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ANOSOV DIFFEOMORPHISMS AND EXPANDING IMMERSIONS. I

BY

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ABSTRACT. The purpose of this paper is to develop a theory for representing Anosov diffeomorphisms by expanding immersions on compact branched manifolds. This theory was motivated by R. F. Williams' study of expanding attractors [**15**, **17**].

0. Introduction. Recall that a *hyperbolic attractor* for a diffeomorphism $g\colon M \to M$ on a smooth compact manifold M is a compact subset Λ of $M - \partial M$ satisfying

0.1. (a) $g\colon \Lambda \to \Lambda$ is a homeomorphism;

(b) the tangent bundle of M splits over Λ, $T(M)_{|\Lambda} = \xi_u \oplus \xi_s$ as the Whitney sum of continuous subbundles each of which is left invariant by dg (the differential of g);

(c) for some Riemannian metric $|\ \ |$ on M, $|dg(v)| > |v|$ for each nonzero $v \in \xi_u$ and $|dg(w)| < |w|$ for each nonzero $w \in \xi_s$;

(d) there is a compact neighborhood U for Λ in M so that $g(U) \subset U$ and $\bigcup_{n=1}^{\infty} g^n(U) = \Lambda$.

The diffeomorphism $g\colon M \to M$ is *Anosov* if $\Lambda = M$.

The attractor Λ is an *expanding attractor* if $g\colon \Lambda \to \Lambda$ has no wandering points and the dimension of the fiber in ξ_u equals the topological dimension of Λ.

In his fundamental papers [**15**, **17**] R. F. Williams reduced the study of expanding attractors to studying expanding immersions on branched manifolds with no boundary. Williams showed that every expanding attractor is obtained by a simple construction from an expanding immersion of a compact branched manifold; and conversely every expanding immersion on a compact branched manifold which satisfies three properties (see 1.5) yields an expanding attractor.

It is the purpose of this paper to develop a theory (similar to Williams') for representing Anosov diffeomorphisms by expanding immersions on compact branched manifolds (without boundary) which are equipped with some additional structure called a *Local Collapsing Structure* (L.C.S.). An L.C.S. for an expanding immersion $f\colon K \to K$ consists of a finite collection of local charts for K with respect to which f satisfies three properties (see 1.2). There is a construction which associates to any Anosov diffeomorphism $g\colon M \to M$ an expanding immersion $f\colon K \to K$ equipped with an L.C.S. (see 1.6).

In §2 a construction is introduced, similar to Williams' solenoid construction [**17**], which associates to any expanding immersion $f\colon K \to K$ equipped with an L.C.S. a homeomorphism $F'\colon \Sigma'(f) \to \Sigma'(f)$. *It is conjectured that* $F'\colon \Sigma'(f) \to \Sigma'(f)$ *is*

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always an Anosov diffeomorphism (see 2.5 below). The truth of this conjecture would open a new approach to constructing exotic Anosov diffeomorphisms. This conjecture will be verified for K of dimension 1 in a later paper [**9**]. If $f\colon K \to K$ is obtained from an Anosov diffeomorphism $g\colon M \to M$ as in 1.6 below, then $F'\colon \Sigma'(f) \to \Sigma'(f)$ is equal to $g\colon M \to M$ (see 2.4).

One recent development in Williams' program studying expanding attractors has been the discovery by F. T. Farrell and L. Jones [**3**, **4**] that this class of attractors is much richer than had been expected. What had been expected was that the existence of an expanding immersion $f\colon K \to K$ on a branched manifold would place very strong restrictions on K, f and so (via Williams' theory) lead to strong restrictions on expanding attractors. This expectation arose from the study of smooth expanding immersions $g\colon N \to N$ on compact smooth manifolds N: it was known that the universal cover of N is Euclidean space (due to M. Shub [**11**]); $\pi_1(N)$ has polynomial growth (due to J. Franks [**5**]); and the topological conjugacy class of $g\colon N \to N$ is determined by the algebraic conjugacy class of $\pi_1(g)\colon \pi_1(N) \to \pi_1(N)$ (due to M. Shub [**11**]). More recently it has been shown that N is an infranil manifold (due to M. Gromov [**6**]). It was thought that similar theorems could be proven for expanding immersions on branched manifolds. Farrell and Jones showed that this could not be done by developing a general technique for constructing expanding immersions on branched manifolds which satisfy Williams' three properties. Their constructions yield expanding immersions on branched manifolds of almost any desired tangential homotopy type (e.g. simply connected with nonvanishing Pontrjagin classes).

It is the author's belief that the existence of an L.C.S. for an expanding immersion $f\colon K \to K$ on a branched manifold is the simplest additional structure which will place restrictions on K and f similar to those restrictions on $g\colon N \to N$ discussed in the last paragraph. In §4 three such possible restrictions on f, K are discussed (see 4.1–4.3). It is shown how these three topological restrictions would imply corresponding restrictions on Anosov diffeomorphisms (see 4.5–4.9). The reader should note that the results of §4 do not depend on the truth or falseness of the conjecture in §2.

In developing the L.C.S. approach to studying Anosov diffeomorphisms, the author has been lead to a theory for studying all hyperbolic attractors, which includes both Williams' theory of expanding attractors and the L.C.S. approach to Anosov diffeomorphisms. This more general theory is developed alongside the L.C.S. theory in the text below. The standing technical assumption for this more general theory, as for the L.C.S. theory, is that the stable foliation is C^r for some $r \geq 1$.

Of the many sources that the author has drawn from while preparing this paper, he has been especially guided by the work of S. Smale [**13**], and R. F. Williams [**15**, **17**]. Conversations with D. Pixton and M. Shub have also been helpful.

1. Definition of an L.C.S. It will be useful to first recall some definitions from [**17**].

A *compact* C^r $(r > 1)$ *branched k-manifold* consists of a compact metrizable space K together with a finite collection of closed subsets $\{U_i\}$, $\{D_{ij}\}$ and mappings $\{\pi_i\colon U_i \to X_i\}$ satisfying the following conditions.

1.1. (a) $\{U_i^\circ\}$ is a covering of K, $U_i^\circ \equiv \text{interior}(U_i)$.

(b) $\bigcup_j D_{ij} = U_i$.

(c) Each X_i is a closed smooth k-dimensional ball in R^k and π_i: $D_{ij} \to X_i$ is a homeomorphism $\forall_{i,j}$.

(d) There is a cocycle of C^r diffeomorphisms α_{il}: $\pi_l(U_l \cap U_i) \to \pi_i(U_l \cap U_i)$ such that $\pi_i = \alpha_{il} \circ \pi_l$.

The *tangent bundle* of K, $T(K)$, is obtained by glueing all the linear bundles $U_i \times R^k \to U_i$ together along the linear isomorphisms, $1 \times d\alpha_{il}$: $(U_i \cap U_l) \times R^k \to (U_i \cap U_l) \times R^k$.

A C^r *immersion* f: $K \to K$ is a continuous mapping such that all the compositions

$$X_i \xrightarrow{(\pi_i/D_{ij})^{-1}} D_{ij} \xrightarrow{f} K \supset U_l \xrightarrow{\pi_l} X_l$$

are C^r immersions, where defined.

Note, any smooth immersion f: $K \to K$ induces a bundle map df: $T(K) \to T(K)$, which is nonsingular on each fiber, by the formula $df_{|Y} \equiv d(\pi_l \circ f \circ \pi_i^{-1})$ where $Y = T(K)_{|U_i \cap f^{-1}(U_l)}$.

A C^r immersion f: $K \to K$ is *expanding* if there exists a Riemannian metric $|\ |$ on $T(K)$, and numbers $a > 0, \lambda > 1$, so that $|(df)^n(v)| > a\lambda^n|v|$ holds for any $v \in T(K)$ and any integer $n > 0$.

We come now to the purpose of this section.

DEFINITION 1.2. The sets $\{U_i\}$, $\{D_{ij}\}$ and mappings $\{\pi_i$: $U_i \to X_i\}$ of 1.1 constitute a *local collapsing structure* (L.C.S.) for the expanding immersion f: $K \to K$ if the following properties are satisfied.

(a) If $x, y \in f^{-}(U_i)$ and $\pi_i(f(x)) = \pi_i(f(y))$, then there is a sequence $x_1, x_2, \ldots, x_l$ in K with $x_1 = x$, $x_l = y$, so each pair $x_j, x_{j+1} \in U_j^\circ$, with $\pi_{j'}(x_j) = \pi_{j'}(x_{j+1})$ for some j'.

(b) There is an integer $q > 0$ so that for each $x \in K$, and each integer $q \le l \le 2q - 1$, there is a U_i so that $f^l(K_x) \subset U_i^\circ$ and $\pi_i(f^l(K_x))$ is a single point. Here $K_x \equiv \{y \in K | x, y \in U_j \text{ and } \pi_j(x) = \pi_j(y) \text{ for some } j\}$.

(c) f: $K \to K$ is onto.

The sets $\{U_i\}$, $\{D_{ij}\}$ and mappings $\{\pi_i$: $U_i \to X_i\}$ of 1.1 constitute a *Weak Local Collapsing Structure* (W.L.C.S.) for the expanding immersion f: $K \to K$ if they satisfy 1.2(b), (c).

EXAMPLE 1.3. Let h: $N \to N$ be an expanding immersion of the C^r compact connected manifold N (without boundary). Let $\{D_{i,1}\}$ denote a finite collection of smooth n-dimensional balls in N ($n = \dim(N)$) such that $U_i D_{i,1}^\circ = N$. Set $U_i \equiv D_{i,1}$ and select π_i: $U_i \to X_i$ to be a diffeomorphism. Then these $\{U_i\}$, $\{D_{i,j}\}$, $\{\pi_i$: $U_i \to X_i\}$ constitute a W.L.C.S. for h: $N \to N$. *This W.L.C.S. is not an L.C.S., because it does not satisfy* 1.2(a). To see this choose $x, y \in N$ so that $x \ne y$ but $h(x) = h(y)$. Then by 1.2(a) there is a sequence $x_1, x_2, \ldots, x_l \in N$ with $x_1 = x$, $x_l = y$, $\pi_{j'}(x_j) = \pi_{j'}(x_{j+1})$ for some j' and all $1 \le j \le l - 1$. Since each $\pi_{j'}$ is one-to-one we conclude $x = x_1 = x_2 = \cdots = x_l = y$ which contradicts $x \ne y$.

The above example motivated the following proposition, which will be proven at the end of this section.

PROPOSITION 1.4. *A C^1-expanding immersion h: $N \to N$ on a C^1 compact connected manifold without boundary never processes an L.C.S.*

EXAMPLE 1.5. Let f: $K \to K$ be an immersion satisfying Williams' three properties (see [**17**, 3.0]): (1) f: $K \to K$ is expanding; (2) for each $x \in K$ there is a neighborhood N of x and an integer $j > 0$ such that $f^j(N)$ is a subset of a C^1 k-cell in K ($k = \dim K$); (3) f: $K \to K$ has no wandering points. Using (2) and a compactness argument (K is compact), one concludes that $\{U_i\}$ and $\{\pi_i\colon U_i \to X_i\}$ of 1.1 can be chosen so that each π_i: $U_i \to X_i$ is C^1 conjugate to f^q: $U_i \to f^q(U_i)$ for some positive integer q. These $\{U_i\}$, $\{\pi_i\colon U_i \to X_i\}$ must then satisfy 1.2(b), (c), and so constitute a W.L.C.S. for f: $K \to K$. *This W.L.C.S. is never an L.C.S. because it does not satisfy* 1.2(a). To see this choose $x, y \in K$ so that $f^q(x) \neq f^1(y)$, but $f^{q+1}(x) = f^{q+1}(y)$. Assuming 1.2(a), it follows from Lemma 1.10 that there are $x_1, x_2, \ldots, x_l \in K$ so that $x_1 = x$, $x_l = y$, $x_j, x_{j+1} \in U_{j'}$ with $\pi_{j'}(x_j) = \pi_{j'}(x_{j+1})$ for all $1 \leq j \leq l-1$. Since $\pi_{j'}(x_j) = \pi_{j'}(x_{j+1}) \Leftrightarrow f^q(x_j) = f^q(x_{j+1})$, we have $f^q(x) = f^q(x_1) = f^1(x_2) = \cdots = f^q(x_l) = f^q(y)$ which contradicts the choice of x, y.

The results 1.3, 1.4 and 1.5 indicate a basic fact of life: it is easy to find expanding immersions with a W.L.C.S., but very difficult to find expanding immersions equipped with an L.C.S. The only method known to the author for constructing expanding immersions with an L.C.S. is contained in the following.

PROPOSITION 1.6. *Let $\Lambda \subset M$ be a hyperbolic attractor for a diffeomorphism g: $M \to M$, such that the stable foliation of Λ is C^r. Then there is a construction which associates to every Markov partition for g: $\Lambda \to \Lambda$ a C^r expanding immersion f: $K \to K$ equipped with a W.L.C.S. This W.L.C.S. is actually an L.C.S. if $\Lambda \cap W^s(x)$ is path connected for each stable leaf $W^s(x)$, $x \in \Lambda$.*

REMARK 1.7. The only known hyperbolic attractors g: $\Lambda \to \Lambda$ such that $\Lambda \cap W^s(x)$ is path connected for each stable leaf $W^s(x)$, $x \in \Lambda$, are topologically conjugate to Anosov diffeomorphisms on infranil manifolds. So the expanding immersions with an L.C.S. that can be derived from 1.6 are a very restricted class (at present). An important problem is to find other methods of obtaining expanding immersions equipped with an L.C.S.

Proposition 1.6 will be proven in §3. However the idea of the proof is contained in the following construction of a one-dimensional expanding immersion with L.C.S. The reader is urged to study this example before reading the proof of 1.6.

EXAMPLE 1.8. Let g: $T^2 \to T^2$ denote a linear Anosov diffeomorphism of the two-dimensional torus, and denote by $\{P_i\}$ a Markov partition of g: $T^2 \to T^2$ consisting of parallelograms in T^2. Let $\partial_u P_i$, $\partial_s P_i$ (see Figure 1.9) denote the arcs in ∂P_i lying in the unstable and stable leaves of g respectively. If $P_i \cap P_j \neq \varnothing$ let A_{ij} denote the arcs in P_i (either $0, 1, 2$ arcs) obtained by continuing the arcs of $\partial_u P_j$ through P_i (see Figure 1.9).

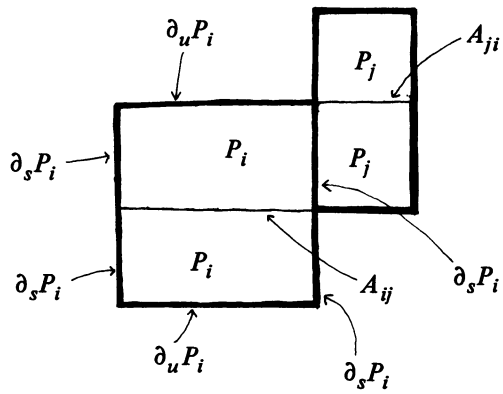

FIGURE 1.9

Define a subset $Y' \subset T^2$ by

$$(T^2 - Y') \cap (P - \partial_s P_i) \equiv (P_i - \partial_s P_i) - (\cup_j A_{ij});$$
$$(T^2 - Y') \cap \partial_s P_i \equiv \partial_s P_i - (\cup_j [\partial_s P_j \cap \partial_u P_j]).$$

For $x, y \in T^2 - Y'$ set $x \sim y$ iff x, y are in the same connected arc of $L_s \cap (T^2 - Y')$ for some stable leaf L_s. Note $g(T^2 - Y') \subset T^2 - Y'$, and $g(x) \sim g(y)$ if $x \sim y$ for $x, y \in T^2 - Y'$ (these follow from the properties of a Markov partition). So an expanding immersion $f\colon K \to K$ can be defined by setting K equal to the quotient space $T^2 - Y'/\sim$, and letting f be the quotient of the map $g\colon T^2 - Y' \to T^2 - Y'$.

To get the $\{\pi_i\colon U_i \to X_i\}$ for K, which will constitute the L.C.S. for $f\colon K \to K$, choose parallelograms $\{V_i\}$ in T^2, one for each P_i, which have much bigger diameters than the $\{P_i\}$. Let X_i denote the unstable factor of V_i (we assume V_i is the cartesian product $X_i \times Y_i$ where $X_i \times y$, $x \times Y_i$ are arcs in unstable and stable leaves). Set $U_i \equiv V_i - Y'/\sim$ and define π_i to be the composition

$$U_i \xrightarrow{\rho^{-1}} V_i \xrightarrow{b_i} X_i,$$

where $\rho\colon T^2 - Y' \to K$ is the quotient map and b_i the projection.

The remainder of this section will be concerned with proving Proposition 1.4. The following lemma will be needed for this.

LEMMA 1.10. *$f\colon K \to K$ denote an expanding immersion with L.C.S. as in* 1.2. *If for some i and $q > 0$ and $x, y \in f^{-q}(U_i)$ we have $\pi_i(f^q(x)) = \pi_i(f^q(y))$, then there is a sequence $x_1, \ldots, x_l$ in K so each pair $x_j, x_{j+1} \in U_j^\circ$, with $\pi_j(x_j) = \pi_{j'}(x_{j+1})$ for some j', and with $x_1 = x$, $x_l = y$.*

PROOF OF 1.10. Note $f^{q-1}(x)$, $f^{q-1}(y) \in f^{-1}(U_i)$ and $\pi_i(f(f^{q-1}(x))) = \pi_i(f(f^{q-1}(y)))$, so by 1.2(a), there is a sequence $y_1, y_2, \ldots, y_t$ in K with $y_1 = f^{q-1}(x)$, $y_t = f^{q-1}(y)$ so that each pair $y_s, y_{s+1} \in U_{s'}$ with $\pi_{s'}(y_s) = \pi_{s'}(y_{s+1})$ for some s'. Choose points $Z_1, \ldots, Z_t \in K$ with $f^{q-1}(Z_j) = y_j$ (see 1.2(c)). Note Z_s, $Z_{s+1} \in f^{-(q-1)}(U_{s'})$ with $\pi_{s'}(f^{q-1}(Z_s)) = \pi_{s'}(f^{q-1}(Z_{s+1}))$; so (assuming 1.10 for $q - 1$)

there is a sequence of points $Z_{s,1}Z_{s,2},\ldots,Z_{s,\phi(s)}$ in K with $Z_{s,1} = Z_s$, $Z_{s,\phi(s)} = Z_{s+1}$ so that each pair $Z_{s,j}Z_{s,j+1} \in U^\circ_{\psi(s,j)}$ with $\pi_{\psi(s,j)}(Z_{s,j}) = \pi_{\psi(s,j)}(Z_{s,j+1})$ for some $\psi(s, j)$. The sequence $Z_{1,1}, Z_{1,2},\ldots, Z_{1,\phi(1)}, Z_{2,1}, Z_{2,2},\ldots, Z_{2,\phi(2)},\ldots, Z_{t,1}, Z_{t,2},\ldots, Z_{t,\phi(t)}$ will satisfy the conclusion of Lemma 1.10, provided $Z_1 = x$ and $Z_t = y$.

Proof of Proposition 1.4. Let $f\colon K \to K$ denote an expanding immersion on a smooth compact manifold with no boundary. Let $\{U_i\}$, $\{D_{i,j}\}$, $\{\pi_i\colon U_i \to X_i\}$ constitute a W.L.C.S. for $f\colon K \to K$. It will suffice to show that this W.L.C.S. cannot satisfy 1.2(a). The main step in the proof of this is to verify the following.

1.11. There is a map $\pi\colon K \to L$ and a number $\gamma > 0$ so that for any $z \in L$ the cardinality of $\pi^{-1}(z)$ is less than γ. Moreover, for any $x, y \in K$ if there is an i so that $x, y \in U_i$ and $\pi_i(x) = \pi_i(y)$, then $\pi(x) = \pi(y)$.

Here is how to complete the proof of 1.4 from 1.11. Choose q large enough so that $f^{-q}(p)$ has cardinality greater than γ of 1.11 for some $p \in K$. Choose i so $p \in U_i$. Then by 1.10 there is for any pair $x, y \in f^{-q}(p)$ a sequence $x_1, x_2,\ldots,x_l \in K$ so that $x_1 = x$, $x_l = y$, and for all $1 \le j < l$ there is j' with $x_j, x_{j+1} \in U_{j'}$ and $\pi_{j'}(x_j) = \pi_{j'}(x_{j+1})$. So by 1.11, and the last two equalities, we have $\pi(x) = \pi(x_1) = \cdots = \pi(x_l) = \pi(y)$. Thus $f^{-q}(p) \subset \pi^{-1}(\pi(x))$. But the cardinality of $f^{-q}(p)$ is greater than γ, and that of $\pi^{-1}(\pi(x))$ is less than γ (by 1.11), which is the desired contradiction.

In the remainder of this proof it will be assumed that

1.12. $q = 1$ in 1.2(b).

There will be no loss of generality in this assumption for if $q > 1$, replace $f\colon K \to K$ by $f^q\colon K \to K$. Note that the L.C.S. for $f\colon K \to K$ is also an L.C.S. for $f^q\colon K \to K$ (by 1.10).

It remains to verify 1.11. Denote the compositions

$$D_{ik} \xrightarrow{\pi_i} \pi_i(U) \xrightarrow{\pi_i^{-1}} D_{il}$$

by $g^i_{kl}\colon D_{ik} \to D_{il}$. We will need the following property of the g^i_{kl}.

1.13. If $x \in D^\circ_{ik} \cap D^\circ_{jq}$ and $g^i_{kl}(x) = g^j_{qp}(x)$, then there is a neighborhood U of x in $D^\circ_{ik} \cap D^\circ_{jq}$ so that $g^i_{kl}(x') = g^j_{qp}(x')$ for all $x' \in U$.

To verify 1.13 note (by 1.1(d), 1.2(b), 1.12) that for some index r and all $x' \in U$ ($U =$ some ε-neighborhood of x) we have:

(a) $f(x'), f(g^i_{kl}(x')), f(g^j_{qp}(x')) \in U^\circ_r$;

(b) $\pi_r(f(x')) = \pi_r(f(g^i_{kl}(x'))) = \pi_r(f(g^j_{qp}(x')))$;

(c) $f \circ g^i_{kl|U}, f \circ g^j_{qp|U}$ are both one-to-one.

Note that U°_r is the *disjoint* union of the $D^\circ_{r1}, D^\circ_{r2},\ldots,$ *because K is a manifold*. So $fg^i_{kl}(U), fg^j_{qp}(U) \subset D^\circ_{rs}$ for some s, because $fg^i_{kl}(x) = fg^j_{qp}(x)$. But $\pi_{r|D_{rs}}$ is one-to-one, so by (b) above we have

(d) $fg^i_{kl|U} = fg^j_{qp|U}$.

Now from (c), (d), and $g^i_{kl}(x) = g^j_{qp}(x)$, we deduce $g^i_{kl|U} = g^j_{qp|U}$, the desired conclusion of 1.13.

We define multivalued maps $G^i_{klm}\colon K \to K$ as follows:

$$y \in G^i_{klm}(x) \quad \text{iff} \quad x = f^m(x'), y = f^m(y'), y' = g^i_{kl}(x'),$$

for some x', y' in K. The following property of the (G^i_{klm}) can be deduced from 1.1(d), 1.2(b), and 1.12, 1.13.

1.14. (a) For sufficiently large m, each $x \in K$ has a neighborhood U so that $G^i_{klm|U} = \bigcup g^j_{qp|U}$, where the union runs over a subset of $\{g^j_{qp}: U \subset D_{jq}\}$.

(b) If for some j, q, p and some $x \in D^\circ_{jq}$ it is true that $g^j_{qp}(x) \in G^i_{klm}(x)$, then $g^j_{qp}(x') \in G^i_{klm}(x')$ for all $x' \in D_{jq}$. Here again this is true only for sufficiently large m.

Choose an integer $n > 0$ (by 1.14) so that for any $m \geq n$ any G^i_{klm} satisfies 1.14. It follows from 1.13, 1.14, that if in the above construction of the G^i_{klm} (and throughout the rest of this proof) f is replaced by f^n, then the following will be true:

1.15. (a) Each G^i_{klm} is equal to one of the $\{G^{i'}_{k'l'1}\}$.

(b) Each G^i_{klm} satisfies 1.14 for all $m \geq 1$.

We also have, by 1.15(a), (b), 1.13 that

(c) Set $G = (\bigcup_{ikl} G^i_{kl1}) \cup 1_K$; i.e. $G(x) = (\bigcup_{ikl} G^i_{kl1}(x)) \cup x$. Then for each $q > 0$ and each $x \in K$, there is a neighborhood U of x in K so that $f^q \circ G_{|U} = \bigcup_{i=1}^q h_i$, where $(h_i: 1 \leq i \leq n_q)$ is a finite set of smooth embeddings satisfying $h_i(U) \cap h_j(U) = \varnothing$ if $i \neq j$.

(d) Note that n_q of 1.15(c) must be independent of $x \in K$, and $n_q \geq n_{q+1}$ for all $q \geq 1$.

Now the key step in verifying 1.11 is the following:

CLAIM 1.16. There is an integer $p > 0$ so that

$$f^p \circ \underbrace{G \circ G \circ G \circ \cdots \circ G}_{k\text{-fold}} = f^p \circ G,$$

for all integers $k > 0$.

To verify 1.16 for $k = 2$, note $G \subset G \circ G$ (because $1_K \subset G$). So $f^p \circ G \subset f^p \circ G \circ G$ for all p. By 1.2(b), 1.12 and 1.15(a), there is $r > 0$ so that $f^r \circ G \circ G \subset G \circ f^r$. Now choose q large enough in 1.15(d) so that $n_q = n_{q+i}$ for all $i > 0$. Set $p = q + r$. Then $f^p \circ G(x)$ has n_p values for all $x \in K$ (see 1.15(d)), and $n_p = n_q$. So to show $f^p \circ G(x) = f^p \circ G \circ G(x)$ for all x it suffices to show the cardinality $|f^p \circ G \circ G(x)| \leq n_q$ (recall $f^p \circ G(x) \subset f^p \circ G \circ G(x)$). Note $f^p \circ G \circ G(x) = f^q \circ f^r \circ G \circ G(x) \subset f^q \circ G \circ f^r(x)$ and $|f^q \circ G(f^r(x))| = n_q$. This completes the verification of 1.16 for $k = 2$. The general case follows by noting

$$f^p \circ \underbrace{G \circ G \circ G \circ \cdots \circ G}_{k\text{-fold}} = \underbrace{f^p \circ G}_{f^p \circ G} \circ \underbrace{G \circ \cdots \circ G}_{(k-2)\text{-fold}} = f^p \circ \underbrace{G \circ G \circ \cdots \circ G}_{(k-1)\text{-fold}}.$$

Finally 1.11 is deduced from 1.16 as follows. Define an equivalence $\sim$ on K by

$$x \sim y \quad \text{iff} \quad f(y) \in \underbrace{G \circ G \circ \cdots \circ G}_{k\text{-fold}}(f(x))$$

for some k. Note if for some i, $x, y \in U_i$ and $\pi_i(x) = \pi_i(y)$, then $x \sim y$. Note also, by 1.16, $x \sim y \Rightarrow f^{p+1}(y) \in f^p \circ G(f(x))$; so there is an upper bound, say $\gamma > 0$, on the cardinality of any one $\sim$-equivalence class of points in K. Set $L = K/\sim$, and let $\pi: K \to L$ denote the quotient map. Then $\pi: K \to L$ satisfies 1.11.

This completes the proof of 1.4.

2. The quotient solenoid construction. Recall that the *solenoid* of a smooth expanding immersion $f\colon K \to K$ (denoted $\Sigma(f)$) is the inverse limit of the maps

$$K \xleftarrow{f} K \xleftarrow{f} K \xleftarrow{f} \cdots$$

(see [**17**]). Any point in $\Sigma(f)$ is an infinite tuple $(a_0, a_1, a_2, a_3, \ldots)$ with $a_i \in K$, $f(a_i) = a_{i-1}$. A continuous map $F\colon \Sigma(f) \to \Sigma(f)$ is defined by $F(a_0, a_1, a_2, \ldots) = (f(a_0), a_0, a_1, a_2, \ldots)$. F is called the *shift map*.

Now suppose $f\colon K \to K$ is equipped with a W.L.C.S. consisting of the charts $\{\pi_i\colon U \to X_i\}$. Define an equivalence relation $\sim$ on $\Sigma(f)$ by $(a_0, a_1, a_2, \ldots) \sim (b_0, b_1, b_2, \ldots)$ if and only if for each $i \geq 0$ there is $U_{i'} \subset K$ with $a_i, b_i \in U_{i'}$ and $\pi_{i'}(a_i) = \pi_{i'}(b_i)$. It follows from 1.2(b) that $\sim$ satisfies the properties of an equivalence relation, and that $\sim$ is preserved by $F\colon \Sigma(f) \to \Sigma(f)$.

DEFINITION 2.1. The *quotient solenoid*, for a smooth expanding immersion $f\colon K \to K$ equipped with a W.L.C.S., is denoted by $\Sigma'(f)$ and is defined as the quotient space $\Sigma(f)/\sim$. The mapping $F'\colon \Sigma'(f) \to \Sigma'(f)$ is defined as the quotient of $F\colon \Sigma(f) \to \Sigma(f)$; F' is called the *shift map*.

EXAMPLE 2.2. For $f\colon K \to K$ as in 1.3 or 1.5 above, the quotient solenoid with shift map $F'\colon \Sigma'(f) \to \Sigma'(f)$ is equal to the solenoid with shift map $F\colon \Sigma(f) \to \Sigma(f)$.

EXAMPLE 2.3. For $f\colon K \to K$ as in 1.8, the shift map $F'\colon \Sigma'(f) \to \Sigma'(f)$ is topologically conjugate to the linear Anosov diffeomorphism $g\colon T^2 \to T^2$ of 1.8. This can be verified directly, or deduced from the following proposition (which will be proven in §3).

PROPOSITION 2.4. *Let $\Lambda \subset M$ denote a hyperbolic attractor for the diffeomorphism $g\colon M \to M$, such that Λ has a C^1 stable foliation. Let $f\colon K \to K$ denote an expanding immersion equipped with W.L.C.S. associated to a Markov partition for $g\colon \Lambda \to \Lambda$ as in* 1.6. *Then the shift map $F'\colon \Sigma'(f) \to \Sigma'(f)$ is topologically conjugate to $g\colon \Lambda \to \Lambda$.*

This last proposition, together with 1.6 and 1.7 in §1, has motivated the following.

CONJECTURE 2.5. If $f\colon K \to K$ is an expanding immersion equipped with an L.C.S., then the shift map on the quotient solenoid $F'\colon \Sigma'(f) \to \Sigma'(f)$ is topologically conjugate to an Anosov diffeomorphism.

In a later paper [**9**] the above conjecture will be verified when $\dim(k) = 1$.

The following question arises at this point: what should $F'\colon \Sigma'(f) \to \Sigma'(f)$ look like if $f\colon K \to K$ is equipped merely with a W.L.C.S. which doesn't satisfy 1.2(a). My guess is the following

CONJECTURE 2.6. $F'\colon \Sigma'(f) \to \Sigma'(f)$ is always a hyperbolic attractor.

The supporting evidence for 2.6 is Proposition 2.4 and Example 2.2.

The conjectures made above, especially 2.5, are very closely related to the following conjecture made by S. Smale (see [**13**, p. 785]).

ATTRACTOR CONJECTURE. Any hyperbolic attractor which has no wandering points is locally homeomorphic to the product of some Euclidean space with some Cantor space.

Note it follows from Propositions 1.6, 2.4 that

2.7. Conjecture 2.5 $\Rightarrow$ Attractor Conjecture

for the class of hyperbolic attractors Λ with C^1 stable foliation such that $\Lambda \cap W^s(x)$ is path connected for each stable leaf $W^s(x)$, $x \in \Lambda$.

In a later paper [**9**] it will be shown that the Attractor Conjecture very nearly implies 2.5, provided 2.6 is known to be true.

Historical Note. Smale originally made his conjecture for any hyperboic set $\Lambda \subset M$ of a diffeomorphism g: $M \to M$, such that g: $\Lambda \to \Lambda$ is a transitive homeomorphism whose periodic points are dense, and Λ has a neighborhood U in M so that $\bigcap_{n \in Z} g^n(U) = \Lambda$. One-dimensional counterexamples to this more general conjecture were given by J. Guckenheimer [**7**]. Later L. Jones [**8**] developed a more general technique for constructing hyperbolic sets with very complicated local topological types. R. F. Williams and Robinson [**10**, **17**] verified the conjecture for expanding attractors. To date no counterexamples exist to Smale's original conjecture which are attractors.

3. Attractors as quotient solenoids. In this section Propositions 1.6 and 2.4 will be proven.

It is useful to review some of the local properties of hyperbolic attractors. For g: $\Lambda \to \Lambda$ as in 0.1, there is a local product structure for Λ (see [**2**, 3.3]) consisting of closed subsets $\{V_i\}$ of Λ and homeomorphisms $\{h_i\colon V_i \to V_i^s \times V_i^u\}$, satisfying

3.1. (a) $\bigcup_i V_i^\circ = \Lambda$.

(b) $h_i^{-1}(p \times V_i^u)$ and $h_i^{-1}(V_i^s \times q)$ are subsets of an unstable leaf and stable leaf, respectively, of g: $\Lambda \to \Lambda$, for any $i, p \in V_i^s$, $q \in V_i^u$.

(c) V_i^u is a smooth k-ball in R^k (k = dimension of unstable leaves).

For notational simplicity any subset $S \subset V_i^s \times V_i^u$ will be identified with $h_i^{-1}(S)$ in the remainder of this section.

Every hyperbolic attractor g: $\Lambda \to \Lambda$ has a Markov partition (see [**2**, 3.12]). A *Markov partition* for g: $\Lambda \to \Lambda$ consists of a finite set of compact subsets $\{P_i\}$ of Λ satisfying the following properties.

3.2. (a) Diameter$(P_i) < 10^{-1}\delta$, where δ is the Lebesgue number of the covering $\{V_i^\circ\}$.

(b) There are subsets $P_i^s \subset V_{i'}^s$ and $P_i^u \subset V_{i'}^u$ with $P_i = P_i^s \times P_i^u$ for some $V_{i'}$.

(c) Closure$(P_i^\circ) = P_i$, where P_i° is the interior of P_i. Moreover $P_i^\circ \cap P_j^\circ = \varnothing$ if $i \neq j$, and $\bigcup_i P_i = \Lambda$.

(d) If $g(q \times (P_i^u)^\circ) \cap p \times P_j^u \neq \varnothing$, then $g(q \times P_i^u) \supset p \times P_j^u$ for any $q \in P_i^s$, $p \in (P_j^s)^\circ$. If $g^{-1}((P_i^s)^\circ \times q) \cap P_j^s \times p \neq \varnothing$, then $g^{-1}(P_i^s \times q) \supset P_j^s \times p$ for any $q \in P_i^u$, $p \in (P_j^u)^\circ$. Thus $g(X) \subset X$ and $g^{-1}(Y) \subset Y$, where $X = \bigcup_i P_i^s \times \partial P_i^u$ and $Y = \bigcup_i \partial P_i^s \times P_i^u$, where $\partial P_i^s(\,) = P_i^s - (P_i^s)^\circ$ and $\partial P_i^u = P_i^u - (P_i^u)^\circ$.

There are the following useful subsets of $\Lambda - Y$.

S_n: $x \in S_n$ if and only if $x \in \Lambda - Y$ and x is contained in exactly n of the partition sets $\{P_i\}$.

$\Lambda - Y'$: for $x \in S_n$ let $\bigcap_{j=1}^n P_{i_j}$ denote the maximal intersection of partition sets $\{P_i\}$ containing x. Let x', $A_1, A_2, \ldots, A_l$ denote the projection to $P_{i_1}^s$ of x and of all

partition sets P_i satisfying $P_i \cap (\bigcup_{j=1}^n P_{i_j}) \neq \varnothing$. Then $x \in \Lambda - Y' \Leftrightarrow x' \in (P_{i_1}^s)^\circ - (\bigcup_{j=1}^l \partial A_j)$, where ∂A_j is the topological boundary of A_j in $P_{i_1}^s$. The sets S_n and $\Lambda - Y'$ satisfy the following by virtue of 3.2(d).

(e) $g(\Lambda - Y') \subset \Lambda - Y'$; $g(\bigcup_{j \geq n} S_j) \subset \bigcup_{j \geq n} S_j \forall n > 0$.

PROOF OF PROPOSITION 1.6. $\{P_i\}$, $Y' \subset \Lambda$ and S_n are as in 3.2. An equivalence $\sim$ is defined on the points of $X - Y'$ be setting $x \sim y$ if and only if the following properties hold.

3.3. (a) $x, y \in S_n$ for some n.

(b) $x, y \in \bigcap_{j=1}^n P_{i_j}$, where $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ are n distinct members of the Markov partition $\{P_i\}$.

(c) Let x', y' denote the projection to $P_{i_1}^s$ of x, y, and let $A_1, A_2, \ldots, A_l$ denote the projection to $P_{i_1}^s$ of all partition sets P_i such that $P_i \cap (\bigcap_{j=1}^n P_{i_j}) \neq \varnothing$. Then $x' \in A_k \Leftrightarrow y' \in A_k$ for all $1 \leq k \leq l$.

(d) x, y project to the same point under $P_{i_1} \to P_{i_1}^u$.

Define K as the quotient space $\Lambda - Y'/\sim$. Define subsets $U_i \subset K$ as the quotient spaces $V_i - Y'/\sim$. In each U_i define a finite collection of subsets $\{D_{ij}\}$ as the distinct images in U_i of all $q \times V_i^u \subset V_i - Y'$ under the quotient map β_i: $V_i - Y' \to U_i$. Set $X_i \equiv V_i^u$. There is a unique map π_i: $U_i \to X_i$ such that the composite

$$V_i - Y' \xrightarrow{\beta_i} U_i \xrightarrow{\pi_i} X_i$$

equals the projection $V_i - Y' \to V_i^u$.

It follows directly from 3.2(a), (b), (c) that the sets $\{U_i\}$, $\{D_{ij}\}$, and maps $\{\pi_i: U_i \to X_i\}$ defined above give to K a branched manifold structure as in 1.1(a), (b), (c). By virtue of the C^r hypothesis for the stable foliation of g: $\Lambda \to \Lambda$, 1.1(d) is also satisfied.

Note if $x, y \in \Lambda - Y'$ and $x \sim y$, then $g(x), g(y) \in \Lambda - Y'$ and $g(x) \sim g(y)$ (see 3.2(d), (e)). Thus the quotient of g: $\Lambda - Y' \to \Lambda - Y'$ (under $\sim$) is a well-defined map f: $K \to K$, which is a C^r expanding immersion by virtue of 3.1(c) and the C^r hypothesis for the stable foliation of g: $\Lambda \to \Lambda$.

Note f: $K \to K$ satisfies 1.2(c) because g: $\Lambda \to \Lambda$ is a homeomorphism. Property 1.2(b) is satisfied by virtue of 3.1(b), 3.2(a).

It remains to verify 1.2(a) when $\Lambda \cap W^s(z)$ is path connected for each stable leaf $W^s(z)$, $z \in \Lambda$. Let $x, y \in K$ be as in 1.2(a), and choose representatives $\bar{x}, \bar{y} \in \Lambda - Y'$ for x, y. Since $f(x), f(y) \in U_i$ with $\pi_i(f(x)) = \pi_i(f(y))$, it must be that $\bar{x}, \bar{y} \in \Lambda \cap W^s(z)$ for some stable leaf $W^s(z)$. Choose a path γ: $[0,1] \to \Lambda \cap W^s(z)$ such that $\gamma(0) = \bar{x}$, $\gamma(1) = \bar{y}$. Choose a sequence of points $t_1 < t_2 < \cdots < t_l \in [0,1]$ so that

3.4. (a) $t_1 = 0$, $t_l = 1$.

(b) $\gamma(t_i), \gamma(t_{i+1}) \in V_{i'}^s \times q$ for some i' and some $q \in V_{i'}^u$, where the distance from $\gamma(t_i)$ and $\gamma(t_{i+1})$ to $\partial V_{i'}^s \times q$ is greater than $10^{-1}\delta$ (see 3.2(a)).

By 3.2(c), 3.4(a), (b) there are points $y_1, y_2, \ldots, y_l$ in Λ satisfying

(c) $y_i, y_{i+1} \in V_{i'}^s \times q - Y'$ for i', q as in (b).

(d) $y_1 = \bar{x}$, $y_l = \bar{y}$.

Now let $x_i \in K$ denote the image of y_i under $\Lambda - Y' \to K$. Then from 3.4(c), (d) it follows that $x_1, x_2, \ldots, x_l$ is the desired sequence in 1.2(a).

This completes the proof of 1.6.

PROOF OF PROPOSITION 2.4. First a mapping $h\colon \Lambda \to \Sigma'(f)$ will be constructed. Let $\{V_i\}$, $\{P_i\}$, Y be as in 3.1, 3.2; $\pi\colon \Lambda - Y' \to K$ will denote the quotient map (see proof of 1.6). Given any $x \in \Lambda$ it follows from 3.2(c), 3.3 that there is a sequence of points $y_0, y_1, y_2, \ldots, y_j, \ldots$ in $\Lambda - Y'$ satisfying

3.5. (a) For each $j > 0$ there is a partition set $P_{j'}$ and $q_j \in P_{j'}^u$ so that y_j, $g^{-j}(x) \in P_{j'}^s \times q_j$.

(b) $\pi(g(y_j)) = \pi(y_{j-1})$.

Define $h\colon \Lambda \to \Sigma'(f)$ by $h(x) = (\pi(y_0), \pi(y_1), \pi(y_2), \ldots)$ for $x, \{y_j\}$ as in 3.5(a), (b). Note 3.5(b) $\Rightarrow f(\pi(y_j)) = \pi(y_{j-1})$, so $h(x)$ lies in $\Sigma'(f)$. To see that $h(x)$ depends only on x, choose another sequence $\{y_j'\}$ in $\Lambda - Y'$ which satisfies 3.5(a), (b) for $\{P_{j''}^s x q_j'\}$. Note $x \in P_{j''} \cap P_{j'}$ (3.5(a)), so the diameter $(P_{j''} \cup P_{j'}) < 2\varepsilon$, where ε is the maximum diameter of any P_i. It has been assumed that 2ε is less than the Lebesgue number of the covering $\{V_i\}$ of Λ (3.2(a)). So for each $j \geq 0$ there is $V_{\phi(j)}$ satisfying

(c) $P_{j''} \cup P_{j'} \subset V_{\phi(j)}$.

Now by 3.5(a), (c) for each $j \geq 0$

$$\pi(y_j), \pi(y_j') \in U_{\phi(j)} \quad \text{and} \quad \pi_{\phi(j)}(\pi(y_j)) = \pi_{\phi(j)}(\pi(y_j')),$$

where $U_{\phi(j)} \equiv \pi(V_{\phi(j)})$. It follows that $(\pi(y_0), \pi(y_1), \pi(y_2), \ldots) \sim (\pi(y_0'), \pi(y_1'), \pi(y_2'), \ldots)$, showing that $h(x)$ is dependent only on x.

The map $h\colon \Lambda \to \Sigma'(f)$ can be seen to be a homeomorphism satisfying $h \circ g = F' \circ h$. The proof is a simple comparison of the definitions of f, F' and h, so is left to the reader.

This completes the proof of Proposition 2.4.

4. Fundamental groups and characteristic classes. Let $f\colon K \to K$ denote a C^1 expanding immersion on a compact branched manifold without boundary, *equipped with an L.C.S.*

QUESTION 4.1. It is true that all points $x \in K$ are "nonwandering" poins of $f\colon K \to K$ in the following *weak* sense: given any neighborhood U of x, there is $\pi_i\colon U_i \to X_i$ (as in 1.2) so that $\pi_i^{-1}(\pi_i(U)) \cap f^n(U) \neq \varnothing$ for some integer $n > 0$?

QUESTION 4.2. Do the real Pontrjagin classes of Tang(K) vanish?

QUESTION 4.3. If $\alpha \in \pi_1(K, x)$ is *generated* by f (see 4.4 for definition) is it true that for no integer $l \geq 0$ does $\pi_1(f^l)(\alpha) = 0$ in $\pi_1(K, f^l(x))$?

DEFINITION 4.4. For any $x \in K$, $\alpha \in \pi_1(K, x)$, the map $f\colon K \to K$ is said to *generate* α if there is a $U_i \subset K$ (of 1.1) and a parametrized path $h\colon [0,1] \to U_i$ satisfying

(a) $\pi_i \circ h\colon [0,1] \to X_i$ is an embedding.

(b) For some $m' > 0$, $f^{m'} \circ h(0) = f^{m'} \circ h(1) = x$, and α is represented by the closed path $f^{m'} \circ h\colon [0,1] \to K$.

REMARK. If the L.C.S. hypothesis of 4.1, 4.2, 4.3 is replaced by W.L.C.S., then the answer to 4.1, 4.2, 4.3 is *no*. Farrell and Jones [**3**, **4**] have constructed expanding immersions $f\colon K \to K$ satisfying Williams' three properties [**17**, 3.0], such that

$\pi_1(K) = 0$ and the real Pontrjagin classes of $\mathrm{Tang}(K)$ do not vanish. These f: $K \to K$ can be equipped with a W.L.C.S. (see 1.5 above), showing the answer to 4.2, 4.3 is *no* (for the W.L.C.S. hypothesis). Any immersion f: $K \to K$ satisfying the first two of Williams' properties, but having wandering points, can be equipped with a W.L.C.S. (as in 1.5), showing the answer to 4.1 is *no* (for W.L.C.S. hypothesis).

The importance of questions 4.1–4.3 is seen in the following results. Let g: $M \to M$ denote an Anosov diffeomorphism, with $\mathrm{Tang}(M) = \xi_s \oplus \xi_u$ as in 0.1. Let ρ: $\hat{M} \to M$ denote the universal cover of M, and $\hat{W}^s(p)$, $\hat{W}^u(q)$ liftings to $\hat{M}$ of a stable and unstable leaf in M.

PROPOSITION 4.5. *Suppose the stable foliation of* g: $M \to M$ *is* C^1, *and the answer to* 4.1 *is yes for all* f: $K \to K$. *Then* g: $M \to M$ *has no wandering points.*

PROPOSITION 4.6. *Suppose the stable and unstable foliation of* g: $M \to M$ *are* C^1, *and the answer to* 4.3 *is yes for all* f: $K \to K$. *Then* $\hat{W}^s(p) \cap \hat{W}^u(q)$ *contains at most one point.*

PROPOSITION 4.7. *Suppose the unstable and stable foliations of* g: $M \to M$ *are* C^1, *and the answer to* 4.2 *is yes for all* f: $K \to K$. *Then the real Pontrjagin classes of* ξ_u *vanish.*

Since there are leaves $W^s(p)$, $W^u(q)$ which intersect in M in an infinite number of points [**5**, 1.9] we deduce from 4.6 the following

COROLLARY 4.8. *Under the hypothesis of* 4.6, $\pi_1(M) \neq 0$.

Note that 4.7 may be applied to both g: $M \to M$ and g^{-1}: $M \to M$ to give the following

COROLLARY 4.9. *Under the hypothesis of* 4.7 *the real Pontrjagin classes of* $\mathrm{Tang}(M)$ *vanish.*

REMARK 4.10. Note that the conclusions of 4.5–4.9 do hold for all known examples of Anosov diffeomorphisms. In fact if g: $M \to M$ is a hyperbolic infranil isomorphism, then it has no wandering points; the stable and unstable foliations of g: $M \to M$ lift to a cartesian product structure $\hat{M} = \hat{W}^s(p) \times \hat{W}^u(q)$; and the bundle ξ_u is a flat bundle with finite structure group, so its real Pontrjagin classes must vanish [**18**].

PROOF OF THEOREM 4.5. Supposing $y \in \Sigma'(f)$ is a wandering point for F': $\Sigma'(f) \to \Sigma'(f)$, a contradiction will be derived. Choose an open neighborhood V' for y so that $V' \cap [F'^j(V')] = \varnothing$ for all $j \neq 0$; choose a representative $(y_0, y_1, y_2, \ldots) \in \Sigma(f)$ for y and let $V \equiv \pi^{-1}(V')$, where π: $\Sigma(f) \to \Sigma'(f)$ is the quotient map. Then V is an open neighborhood of $(y_0, y_1, y_2, \ldots)$ satisfying

(a) $V \cap F^j(V) = \varnothing$ for all integers $j \neq 0$.

It follows from the definitions of V and π: $\Sigma(f) \to \Sigma'(f)$ that there is an integer $r > 0$ and a neighborhood U of y_r in K which satisfies

(b) if $(a_0, a_1, a_2, \ldots) \in \Sigma(f)$ satisfies $a_r \in \pi_i^{-1}(\pi_i(U))$ for some π_i: $U_i \to X_i$, then $(a_0, a_1, a_2, \ldots) \in V$.

Assuming the answer to 4.1 is yes, we have that for $n > 0$, and some i, $\pi_i^{-1}(\pi_i(U)) \cap f^n(U) \neq \varnothing$. So there is $x \in U$ so that $f^n(x) \in \pi_i^{-1}(\pi_i(U))$. Define $(y_0', y_1', y_2', \ldots) \in \Sigma(f)$ by $y_r' = x$, $y_{r-j}' = f^j(y_r')$, $0 \leq j \leq r$, and for $j > 0$, y_{r+j}' are any values satisfying $f(y_{r+j}') = y_{r+j-1}'$ (see 1.2(c)). Then $(y_0', y_1', y_2', \ldots) \in V$, because $y_r' \in U$, and $F^n(y_0', y_1', y_2', \ldots) \in V$ because $f^n(y_r') \in \pi_i^{-1}(\pi_i(U))$ (see (b) above). Thus $F^n(V) \cap V \neq \varnothing$, contradicting (a) above.

This completes the proof of 4.5.

The following lemma, which is proven at the end of this section, will be needed to prove 4.6, 4.7. Recall that a zero-dimensional subset $A \subset R^n$ is called *tame* if there are arbitrarily small closed neighborhoods of A in R^n which are the disjoint union of closed n-balls (not necessarily differentiable).

LEMMA 4.11. *Let g: $M \to M$ be an Anosov diffeomorphism which has a C^1 unstable foliation. Given $\varepsilon > 0$, and $\{V_i\}$ as in* 3.1, *there is an integer $b > 0$ and a Markov partition $\{P_i\}$ for g^b: $M \to M$ as in* 3.2 *which satisfies the following properties.*

(a) *The diameter of each P_i is less than ε.*

(b) *Set $S \equiv \bigcap_{i \geq 0} g^{-ib}(\bigcup_j (\partial P_j^s) \times P_j^u)$. For each V_i there is a tame zero-dimensional closed subset $A_i \subset V_i^u$ such that $V_i \cap S \subset V_i^s \times A_i$.*

PROOF OF 4.6. It will suffice to prove 4.6 under the added assumption that $\dim(\hat{W}^u(q)) \geq 3$. For if $\dim(\hat{W}^u(q)) < 3$, let H: $T^l \to T^l$ be any linear Anosov diffeomorphism on the l-dimensional torus which as expanding leaves of dimension ≥ 3. Then if the theorem is true for both $H \times g$: $T^l \times M \to T^l \times M$ and H: $T^l \to T^l$, it must be true for g: $M \to M$.

Let f: $K \to K$ be the expanding immersion associated to the Markov partition $\{P_i\}$ of 4.11, as in 1.6; π: $M - Y' \to K$ is the quotient map (see proof of 1.6).

In this proof there will be no loss of generality in assuming that the $\{P_i\}$ of 4.11 are a Markov partition for g: $M \to M$ (i.e. $b = 1$ in 4.11). Let Y' be as in 3.2(e) and set $S' \equiv \bigcap_{i \geq 0} g^{-i}(Y')$. Let S be as in 4.11(b). As a direct consequence of the construction of S, S' we have

4.12. (a) $S = S'$.

In fact, we have

(b) $S \subset g^{-i}(Y') \subset g^{-i+1}(Y')\ \forall i > 0$.

(c) the maximal distance from a point of $g^{-i}(Y')$ to S is less than ε_i, where $\lim_{i \to \infty} \varepsilon_i = 0$.

Suppose for some $\hat{W}^s_{(p)}$, $\hat{W}^u_{(q)}$ there is $\hat{x}, \hat{y} \in \hat{W}^s(p) \cap \hat{W}^u(q)$ with $\hat{x} \neq \hat{y}$. Choose embeddings r: $[0,1] \to W^u(q)$, s: $[0,1] \to W^s(p)$ satisfying $r(0) = x = s(1)$, $r(1) = y = s(0)$, where $\rho(\hat{x}) = x$, $\rho(\hat{y}) = y$. There will be no loss of generality in assuming that $\{P_i\}$, $\hat{W}^s(p)$, $\hat{W}^u(q)$, $\hat{x}$, $\hat{y}$, r, s have been chosen to satisfy the following.

4.13. There is an integer $m > 0$, and a $V_j \subset M$ from 3.1, so that

$$g^{-m}(r[0,1]) \subset V_j \cap (M - Y'), \qquad g^m(s[0,1]) \subset M - Y'.$$

Define h: $[0,1] \to K$ to be the composition $[0,1] \xrightarrow{r} M \xrightarrow{g^{-m}} M \supset M - Y' \xrightarrow{\pi} K$. It follows from 4.13 that image$(h) \subset U_j$ and h: $[0,1] \to U_j$ satisfies 4.4(a), (b) for

$m' = 2m$. So $f^{m'} \circ h$: $[0,1] \to K$ represents $\alpha \in \pi_1(K, f^{m'} \circ g(0))$, which is generated by f.

The proof of 4.6 will be completed by deriving the contradiction that $f^l(\alpha) = 0$ in $\pi_1(K, f^{m'+l} \circ g(0))$ for some integer $l > 0$.

Set $s'(t) \equiv g^m(s(t))$, $r'(t) \equiv g^m(r(t))$. The union $s' \cup r'$ is a closed loop representing some $\beta \in \pi_1(M - Y', s'(0))$. For each $l \geq 0$, $g^l(\beta)$ is mapped to $f^l(\alpha)$ by π: $M - Y' \to K$. So it will suffice to show $g^l(\beta) = 0$ in $\pi_1(M - Y', g^l(s'(0)))$ for some $l \geq 0$. Note the closed loop $s' \cup r'$ lifts to a closed loop in $\hat{M}$; so $\beta = 0$ in $\pi_1(M, s'(0))$. Let G: $D^2 \to N$ be a null homotopy for $s' \cup r'$. Because of the dimension assumption $\dim(\hat{W}^u(q)) \geq 3$ and the conditions on A_i in 4.11(b), G may be varied slightly in the expanding direction of g: $M \to M$ to obtain a new function G': $D^2 \to N$ satisfying

4.14. (a) $\text{image}(G') \cap (V_i^s \times A_i) = \varnothing \ \forall i$.

(b) $G'_{|\partial D^2|}$ is homotopic to $s' \cup r'$ in $M - Y'$.

In particular $\text{image}(G') \cap S = \varnothing$ (4.11(b)). So, by a compactness argument, and 4.12(a)–(c), we have $\text{image}(G') \cap g^{-l}(Y') = \varnothing$ for some $l \geq 0$. Thus $(\text{image}(g^l \circ G')) \cap Y' = \varnothing$, showing that $g^l \circ G'_{|\partial D^2|}$ is null homotopic in $M - Y'$. Now, by 4.14(b), $g^l(\beta) = 0$ in $\pi_1(M - Y'), \ g^l(s'(0)))$ as desired.

This completes the proof of 4.6.

Proof of Proposition 4.7. We use the same notation as in the proof of 4.6. Let $P_i(\xi_u) \in H^{4i}(M, R)$ denote the Pontrjagin class of ξ_u of dimension $4i$ ($i > 0$). It will suffice to show $P_i(\xi_u) \cap \alpha = 0$ for any $\alpha \in H_{4i}(M, Z)$. This will be proven first for all i such that $4i < \dim(K)$.

Note that π: $M - Y' \to K$ pulls $\text{Tang}(K)$ back to $\xi_{u|_{M-Y'}}$. Since $P_i(\text{Tang}(K)) = 0$ (hypothesis of 4.7), we must have

4.15. $P_i(\xi_{u|_{M-Y'}}) = 0$.

It follows from 4.11(b), 4.12(a) and $4i < \dim(K)$, that the inclusion induced map $H_{4i}(M - S', Z) \to H_{4i}(M, Z)$ is onto. Choose $\beta \in H_{4i}(M - S', Z)$ mapping to α. Note by 4.12(b), (c), $\lim_{i \to \infty} H_{4i}(M - g^{-i}(Y'), Z) = H_{4i}(M - S', Z)$. So $\beta \in H_{4i}(M - g^{-m}(Y'), Z)$ for some integer $m > 0$, and $g^m(\beta) \in H_{4i}(M - Y', Z)$. Applying 4.15, we have

$$P_i\left(\xi_{u|_{M-dY'}}\right) \cap g^m(\beta) = 0.$$

But

$$P_i(\xi_u) \cap \alpha = P_i(\xi_u) \cap \beta = P_i\left(\xi_{u_{|M-Y'}}\right) \cap g^m(\beta).$$

So $P_i(\xi_u) \cap \alpha = 0$ as desired.

In order to drop the hypothesis $4i < \dim(K)$, replace g: $M \to M$ by $g \times h$: $M \times T^n \to M \times T^n$, where h is a linear Anosov diffeomorphism on a torus. Note $P_i(\xi_u(g \times h)) = 0$ if $4i > \dim(M)$; and by the argument of the previous paragraph, $P_i(\xi_u(g \times h)) = 0$ if $4i < a$, where a is the dimension of an unstable leaf of $g \times h$: $M \times T^n \to M \times T^n$. So if h: $T^n \to T^n$ is selected so that $a > \dim(M)$, we have $P_*(\xi_u(g \times h)) = 0$. This implies $P_*(\xi_u(g)) = 0$ as desired.

This completes the proof of 4.7.

PROOF OF LEMMA 4.11. Fix a Riemannian metric $\langle\ ,\ \rangle$ on M. Distance in each leaf $W^s(x)$, $W^u(x)$ will mean the restriction of $\langle\ ,\ \rangle$ to that leaf. Let $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ denote the ε-ball in $W^s(x)$, $W^u(x)$ centered at x. For sufficiently small ε the product $W_\varepsilon(x) \equiv W^s_\varepsilon(x) \times W^u_\varepsilon(x)$ is a well-defined neighborhood of x in M and each of $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ is diffeomorphic to the unit ball in a Euclidean space.

Now the proof of the lemma is broken into the following four steps.

Step 1. Given any integer $n > 0$ there is a number δ, with $1 > \delta > 0$, which depends only on n and $g\colon M \to M$, so that for any sufficiently small $\varepsilon > 0$, M can be covered by a finite set $\{W_\varepsilon(x)\colon x \in I\}$ of product balls (I is a finite subset of M) so that the following properties hold.

4.16. (a) For any $x \in M$ define $I_x \subset I$ by $y \in I_x \Leftrightarrow W_{\varepsilon+\delta\varepsilon}(y) \subset W_{n\varepsilon}(x)$. For each $y \in I_x$ choose $t_y \in [-\delta, \delta]$. Then the projection to $W^s_{n\varepsilon}(x)$ of all $\{W_{\varepsilon+t_y\cdot\varepsilon}(y)\colon y \in I_x\}$ is a finite set of smooth balls in transverse position.

(b) The projection to $W^s_{n\varepsilon}(x)$ of all $\{W_{\varepsilon+t_y\cdot\varepsilon}(y)\colon y \in I_x\}$ is diffeomorphic to the projection to $W^s_{n\varepsilon}(x)$ of all $\{W_\varepsilon(y)\colon y \in I_x\}$.

(c) There is a number $N > 0$, depending only on $g\colon M \to M$, so that for any $x \in M$ the cardinality of $\{y\colon y \in I,\ W_\varepsilon(y) \cap W_\varepsilon(x) \neq \varnothing\}$ is less than N.

The verification of 4.16 is routine, so is left to the reader. The hypothesis that the unstable foliation of $g\colon M \to M$ is C^1 is needed here.

Step 2. In this step the construction of a Markov partition for $g\colon M \to M$ is carried out.

Let $\{W_\varepsilon(y)\colon y \in I\}$ be a covering of M as in 4.16. For a fixed power $g^b\colon M \to M$ of $g\colon M \to M$, a *carrier* for $g^b(W_\varepsilon(y))$ is any subset $C_{y,\varepsilon,b} \subset I$ satisfying the following.

4.17. (a) $x \in C_{y,\varepsilon,b} \Rightarrow W_\varepsilon(x) \cap g^b(W_\varepsilon(y)) \neq \varnothing$.

(b) $g^b(W_\varepsilon(y)) \subset \bigcup (W_\varepsilon(x))^0$, where the union runs over all $x \in C_{y,\varepsilon,b}$.

Define $\partial C_{y,\varepsilon,b} \subset C_{y,\varepsilon,b}$ by

(c) If $b > 0$, then $x \in \partial C_{y,\varepsilon,b}$ if and only if $x \in C_{y,\varepsilon,b}$ but the projection of $g^{-b}(W_\varepsilon(x))$ to $W^u_{2\varepsilon}(y)$ is not contained in $(W^u_\varepsilon(y))^0$.

(d) If $b < 0$, then $x \in \partial C_{y,\xi,b}$ if and only if $x \in C_{y,\varepsilon,b}$ but the projection of $g^{-b}(W_\varepsilon(x))$ to $W^s_{2\varepsilon}(y)$ is not contained in $(W^s_\varepsilon(y))^0$.

A sequence of coverings for M, $\{W_{\varepsilon,i}(y) = y \in I\}$, $i = 1, 2, 3, \ldots$, is constructed as follows. Given an integer $b > 0$, choose a carrier $C_{y,\varepsilon,b}$ for each $g^b(W_\varepsilon(y))$ and a carrier $C_{y,\varepsilon,-b}$ for each $g^{-b}(W_\varepsilon(y))$. Set $W_{\varepsilon,1}(y) \equiv W_\varepsilon(y)$, and construct the $W_{\varepsilon,i}(y)$, $i \geq 2$, inductively by the formulae

4.18. (a) $W_{\varepsilon,i}(y) = W^s_{\varepsilon,i}(y) \times W^u_{\varepsilon,i}(y)$.

(b) $W^u_{\varepsilon,i+1}(y) \equiv \bigcup \rho(g^{-b}(W_{\varepsilon,i}(x)))$, where ρ denotes the projection to $W^u_{2\varepsilon}(y)$ and the union runs over all x in $C_{y,\varepsilon,b}$.

(c) $W^s_{\varepsilon,i+1}(y) \equiv \bigcup \rho(g^b(W_{\varepsilon,i}(x)))$, where ρ denotes projection to $W^s_{2\varepsilon}(y)$ and the union runs over all x in $C_{y,\varepsilon,-b}$.

Now define $W_{\varepsilon,\infty}(y)$ to be the closure in M of $\bigcup_{i>0} W_{\varepsilon,i}(y)$. Note $W_{\varepsilon,\infty}(y) = W^s_{\varepsilon,\infty}(y) \times W^u_{\varepsilon,\infty}(y)$. Let $Z_i \subset M$ denote all points of M which are contained in at least i distinct $W_{\varepsilon,\infty}(y)$, $y \in I$. Note

$$M = Z_1 \supset Z_2 \supset \cdots \supset Z_N = \varnothing,$$

where N comes from 4.16(c). For each $1 \le j \le N-1$ define a partition $\{P_{i,j}\}$ of $\overline{Z_j - Z_{j+1}}$ as follows. For each nonempty intersection $\bigcap_{k=1}^{j} W_{\varepsilon,\infty}(x_k)$ of j-distinct sets in $\{W_{\varepsilon,\infty}(y): y \in I\}$ let $\{P_i^x\}$ denote the projection to $W_{10\varepsilon}^x(x_1)$ (here $x = s$ or u) of all $W_{\varepsilon,\infty}(y)$, $y \in I$, such that $W_{\varepsilon,\infty}(y) \cap W_{\varepsilon,\infty}(x_k) \neq \varnothing$ for some $1 \le k \le j$. Let $\{Q_t^x\}$ denote the collection of all maximal intersections of sets in $\{P_i^x\}$ and in $\{\overline{W_{10\varepsilon}^x(x_1) - P_i^x}\}$. Now the restriction of $\{P_{i,j}\}$ to

$$\overline{\bigcap_{k=1}^{j} W_{\varepsilon,\infty}(x_k) - Z_{j+1}}$$

is defined as the collection of all $Q_t^s \times Q_{t'}^u$ whose interiors lie in $Z_j - Z_{j+1}$.

Define a partition $\{P_i\}$ for all of M by

$$\{P_i\} \equiv \bigcup_{j=1}^{N-1} \{P_{i,j}\}.$$

The following is proven as in [**12**].

4.19. For sufficiently large b (as in 4.17, 4.18) the partition $\{P_i\}$ is a Markov partition for $g: M \to M$ for all sufficiently small ε.

Step 3. The purpose of this step is to show that for sufficiently large b in 4.17, 4.18, and sufficiently small ε, the following will be true.

4.20. Let $W_\varepsilon(x_j) = j = 1, 2, \ldots, m$ ($m = \dim(M)$) be a sequence of *distinct* sets $\{W_\varepsilon(y): y \in I\}$ such that for some $x \in M$ we have $\bigcup_{j=1}^{m} W_\varepsilon(x_j) \subset W_{n\varepsilon}(x)$ (where n comes from 4.16). Let $V_r^s \times V_r^u \subset M$ be one of the local product sets of 3.1, and q any point in V_r^s. Then for some $t \in \{1, 2, \ldots, m\}$ we will have

$$q \times V_r^u \cap \left[(\partial W_{\varepsilon,\infty}^s(x_t)) \times W_{10\varepsilon}^u(x_t)\right] = \varnothing.$$

Now 4.20 will be verified. Denote by $W_\varepsilon'(x_k)$ the projection of $W_\varepsilon(x_k)$ to $W_{n\varepsilon}^s(x)$. The $\{\partial W_\varepsilon'(x_k): 1 \le k \le m\}$ are a collection of m smooth spheres intersecting transversely in $W_{n\varepsilon}^s(x)$ (see 4.16). Since $\dim(W_{n\varepsilon}^s(x)) < m$ we have $\bigcap_{k=1}^{m} \partial W_\varepsilon'(x_k) = \varnothing$; moreover it follows from 4.16(a), (b) that $\bigcap_{k=1}^{m} \partial W_{\varepsilon+t_k\varepsilon}'(x_k) = \varnothing$ for any $t_k \in [-\delta, \delta]$, where $W_{\varepsilon+t_k\varepsilon}'(x_k)$ is the projection of $W_{\varepsilon+t_k\varepsilon}(x_k)$ onto $W_{n\varepsilon}^s(x)$. It follows there is $t \in \{1, 2, 3, \ldots, m\}$ so that the distance from $q \times V_r^u$ to $(\partial W_\varepsilon^s(x_t)) \times W_{10\varepsilon}^u(x_t)$ is greater than $\frac{1}{4} \cdot \delta \cdot \varepsilon$ (for sufficiently small ε).

Now choose $b > 0$ in 4.17–4.19 sufficiently large to insure that for any $W_\varepsilon^s(x)$ with $x \in I$, and any $p \in W_{\varepsilon,\infty}^s(x)$, the distance from p to $W_\varepsilon^s(x)$ is less than $10^{-2} \cdot \delta \cdot \varepsilon$. It will then follow from the results of the last paragraph that the distance in M from $q \times V_r^u$ to $(\partial W_{\varepsilon,\infty}^s(x_t)) \times W_{10\varepsilon}^u(x_t)$ is greater than $\frac{1}{8}\delta\varepsilon$.

This completes the verification of 4.20.

Step 4. In this step the proof of Lemma 4.11 is completed.

For any open set $V_r^s \times V_r^u \subset M$ as in 3.1, define an equivalence relation $\sim$ on points $q, p \in V_r^s$ as follows: $q \sim p$ if and only if for any partition set P_j either both $q \times V_r^u \cap P_j = \varnothing$ and $p \times V_r^u \cap P_j = \varnothing$, or both $q \times V_r^u \cap P_j \neq \varnothing$ and $p \times V^u \cap P_j \neq \varnothing$. There are only a finite number of such equivalence classes in V_r^s. For each $q \in V_r^s$ set $A_{r,q} \equiv \rho_r[(q \times V_r^s) \cap S]$, where $\rho_r: V_r^s \times V_r^u \to V_r^u$ is the projection and $S = \bigcap_{j>0} g^{-jb}[\bigcup_k \partial P_k^s \times P_k^u]$. Note if $q \sim p$, then $A_{r,q} = A_{r,p}$. So to complete the

proof of 4.11 it will suffice to show that each $A_{r,q}$ is a tame zero-dimensional subset of V_r^u (here we use that the union of a finite number of tame zero-dimensional subsets is a tame zero-dimensional subset).

Set $C = \bigcup_{x \in I}(\partial W^s_{\varepsilon,\infty}(x)) \times W^u_{10\varepsilon}(x)$. Note $S \subset g^{-ib}(C)$ for all integers $i > 0$. So to show that $A_{r,q} \subset V_r^u$ is tame and zero-dimensional it will suffice to show that there are closed neighborhoods $B_{r,q,i}$ for each $q \times V_r^u \cap g^{-ib}(C)$ in $q \times V_r^u$ such that each $B_{r,q,i}$ is the disjoint union of closed balls of radius less than i^{-1} (here $i = 1, 2, 3, \ldots$). Note it follows from 4.20 there is a closed neighborhood $B'_{r,q,i}$ for $g^{ib}(q \times V_r^u \cap g^{-ib}(C))$ in $g^{ib}(q \times V_r^u)$ such that $B'_{r,q,i}$ is the disjoint union of closed balls of radius less than $2m^m\varepsilon$ (here we must assume that the integer n of 4.16, 4.20 is greater than $10m^m\varepsilon$). Now set $B_{r,q,i} = g^{-ib}(B'_{r,q,i})$.

This completes the proof of Lemma 4.11.

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