) \in \mathcal{A}_{i+1} \).

THEOREM 2.5. If \( A(\Omega) \) has just one orbital, the following are equivalent:

(i) There exist at least \( n \) nontrivial primitive components or some primitive component is not regular.

(ii) \( \text{Wr}^n \mathbb{Z} \subseteq A(\Omega) \).

(iii) \( A^n \subseteq l\text{-var } A(\Omega) \).

(iv) \( \mathfrak{S}_n \subseteq g\text{-var } A(\Omega) \).
PROOF. That (i) implies (ii) follows from Lemmas 2.1 and 2.2, while (ii) implies both (iii) and (iv) by Lemma 2.3. If there are no more than \( m < n \) covering pairs of natural congruences and all primitive components are regular, then Lemma 2.4 implies \( A(\Omega) \) can be embedded in the iterated wreath product of \( m \) regular, hence abelian, \( l \)-groups, and so \( A(\Omega) \in \mathcal{A}^m \subseteq \mathcal{S}_m \), contradicting both (iii) and (iv). Hence (iii) implies (i), and (iv) implies (i).

COROLLARY 2.6. If \( A(\Omega) \) has just one orbital then the following are equivalent:

(i) There exist exactly \( n \) nontrivial primitive components and all are regular.

(ii) \( \text{Wr}^n Z \subseteq A(\Omega) \) but \( \text{Wr}^{n+1} Z \not\subseteq A(\Omega) \).

(iii) \( l\text{-var} A(\Omega) = \mathcal{A}^n \).

(iv) \( g\text{-var} A(\Omega) = \mathcal{S}_n \).

COROLLARY 2.7. For any \( A(\Omega) \), \( l\text{-var} A(\Omega) = \mathcal{A}^n \) iff \( g\text{-var} A(\Omega) = \mathcal{S}_n \).

COROLLARY 2.8. For any \( A(\Omega) \), \( l\text{-var} A(\Omega) = \mathcal{N} \) or \( \mathcal{L} \) iff \( g\text{-var} A(\Omega) = \mathcal{S} \).

PROOF. \( \mathcal{N} \) is the join of all \( \mathcal{A}^n \) and \( \mathcal{L} \) covers \( \mathcal{N} \). Hence \( l\text{-var} A(\Omega) = \mathcal{N} \) or \( \mathcal{L} \) is equivalent to the statement: for all \( n \) there is an orbital \( \Omega_i \) such that \( l\text{-var} A(\Omega_i) \supseteq \mathcal{A}^n \). In turn, this is equivalent to \( g\text{-var} A(\Omega_i) \supseteq \mathcal{S}_n \); and the corollary follows because \( \mathcal{S} \) is the join of all \( \mathcal{S}_n \) (any free group is a subdirect product of solvable groups).

THEOREM 2.9. \( l\text{-var} A(\Omega) \) is either \( \mathcal{L} \), \( \mathcal{N} \), or \( \mathcal{A}^n \) for some \( n \).

PROOF. Since the named varieties form a complete tower, and since \( A(\Omega) \) belongs to a variety \( \mathcal{V} \) iff the restriction of \( A(\Omega) \) to each of its orbitals belongs to \( \mathcal{V} \), it suffices to prove the theorem assuming \( A(\Omega) \) has just one orbital. If some primitive component is not regular, then \( l\text{-var} A(\Omega) = \mathcal{L} \) [5]. If all primitive components are regular then \( A(\Omega) \in \mathcal{N} \) (Read [11]). The theorem now follows from Corollary 2.6 and the fact that \( \mathcal{N} \) is the join of all \( \mathcal{A}^n \).

THEOREM 2.10. \( g\text{-var} A(\Omega) \) is either \( \mathcal{S} \) or \( \mathcal{S}_n \) for some \( n \).

PROOF. Immediate from Theorem 2.9 and Corollaries 2.7 and 2.8.

COROLLARY 2.11. \( A(\Omega) \in \mathcal{A}^n \) iff \( A(\Omega) \in \mathcal{S}_n \).

It follows from Theorem 2.5 that if \( l\text{-var} A(\Omega) = \mathcal{N} \) then, for each \( n \), \( A(\Omega) \) contains an \( l \)-subgroup isomorphic to \( \text{Wr}^n Z \). In \$5\$ we need the following strong version of this fact.

LEMMA 2.12. If \( l\text{-var} A(\Omega) = \mathcal{N} \) then \( A(\Omega) \) contains \( l \)-subgroups \( G_n \), \( n = 1,2,\ldots \), such that \( G_n \approx \text{Wr}^n Z \) and \( \text{supp} G_i \cap \text{supp} G_j = \emptyset \) if \( i \neq j \).

PROOF. Case 1. Suppose, for every orbital \( \Omega_\lambda \), \( l\text{-var} A(\Omega_\lambda) \not= \mathcal{N} \). Inductively, suppose \( \text{Wr}^i Z \approx G_i \subseteq A(\Omega_\lambda) \), \( i = 1,2,\ldots,n \), with \( \Omega_\lambda_i \not= \Omega_\lambda_j \) when \( i \neq j \). Then \( l\text{-var} A(\Omega_\lambda) = \mathcal{A}^{m_i} \), so there exists \( \Omega_{\lambda_{n+1}} \) such that \( l\text{-var} A(\Omega_{\lambda_{n+1}}) \not\supseteq \mathcal{A}^{m_i} \), \( i = 1,2,\ldots,n \). Hence \( \Omega_{\lambda_{n+1}} \) is different from all \( \Omega_\lambda_i \), \( i \leq n \), and \( A(\Omega_{n+1}) \) contains an \( l \)-subgroup \( G_{n+1} \) isomorphic to \( \text{Wr}^{n+1} Z \) by Theorem 2.5.

Case 2. Suppose \( l\text{-var} A(\Omega_\lambda) = \mathcal{N} \) for some orbital \( \Omega_\lambda \).
Subcase 2a. Suppose $\Omega_\lambda$ has no largest proper natural congruence. Inductively, suppose $W^r Z \approx G_i \subseteq A(\Omega_\lambda)$, $i = 1, 2, \ldots, n$, with each $G_i$ having a join-irreducible generator $g_i$ for its uppermost copy of $Z$. Then $\supp g_i$ is a congruence class $B_i$ (because $A(\Omega_\lambda) \in \mathcal{N}$; see [11]). There exists a join irreducible $g_{n+1}'$ which moves $B_n$. Let $\supp g_{n+1}' = B_{n+1}'. \text{Then the conjugates of } G_n \text{ by powers of } g_{n+1}' \text{ have disjoint support, so } A(\Omega_\lambda) \text{ contains their full direct product which, together with } g_{n+1}', \text{ generates an } l\text{-subgroup } G'_{n+1} \text{ isomorphic to } W^r Z_{n+1}. \text{ There is a proper congruence class } B \supseteq B_{n+1}'. \text{ Let } g \text{ be any element of } A(\Omega_\lambda) \text{ which moves } B. \text{ Let } G_{n+1} = g^{-1}G_{n+1}'g.$

Subcase 2b. Suppose $\Omega_\lambda$ has a largest proper natural congruence $C$. Then $A(\Omega)$ induces $A(C)$ on each $C$ class $C$. It cannot be that $l\text{-var } A(C) \subseteq A^n$ for a fixed $n$ and all $C$, for then $A(\Omega_\lambda) \subseteq A^n+1$. Hence for each $n$ there exists a $C$ class $C_n$ such that $W^r Z \subseteq A(C_n)$, and we may assume $C_n \neq C_m$ when $n \neq m$, since each $C$ class has infinitely many isomorphic translates. The proof can now be completed just as in Case 1.

3. The structure of transitive $A(\Omega) \subseteq A^n$. Ohkuma [10] showed that if $A(\Omega)$ is uniquely transitive (for each $\alpha, \beta \in \Omega$ there exists a unique $g \in A(\Omega)$ such that $ag = \beta$), then $A(\Omega)$ is isomorphic to a subgroup of the additive ordered group of real numbers and $\Omega \approx A(\Omega)$ as ordered sets. Nonzero subgroups of the reals with this property have since been called Ohkuma groups and extensively studied ([3]; see also Glass [2]). Since it is easily seen that a transitive abelian $A(\Omega)$ must be uniquely transitive, Ohkuma’s result may be stated as follows: if $A(\Omega)$ is transitive and nontrivial, then $l\text{-var } A(\Omega) = A$ iff $\Omega$ is an Ohkuma group, and in this case $A(\Omega) \approx \Omega$. In this section we study the analogous problem when $l\text{-var } A(\Omega) = A^n$.

The heart of the main theorem of this section is contained in the following technical lemma.

**Lemma 3.1.** Let $O_1, O_2, \ldots, O_n$ be Ohkuma groups, and $\Omega = O_1 \times O_2 \times \cdots \times O_n$, ordered lexicographically from the right. Then (i) $\Omega$ is not isomorphic to any proper segment of itself, and (ii) no proper initial segment of $\Omega$ is isomorphic to any proper final segment of $\Omega$.

**Proof.** We deal first with the case $n = 1$. The result is clearly true if $\Omega = O_1 \approx Z$. Since all other (noncyclic) subgroups of the reals are dense, we assume $\Omega = O_1$ is a dense subgroup of the reals. If $\phi: \Omega \rightarrow \Omega'$ is an isomorphism (of ordered sets) where $\Omega'$ is a proper segment of $\Omega$, let $f$ be any nontrivial member of $A(\Omega)$. Then $\phi^{-1}f\phi$ is a nontrivial member of $A(\Omega')$, which can be extended to $f' \in A(\Omega)$, fixing all points of $\Omega \setminus \Omega'$. This contradicts the unique transitivity of $A(\Omega)$, hence (i) holds.

Since $\Omega$ is a dense subset of the real numbers, each initial segment of $\Omega$ has the form $(-\infty, \alpha) = \{x \in \Omega: x < \alpha\}$, where $\alpha$ is a real number (not necessarily in $\Omega$). Likewise, each final segment has the form $(\beta, \infty)$. Suppose $\psi: (-\infty, \alpha) \approx (\beta, \infty)$. Let $g$ be any negative member of $A(\Omega)$. Then $(-\infty, \alpha) \approx (-\infty, ag)$, and if we let $\beta' = ag\psi$, there is an isomorphism $\phi: (\beta, \beta') \approx (\beta, \infty)$, which can be extended to an isomorphism of the real intervals $(\beta, \beta') \approx (\beta, \infty)$. Because $\Omega$ is dense, there exists a positive $t \in A(\Omega)$ such that $\beta < \beta t < \beta'$. We may also consider $t$ extended to a map from the reals to the reals. Because both $t$ and $\phi$ preserve order, their extensions are continuous. Since $\beta t < \beta' = \beta \phi$ and $\beta' t < \infty = \beta' \phi$, there exists a
real number $\gamma$ such that $\gamma t = \gamma \phi$ and $\beta < \gamma < \beta'$. Now define $f: \Omega \to \Omega$ by

$$xf = \begin{cases} x & \text{if } x < \beta, \\ x\phi & \text{if } \beta \leq x \leq \beta', \\ xt & \text{if } \beta' < x. \end{cases}$$

Then $f \in A(\Omega)$. But $f$ fixes some points without fixing all, denying unique transitivity of $A(\Omega)$. Hence (ii) is satisfied. This takes care of the case $n = 1$.

Now let $\Omega = O_1 \times O_2 \times \cdots \times O_n$, $n > 1$, and suppose the result is true for $\Lambda = O_1 \times O_2 \times \cdots \times O_{n-1}$. Let $\mathcal{C}_n$ be the equivalence relation on $\Omega$ given by $(a_1, a_2, \ldots, a_n) \mathcal{C}_1 (b_1, b_2, \ldots, b_n)$ iff $a_n = b_n$. If $\phi: \Omega \to \Omega'$ is an isomorphism of $\Omega$ onto a proper segment of itself and $\phi$ preserves $\mathcal{C}_n$ classes, then $\phi$ induces an isomorphism of $O_n$ onto a proper segment of itself, contradicting the result for $n = 1$. Hence for some $\mathcal{C}_n$ class $x \mathcal{C}_n$, $x\phi \mathcal{C}_1 \neq x \mathcal{C}_n \phi$. Each of these two classes is a convex segment of $\Omega$ and is isomorphic to $\Lambda$. Further, $x\phi$ belongs to both. Hence either one is contained in the other, or they overlap so that an initial segment of one is a final segment of the other. But each of these possibilities contradicts the inductive assumption about $\Lambda$. Hence (i) is satisfied by $\Omega$. A similar argument shows that (ii) is satisfied, completing the proof of the lemma.

**Theorem 3.2.** If $A(\Omega)$ is transitive,

$$l\text{-}\text{var } A(\Omega) = \mathcal{A}^n$$

iff $\Omega \cong O_1 \times O_2 \times \cdots \times O_n$,

a lexicographically ordered product of Ohkuma groups; in this case $A(\Omega) \cong O_1 \text{ Wr } O_2 \text{ Wr } \cdots \text{ Wr } O_n$.

**Proof.** If $l\text{-}\text{var } A(\Omega) = \mathcal{A}^n$, then there are exactly $n$ primitive components $O_i$ and are all regular by Corollary 2.6. But a transitive regular primitive component must be an Ohkuma group (see Glass [2]). Hence $A(\Omega)$ can be embedded in the wreath product $O_1 \text{ Wr } O_2 \text{ Wr } \cdots \text{ Wr } O_n$, which acts as an ordered permutation group on the set $O_1 \times O_2 \times \cdots \times O_n$ ordered lexicographically from the right. In the embedding $\Omega \cong O_1 \times O_2 \times \cdots \times O_n$, however, so $A(\Omega) \cong O_1 \text{ Wr } O_2 \text{ Wr } \cdots \text{ Wr } O_n$.

For the converse, suppose that $O_1, O_2, \ldots, O_n$ are given Ohkuma groups and $\Omega = O_1 \times O_2 \times \cdots \times O_n$. We claim that each of the relations $\mathcal{C}_i$, given by $(a_1, a_2, \ldots, a_n) \mathcal{C}_i (b_1, b_2, \ldots, b_n)$ iff $a_j = b_j$ for all $j \geq i$, is a congruence for $A(\Omega)$. This is the case because each of the $\mathcal{C}_i$ classes is isomorphic to the set $\Omega = O_1 \times O_2 \times \cdots \times O_{i-1}$, to which Lemma 3.1 applies. It follows that the primitive components of $A(\Omega)$ are $A(O_i) \cong O_i$, each of which is nontrivial and regular, so $l\text{-}\text{var } A(\Omega) = \mathcal{A}^n$ by Corollary 2.6.

The case of transitive $A(\Omega)$ with $l\text{-}\text{var } A(\Omega) = \mathcal{A}$ appears to be much more complicated. It is still true, just as before, that $\Omega$ is embedded in a lexicographically ordered product of Ohkuma groups, but the embedding need not be onto. Worse yet, even the full product of Ohkuma groups need not have the natural congruences. For example, $\cdots \times Z \times Z \times Z$ is isomorphic to the set of irrational real numbers, which has no proper natural congruence.

**4. The group structure of arbitrary $A(\Omega) \in \mathcal{A}^n$.** A theorem of Chang and Ehrenfeucht [1] states that $A(\Omega) \in \mathcal{A}$ iff $A(\Omega)$, as a group, is a full direct product of subgroups of the real numbers $R$. In order to extend this to higher powers, we define a group $G$ to be of $w\text{ real height} \leq n$ inductively as follows. If $n = 0$, $G = \{e\}$;
in general, $G \cong \{G_\lambda\} \Wr (H, \Lambda)$, where $H$ is a subgroup of $R$ acting regularly on $\Lambda$, a union of orbits of $H$ in $R$, $G_\lambda = \prod_k G_{\lambda,k}$, and each $G_{\lambda,k}$ is a group of wreal height $\leq n - 1$. Note that $G$ has wreal height $\leq 1$ iff $G$ is isomorphic to a subgroup of $R$. We will show that $A(\Omega) \in \mathcal{A}^n$ iff $A(\Omega)$ if a full direct product of groups of wreal height $\leq n$.

**Theorem 4.1.** If $A(\Omega) \in \mathcal{A}^n$, then $A(\Omega)$ is a full direct product of groups of wreal height $\leq n$.

**Proof.** It suffices to prove that if $A(\Omega)$ has just one orbital and belongs to $\mathcal{A}^n$, then $A(\Omega)$ is a group of wreal height $\leq n$. The theorem is trivially true if $n = 0$. In the general case, by Corollary 2.6 and Theorem 2.9, $A(\Omega)$ has exactly $m \leq n$ nontrivial primitive components, and each is regular. If $H$ is the component corresponding to the uppermost covering pair of natural congruences $(C_i, C^i)$, then $H$ is a subgroup of $R$ acting on $\Lambda$, a union of its orbits. On each of the $C_i$ classes $\theta_\lambda$, $A(\Omega)$ induces $A(\theta_\lambda) \in \mathcal{A}^{n-1}$, and so by induction $A(\theta_\lambda)$ is a product of groups of wreal height $\leq n - 1$. Since $A(\Omega) \cong \{A(\theta_\lambda)\} \Wr (H, \Lambda)$, $A(\Omega)$ has wreal height $\leq n$.

**Theorem 4.2.** If $G$ is a product of groups of wreal height $\leq n$, then, for some $\Omega$, $G \cong A(\Omega)$ and $A(\Omega) \in \mathcal{A}^n$.

**Proof.** The theorem is trivial for $n = 0$. We proceed by induction. We may assume $G$ has wreal height $\leq n$, for if $G = \prod P_i$ and each $P_i$ has wreal height $\leq n$, and if we know $P_i \cong A(\Omega_i) \in \mathcal{A}^n$, let $\{\pi_i\}$ be ordinals each greater than $\bigvee_i |P_i|$, and all different, and let $\Omega_i' = \pi_i \times \Omega_i$ ordered lexicographically from the right. Let $\Omega$ be the disjoint union of all $\Omega_i'$ ordered in any way so that each $\Omega_i'$ (as previously ordered) is convex. Then $A(\Omega) = \prod A(\Omega_i') \cong \prod A(\Omega_i) = G$. Hence, we now assume $G$ has wreal height $\leq n$ and is a wreath product as in the definition. By induction, each $G_\lambda \cong A(\theta_\lambda) \in \mathcal{A}^{n-1}$. Let $\Omega^* = \{\sigma, \lambda\} \in (\bigcup \theta_\lambda) \times \Lambda: \sigma \in \theta_\lambda\}$. Then

$$G = \{g \in A(\Omega^*): (\sigma, \lambda)g = (\sigma g_\lambda, \bar{g}), g_\lambda \in A(\theta_\lambda), \bar{g} \in H\}.$$ 

Suppose, in the first case, that $H$ is a proper subgroup of $R$. We consider the regular action $(H, R)$. Following Chang and Ehrenfeucht, for each orbit (coset) $J_i$ of $H$ in $R$, we let $\pi_i$ be an ordinal greater than $\bigvee_i |P_i|$ and all $\pi_i$ different. If we replace each real number $r$ by the ordinal $\pi_i$ when $r \in J_i$, we get a totally ordered set $T$ and a natural action $(H, T)$ such that $H = A(T)$ (see [1]). Moreover, we can assume $\Lambda \subseteq R \subseteq T$. For $\lambda \in \Lambda$ we already have $\theta_\lambda$ defined. For the other $t \in T \setminus \Lambda$, let $\theta_t$ be a single point, so $A(\theta_t) = \{e\}$. Then

$$G \cong \{A(\theta_t): t \in T\} \Wr (A(T), T).$$ 

Let $\Lambda = \{(\sigma, t) \in (\bigcup \theta_t) \times T: \sigma \in \theta_t\}$ be the set acted on by this wreath product. Clearly each $\theta_t \times \{t\}$ is a congruence class for $A(\Omega)$ because if $x < y$ with $x, y \in \theta_t \times \{t\}$, then $|(x, y)| < \pi_i$ for all $i$, while if $x \in \theta_t \times \{t\}$ and $y \in \theta_{t'} \times \{t'\}$ with $t \neq t'$, then some ordinal $\pi_i$ lies between $x$ and $y$. It follows that

$$A(\Omega) \cong \{A(\theta_\lambda)\} \Wr (A(T), T) \cong G.$$ 

In the second case, $H = R$. Then $H$ is isomorphic to a proper subgroup $H'$ of $R$, and as permutation groups (thought not as ordered permutation groups), $(H, H) \cong (H', H')$. It follows that $G \cong \{G_h\} \Wr (H, H) \cong \{G_{h'}\} \Wr (H', H')$, and we proceed as in the first case.
Finally, observe that it follows from induction that any group of wreal height \( \leq n \) belongs to the solvable variety \( \mathcal{G}_n \). Hence, by Corollary 2.11, if \( A(\Omega) \) is a product of groups of wreal height \( \leq n \), then \( A(\Omega) \in \mathcal{A}_n \).

**COROLLARY 4.3.** If \( G \) is a group, there exists \( \Omega \) such that \( G \cong A(\Omega) \in \mathcal{A}_n \) iff \( G \) is a full direct product of groups of wreal height \( \leq n \).

**COROLLARY 4.4** (CHANG AND EHRENFEUCHT [1]). If \( G \) is a group, there exists \( \Omega \) such that \( G \cong A(\Omega) \in \mathcal{A} \) iff \( G \) is a full direct product of subgroups of the real numbers.

5. **Free subgroups of** \( A(\Omega) \). The following lemma is only a slight modification of a theorem of J. Mycielski applied to the special case of \( l \)-groups. We invoke Mycielski's theorem in the proof.

**LEMMA 5.1.** Let \( \mathcal{V} \) be a variety of groups (or of \( l \)-groups). Suppose there are subgroups (or \( l \)-subgroups) \( G_i \subseteq A(\Omega) \), \( i = 1, 2, \ldots \), such that \( \text{supp} G_i \cap \text{supp} G_j = \emptyset \) if \( i \neq j \), and whenever \( (w = e) \) is not a law of \( \mathcal{V} \) (where \( w \) is a word in the free group (or free \( l \)-group) on \( x_1, x_2, \ldots \)), then there exists \( i = i(w) \) such that if \( j \geq i \), \( G_j \) does not satisfy \( (w = e) \). Then \( A(\Omega) \) contains \( \prod G_i \) which contains a free \( \mathcal{V} \) group (or \( l \)-group) on \( 2^{\aleph_0} \) generators.

**PROOF.** Clearly \( A(\Omega) \supseteq \prod G_i \). We first show \( \prod G_i \) has a free subgroup (\( l \)-subgroup) on \( \aleph_0 \) generators. List all those words \( w_1, w_2, \ldots \) such that \( (w_i = e) \) is not a law of \( \mathcal{V} \). Inductively, there exists \( i(n) > n(n - 1) \) such that \( G_{i(n)} \) does not satisfy \( (w_n = e) \); that is, there are \( g(n, 1), g(n, 2), \ldots \in G_{i(n)} \) such that \( w_n(g(n, -)) \neq e \). Let \( g_i = (g(1, i), g(2, i), \ldots) \in \prod G_i \). Then \( \{g_1, g_2, \ldots\} \) is free.

Next, partition \( \mathbb{Z}^+ = P_1 \cup P_2 \cup \ldots \) into an infinite number of infinite sets. Then for each \( j \), \( \{G_i : i \in P_j\} \) satisfies the hypotheses of the lemma. Hence \( G_i' = \prod_{i \in P_j} G_i \) contains a free \( \mathcal{V} \) subgroup (or \( l \)-subgroup) on a countable set. Moreover, \( \text{supp} G_j' \cap \text{supp} G_k' = \emptyset \) when \( j \neq k \), so \( A(\Omega) \supseteq \prod G_j' \). Mycielski's theorem ([8, Corollary 3] and the ensuing discussion) implies that \( \prod G_j' \) contains a free \( \mathcal{V} \) subgroup (or \( l \)-subgroup) on \( 2^{\aleph_0} \) generators.

**THEOREM 5.2.** If \( l \)-var \( A(\Omega) = \mathcal{A}_n \) then \( A(\Omega) \) contains a free \( \mathcal{A}_{n-1} \) \( l \)-subgroup on \( 2^{\aleph_0} \) generators.

**PROOF.** We may suppose \( A(\Omega) \) has just one orbital and a maximal proper natural congruence \( \mathcal{C} \) (Corollary 2.6). For some \( \mathcal{C} \) class \( C \), \( l \)-var \( A(C) = \mathcal{A}_{n-1} \) (otherwise \( A(\Omega) \in \mathcal{A}_{n-1} \)). There are infinitely many disjoint translates of \( C \), so \( A(\Omega) \) contains \( l \)-subgroups \( G_i \cong Wr^{n-1} \mathcal{Z} \) satisfying the hypotheses of Lemma 5.1 for \( \mathcal{V} = \mathcal{A}_{n-1} \) (by Lemma 2.3).

**THEOREM 5.3.** If \( g \)-var \( A(\Omega) = \mathcal{G}_n \), then \( A(\Omega) \) contains a free \( \mathcal{G}_{n-1} \) subgroup on \( 2^{\aleph_0} \) generators.

**PROOF.** The hypothesis is equivalent to that of Theorem 5.2 (by Corollary 2.6).

In the proof of Theorem 5.2, since \( l \)-var \( A(C) = \mathcal{A}_{n-1} \), \( g \)-var \( A(C) = \mathcal{G}_{n-1} \) by Corollary 2.6, and the hypotheses of Lemma 5.1 are satisfied for \( \mathcal{V} = \mathcal{G}_{n-1} \).

**THEOREM 5.4.** If \( l \)-var \( A(\Omega) = \mathcal{N} \) then \( A(\Omega) \) contains a free \( \mathcal{N} \) \( l \)-subgroup on \( 2^{\aleph_0} \) generators.
PROOF. $\mathcal{N}$ is generated by the set of all $\text{Wr}^n Z$ since $\mathcal{N}$ is the join of all $\mathcal{A}^n$ (and using Lemma 2.3). From Lemma 2.12 the hypotheses of Lemma 5.1 are satisfied with $\mathcal{V} = \mathcal{N}$.

THEOREM 5.5. If $l$-var $A(\Omega) = L$ then $A(\Omega)$ contains a free $l$-subgroup on $2^{\aleph_0}$ generators.

PROOF. Some primitive component of $A(\Omega)$ is not regular and so is 0-2-transitive on one of its orbits (see remarks preceding Lemma 2.1). Let $\alpha < \beta$ be points of that orbit. Then the restriction of $A(\Omega)$ to the interval $(\alpha, \beta)$ satisfies no nontrivial $l$-group law [5]. Hence, choosing disjoint intervals, we have $l$-subgroups $G_i \subseteq A(\Omega)$ satisfying the hypotheses of Lemma 5.1 for $\mathcal{V} = L$.

THEOREM 5.6. If $g$-var $A(\Omega) = \emptyset$, then $A(\Omega)$ contains a free subgroup on $2^{\aleph_0}$ generators.

PROOF. We have either $l$-var $A(\Omega) = \mathcal{N}$ or $l$-var $A(\Omega) = L$ by Corollary 2.8. In either case $A(\Omega)$ has subgroups $G_n \approx \text{Wr}^n Z$ satisfying the hypotheses of Lemma 5.1 for $\mathcal{V} = \emptyset$ (by Lemmas 2.2, 2.3 and 2.12).

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