FIXED POINTS AND CONJUGACY CLASSES
OF REGULAR ELLIPTIC ELEMENTS IN $\text{Sp}(3, \mathbb{Z})$

BY

MINKING EIE AND CHUNG-YUAN LIN

Abstract. In this paper, we obtain 13 isolated fixed points (up to a $\text{Sp}(3, \mathbb{Z})$-equivalence) and 86 conjugacy classes of regular elliptic elements in $\text{Sp}(3, \mathbb{Z})$. Hence the contributions from regular elliptic conjugacy classes in $\text{Sp}(3, \mathbb{Z})$ to the dimension formula computed via the Selberg trace formula can be computed explicitly by the main theorem of [4 or 5].

Introduction. In [6 and 7], E. Gottschling studied the fixed points and their isotropy groups of finite order elements in $\text{Sp}(2, \mathbb{Z})$. He finally obtained six $\text{Sp}(2, \mathbb{Z})$-inequivalent isolated fixed points as follows:

(1) $Z_1 = \text{diag}[i, i]$, 
(2) $Z_2 = \text{diag}[\rho, \rho]$, $\rho = e^{\pi i/3}$, 
(3) $Z_3 = \text{diag}[i, \rho]$, 
(4) $Z_4 = \frac{i}{\sqrt{3}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, 
(5) $Z_5 = \begin{bmatrix} \eta & (\eta - 1)/2 \\ (\eta - 1)/2 & \eta \end{bmatrix}$, $\eta = \frac{1}{3} + \frac{2\sqrt{2}i}{3}$, 
(6) $Z_6 = \begin{bmatrix} \omega & \omega + \omega^{-2} \\ \omega + \omega^{-2} & -\omega^{-1} \end{bmatrix}$, $\omega = e^{2\pi i/5}$.

The isotropy subgroups at $Z_i$ ($i = 1, 2, 3, 4, 5, 6$) are groups of order 16, 36, 12, 12, 24 and 5, respectively.

By the argument of [9], these fixed points can be obtained from symplectic embeddings of $Q(i) \oplus Q(i), Q(\rho) \oplus Q(\rho), Q(i) \oplus Q(\rho), Q(e^{\pi i/6}), Q(e^{\pi i/4}), Q(e^{2\pi i/5})$, into $M_4(Q)$. In this paper, we shall combine the reduction theory of symplectic matrices [2, 3] with the arguments of [8, 9] and obtain all $\text{Sp}(3, \mathbb{Z})$-inequivalent isolated fixed point and conjugacy classes of regular elliptic elements in $\text{Sp}(3, \mathbb{Z})$. A table for all representatives and their centralizer in $\text{Sp}(3, \mathbb{Z})/\{\pm 1\}$ of regular elliptic conjugacy classes in $\text{Sp}(3, \mathbb{Z})$ is given.
1. Notations and basic results. Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the fields of rational, real and complex numbers, respectively. The real symplectic matrices of degree $n$,

$$\text{Sp}(n, \mathbb{R}) = \left\{ M \in M_{2n}(\mathbb{R}) | \text{tr}M = J, J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix} \right\},$$

act on the generalized half space $H_n$ defined by

$$H_n = \left\{ Z \in M_n(\mathbb{C}) | Z = Z^*, \text{Im}Z > 0 \right\}.$$

Here $M_{2n}(\mathbb{R})$ is the $2n \times 2n$ matrix ring over $\mathbb{R}$, $M_n(\mathbb{C})$ is the $n \times n$ matrix ring over $\mathbb{C}$, $E_n$ is the identity of $M_n(\mathbb{C})$ and $Z^*$ is the transpose of $Z$.

A point $Z_0$ in $H_n$ is called an isolated fixed point of $\text{Sp}(3, \mathbb{Z})$ if there exists $M = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ in $\text{Sp}(3, \mathbb{Z})$ such that $Z_0$ is the unique solution of the equation,

$$AZ + B = Z(CZ + D), \quad Z \in H_n.$$

An element $M$ of $\text{Sp}(3, \mathbb{Z})$ is regular elliptic if $M$ has an isolated fixed point (see [4]). Now suppose $M$ is a regular elliptic element of $\text{Sp}(3, \mathbb{Z})$; then by the discreteness of $\text{Sp}(3, \mathbb{Z})$ and the property that $\text{Sp}(3, \mathbb{Z})$ acts transitively on $H_3$, we conclude that

1. $M$ is an element of finite order,
2. $M$ is conjugate in $\text{Sp}(3, \mathbb{R})$ to $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ with $A + Bi = \text{diag}[\lambda_1, \lambda_2, \lambda_3], \lambda_i (i = 1, 2, 3)$ root of unity and $\lambda_i, \lambda_j \neq 1$ for all $i, j$,
3. the centralizer of $M$ in $\text{Sp}(3, \mathbb{Z})$ is a group of finite order.

By property (1), we see that the minimal polynomial of $M$ is a product of different cyclotomic polynomials as follow: $X^2 + 1, X^2 - X + 1, X^2 + X + 1, X^4 + 1, X^4 - X^2 + 1, X^4 + X^3 + X^2 + X + 1, X^4 - X^3 + X^2 - X + 1, X^6 - X^3 + 1, X^6 + X^5 + X^4 + X^3 + X^2 + X + 1, X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$.

For our convenience, we identify $\text{Sp}(n_1, \mathbb{R}) \times \text{Sp}(n_2, \mathbb{R})$ as a subgroup of $\text{Sp}(n_1 + n_2, \mathbb{R})$ via the embedding

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 & B & 0 \\ 0 & P & 0 & Q \\ C & 0 & D & 0 \\ 0 & R & 0 & S \end{bmatrix}.$$
Now suppose the characteristic polynomial \( P(X) \) of \( M \) is reducible over \( \mathbb{Z}[X] \); then we obtain the following ten possible fixed points for \( M \) simply from fixed points of regular elliptic elements of \( \text{SL}_2(\mathbb{Z}) \) and \( \text{Sp}(2, \mathbb{Z}) \).

1. \( Z_{01} = \text{diag}[i, i, i] \),
2. \( Z_{02} = [\rho, \rho, \rho] \),
3. \( Z_{03} = \text{diag}[\rho, i, i] \),
4. \( Z_{04} = [i, \rho, \rho] \),
5. \( Z_{05} = \begin{bmatrix} i & 0 & 0 \\ 0 & \eta & (\eta - 1)/2 \\ 0 & (\eta - 1)/2 & \eta \end{bmatrix} \), \( \eta = \frac{1}{3} + \frac{2\sqrt{2}i}{3} \),
6. \( Z_{06} = \frac{i}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \),
7. \( Z_{07} = \begin{bmatrix} i & 0 & 0 \\ 0 & \omega & \omega + \omega^{-2} \\ 0 & \omega + \omega^{-2} & -\omega^{-1} \end{bmatrix} \), \( \omega = e^{2\pi i/5} \),
8. \( Z_{08} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \eta & (\eta - 1)/2 \\ 0 & (\eta - 1)/2 & \eta \end{bmatrix} \),
9. \( Z_{09} = \frac{i}{\sqrt{3}} \begin{bmatrix} 1 + \bar{\rho} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \),
10. \( Z_{10} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \omega & \omega + \omega^{-2} \\ 0 & \omega + \omega^{-2} & -\omega^{-1} \end{bmatrix} \).

Let \( G_i (i = 01, 02, 03, 04, 05, 06, 07, 08, 09, 10) \) be the isotropy group of \( \text{Sp}(3, \mathbb{Z})/\{ \pm 1 \} \) at \( Z_i (i = 01, 02, 03, 04, 05, 06, 07, 08, 09, 10) \), respectively. Then a direct calculation shows that the order of \( G_i (i = 01, 02, \ldots, 10) \) are 192, 648, 96, 144, 96, 48, 20, 144, 72, 30, respectively. By considering conjugacy classes in \( G_i (i = 01, 02, \ldots, 10) \), we get 72 conjugacy classes of regular elliptic elements of \( \text{Sp}(3, \mathbb{Z}) \) as shown in the table.

Now we shall show that every regular elliptic element with reducible characteristic polynomial is conjugate in \( \text{Sp}(3, \mathbb{Z}) \) to one of these 72 conjugacy classes. First we need

**Lemma 1.** Suppose \( M \in \text{Sp}(n, \mathbb{Z}) \) with characteristic polynomial \( P(X) \) satisfying

1. \( P(X) \) is a product of two relative prime polynomials \( P_1(X) \) and \( P_2(X) \) with integral coefficients of degrees \( 2n_1 \) and \( 2n_2 \) \((n_1 + n_2 = n)\), respectively,
2. \( P_i(X) = X^{2n}P_i(1/X), i = 1, 2 \).

Then there exists \( R \in \text{Sp}(n, \mathbb{Q}) \) such that \( R^{-1}MR = M_1 \times M_2 \in \text{Sp}(n_1, \mathbb{Q}) \times \text{Sp}(n_2, \mathbb{Q}) \). Furthermore, the characteristic polynomial of \( M_1 \) (resp. \( M_2 \)) is \( P_1(X) \) (resp. \( P_2(X) \)).

**Proof.** (See Lemmas 1 and 2 of [2].)
Lemma 2. Let $M \in \text{Sp}(n, \mathbb{Z})$. Suppose that there exists $R \in \text{Sp}(n, \mathbb{Q})$ such that
\[
R^{-1}MR = \begin{bmatrix}
A & 0 & B & \ast \\
\ast & U & \ast & \ast \\
C & 0 & D & \ast \\
0 & 0 & 0 & U^{-1}
\end{bmatrix}.
\]
Then there exists $\tilde{R} \in \text{Sp}(n, \mathbb{Z})$ such that $\tilde{R}^{-1}M\tilde{R}$ has the same form as $R^{-1}MR$.

Proof. (See Satz 2 of [3].)

Theorem 1. Suppose $M$ is a regular elliptic element of $\text{Sp}(3, \mathbb{Z})$ with a reducible characteristic polynomial $P(X)$. Then $M$ is conjugate in $\text{Sp}(3, \mathbb{Z})$ to an element of $\bigcup_{i=0}^{10} G_i$.

Proof. Here we only prove three special cases, other cases follow with similar arguments.

1. $P(X) = (X^2 + 1)^3$. A representative of $M$ in $U(3)$ is $\text{diag}[i, i, i]$. Thus $M$ is conjugate in $\text{Sp}(3, \mathbb{R})$ to $J = [-1 0 \varepsilon 1]$, i.e. there exists $L \in \text{Sp}(3, \mathbb{R})$ such that $M = L^{-1}JL$. With the Iwasawa decomposition of $\text{Sp}(3, \mathbb{R})$, we can write
\[
L = \begin{bmatrix}
A & B \\
-B & A
\end{bmatrix} \begin{bmatrix}
U & S'U^{-1} \\
0 & U^{-1}
\end{bmatrix}, \quad A + Bi \in U(3).
\]
Since $J$ commutes with $[-A B 1]$, it follows
\[
M = \begin{bmatrix}
U & S'U^{-1} \\
0 & U^{-1}
\end{bmatrix} J = \begin{bmatrix}
U & S'U^{-1} \\
0 & U^{-1}
\end{bmatrix}.
\]
This forces $U, U^{-1}, S \in \text{GL}(3, \mathbb{Z})$. Hence $M$ is conjugate in $\text{Sp}(3, \mathbb{Z})$ to $J = [0 \varepsilon 0]$.

2. $P(X) = (X^2 - X + 1)^3$. A representative of $M$ in $U(3)$ is $\text{diag}[\rho, \rho, \rho]$ or $\text{diag}[^{\rho^2} \rho^2, \rho^2]$. On the other hand, $Q(M) = Q(\rho)$ as fields and the class number of $Q(\rho)$ is 1 by Theorem 11.1 in Chapter 11 of [10]. Hence the number of conjugacy classes of regular elliptic elements with $X^2 - X + 1$ as minimal polynomial is 2 by [8 or 9]. Thus $M$ is conjugate in $\text{Sp}(3, \mathbb{Z})$ to $[^1 1 0] \text{ or } \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}$.

3. $P(X) = (X^2 + 1)(X^4 + X^3 + X^2 + X + 1)$. Note that $M$ can be represented in $U(3)$ as
\[
e[1/2, 1/2, 1/2] \text{ or } e[1/2, 1/2, 1/2] \text{ or } e[1/2, 1/2, 1/2] \text{ or } e[1/2, 1/2, 1/2].
\]
$e[a, b, c]$ stands for $[e^{nia}, e^{nib}, e^{nic}]$.

In particular, $M^5$ can be represented in $U(3)$ as $\text{diag}[i, 1, 1] \text{ or } [-i, 1, 1]$ and has characteristic polynomial $(X^2 + 1)(X - 1)^4$. By Lemmas 1 and 2, there exists $R \in \text{Sp}(3, \mathbb{Z})$ such that
\[
R^{-1}M^5R = \begin{bmatrix}
0 & 0 & 1 & \ast \\
\ast & E_2 & \ast & \ast \\
-1 & 0 & 0 & \ast \\
0 & 0 & 0 & E_2
\end{bmatrix} \text{ or } \begin{bmatrix}
0 & 0 & -1 & \ast \\
\ast & E_2 & \ast & \ast \\
1 & 0 & 0 & \ast \\
0 & 0 & 0 & E_2
\end{bmatrix}
\]
which is conjugate in $\text{Sp}(3, \mathbb{Z})$ to
\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \times E_4 \text{ or } \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \times E_4.$
Hence we may assume

\[ R^{-1}M^2R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times E_4 \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times E_4. \]

Note that the isolated fixed point of \( R^{-1}MR \) is contained in the set of fixed points of \( R^{-1}M^2R \), i.e. the set

\[ Z = \begin{bmatrix} i & 0 & 0 \\ 0 & z_2 & z_{23} \\ 0 & z_{23} & z_3 \end{bmatrix}, \quad \text{Im } Z > 0. \]

Now it is easy to see that the isolated fixed point of \( R^{-1}MR \) is \( (\text{SL}_2(\mathbb{Z}) \times \text{Sp}(2, \mathbb{Z})) \)-equivalent to \( Z_{07} \) and \( M \) is conjugate in \( \text{Sp}(3, \mathbb{Z}) \) to an element of \( G_{07} \). Q.E.D.

In the sections following, we shall determine conjugacy classes of regular elliptic elements of orders 9 and 7.

3. Symplectic embeddings of \( \mathbb{Q}(e^{2\pi i/9}) \) and \( \mathbb{Q}(e^{2\pi i/7}) \). For our convenience, we denote \( e^{2\pi i/9} \) by \( \xi \). Note that \( \mathbb{Q}(\xi) \) is the splitting field of the cyclotomic polynomial \( X^6 + X^3 + 1 \) and contains the total real number field \( \mathbb{Q}(\xi + \xi^{-1}) \) which is the splitting field of \( X^3 - 3X + 1 \). By a symplectic embedding of \( \mathbb{Q}(\xi) \) into \( M_6(\mathbb{Q}) \), we mean an injection from \( \mathbb{Q}(\xi) \) into \( M_6(\mathbb{Q}) \) such that \( \xi \) is mapped into a symplectic matrix \( M \) and \( \mathbb{Q}(\xi) \equiv \mathbb{Q}(M) \) as fields [9].

**Lemma 3.** Let \( M \) be an element of \( \text{Sp}(3, \mathbb{Z}) \) of order 9. Then \( M \) is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to one of the following: \( [\xi, \xi^4, \xi^7], [\xi, \xi^2, \xi^4], [\xi, \xi^5, \xi^7], [\xi^2, \xi^4, \xi^8], [\xi^2, \xi^5, \xi^8], [\xi^4, \xi^7, \xi^8], [\xi^5, \xi^7, \xi^8] \).

**Proof.** The minimal polynomial of \( M \) is \( X^6 + X^3 + 1 \) which can be factored into

\[
(X - \xi)(X - \xi^2)(X - \xi^4)(X - \xi^5)(X - \xi^7)(X - \xi^8) = [X^2 + (\xi + \xi^{-1})X + 1][X^2 - (\xi^2 + \xi^{-2})X + 1][X^2 - (\xi^4 + \xi^{-4})X + 1].
\]

Hence \( M \) is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to

\[
\begin{bmatrix} \cos \theta & \pm \sin \theta \\ \mp \sin \theta & \cos \theta \end{bmatrix} \times \begin{bmatrix} \cos 2\theta & \pm \sin 2\theta \\ \mp \sin 2\theta & \cos 2\theta \end{bmatrix} \times \begin{bmatrix} \cos 4\theta & \pm \sin 4\theta \\ \mp \sin 4\theta & \cos 4\theta \end{bmatrix}, \quad \theta = \frac{2\pi}{9}.
\]

Note that the above eight elements of \( \text{Sp}(3, \mathbb{R}) \) are represented by the prescribed elements in \( U(3) \) as in our lemma.

**Lemma 4.** The number of conjugacy classes of regular elliptic elements of order 9 in \( \text{Sp}(3, \mathbb{Z}) \) is 8.

**Proof.** The ideal class number of \( \mathbb{Q}(\xi) \) is 1 by Theorem 11.1 of [9], hence the number of conjugacy classes of regular elliptic elements of order 9 is given by \( [E_0 : N(E)] \), where

\[
E: \text{the group of units in } \mathbb{Q}(\xi),
E_0: \text{the group of units in } \mathbb{Q}(\xi + \xi^{-1}),
N(E) = \{ u\bar{u} | u \in E \},
\]

according to the argument of [8 or 9].

The group of units for cyclotomic fields is determined in Chapter 8 of [10]. Applying this to our case, we get \( [E_0 : N(E)] = 8 \) when the cyclotomic field is \( \mathbb{Q}(\xi) \).
There are two conjugacy classes of elements of order 9 appearing in the isotropy group $G_{02}$ of $Z_{02} = \text{diag}[\rho, \rho, \rho]$. Indeed, if we let $M = [\xi \beta^2 \gamma^2]$ with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then it is a direct verification to show that

1. $M$ is an element of order 9.
2. $M$ can be represented in $U(3)$ as $[\xi, \xi^4, \xi^7]$ or $[\xi^2, \xi^8, \xi^5]$.
3. $M^3 = [\xi^{-1} - \xi^2]$ has an isolated fixed point at $Z_{02}$.

Now we begin to look for the other six conjugacy classes of regular elliptic elements of order 9 in $\text{Sp}(3, \mathbb{Z})$.

**Theorem 2.** Suppose $\alpha, \beta, \gamma$ are distinct roots of the equation $X^3 - 3X + 1 = 0$ (or more precisely, $\alpha = 2\cos 2\pi/9, \beta = 2\cos 4\pi/9, \gamma = 2\cos 8\pi/9$),

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \Omega = \frac{1}{3} \begin{bmatrix} -3 + \alpha + \alpha^2 & -3 + \beta + \beta^2 & -3 + \gamma + \gamma^2 \\ -1 + \alpha^2 & -1 + \beta^2 & -1 + \gamma^2 \\ 1 + \alpha & 1 + \beta & 1 + \gamma \end{bmatrix},$$

and

$$M = \begin{bmatrix} A & E \\ -E & 0 \end{bmatrix},$$

then

1. $M$ is an element of order 9 in $\text{Sp}(3, \mathbb{Z})$ and has an isolated fixed point at

$$Z_{11} = -\frac{1}{3} A + i\Omega \left(E - \frac{1}{3} \Omega A^2 \Omega \right)^{1/2},$$

2. $M$ is conjugate in $\text{Sp}(3, \mathbb{R})$ to $[\xi, \xi^2, \xi^3]$ of $U(3)$,
3. the centralizer of $M$ in $\text{Sp}(3, \mathbb{Z})/\{\pm 1\}$ is a group of order 9.

**Proof.** (1) Since the characteristic polynomial of $M$ is $X^6 + X^2 + 1$, it follows that $M$ is an element of order 9 in $\text{Sp}(3, \mathbb{Z})$. Note that $\frac{1}{3}[-3 + \alpha + \alpha^2, -1 + \alpha^2, 1 + \alpha]$ is the normalized eigenvector of $A$ corresponding to the eigenvalue $\alpha$. It follows that $\Omega A \Omega = \text{diag}[\alpha, \beta, \gamma]$ and

$$\left(E - \frac{1}{3} \Omega A \Omega \right)^{1/2} = \text{diag}\left[(1 - \alpha^2/4)^{1/2}, (1 - \beta^2/4)^{1/2}, (1 - \gamma^2/4)^{1/2}\right].$$

Now it is a direct verification to show that $AZ_{11} = Z_{11} A$ and $Z_{11}^2 + AZ_{11} + E = 0$. Thus $Z_{11} = -\frac{1}{3} A + i\Omega (E - \frac{1}{3} \Omega A^2 \Omega)^{1/2}\Omega$ is a fixed point of $M$. But $M$ has exactly one fixed point by Lemma 3, hence $Z_{11}$ is the unique isolated fixed point of $M$.

(2) Let $R = [\alpha \beta \gamma]$. Then $R \in \text{Sp}(3, \mathbb{R})$ and

$$R^{-1}MR = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} \beta & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} \gamma & 1 \\ -1 & 0 \end{bmatrix}.$$
because \( \left[ \begin{array}{cc} 2 \cos \mu & 1 \\ -1 & 0 \end{array} \right] \) is conjugate in \( SL_2(\mathbb{R}) \) to \( \left[ \begin{array}{cc} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{array} \right] \). This proves our assertion in (2).

(3) Let \( C(M, \mathbb{Z}) \) be the centralizer of \( M \) in \( \text{Sp}(3, \mathbb{Z})/\{ \pm 1 \} \). Suppose \( \gamma \) is an element of \( C(M, \mathbb{Z}) \). Then

\[
M(\gamma(Z_{11})) = \gamma(M(Z_{11})) = \gamma(Z_{11}).
\]

Since \( Z_{11} \) is the only fixed point of \( M \), this forces \( \gamma(Z_{11}) = Z_{11} \).

Note that \( \Omega Z_{11} \Omega = R(Z_{11}) = \text{diag}[-\xi, -\xi^2, -\xi^4] \). Here \( R = [\xi_0 \, \xi_1] \) as in (2). From \( \gamma(Z_{11}) = Z_{11} \), we get

\[
R \gamma R^{-1}(\Omega Z_{11} \Omega) = \Omega Z_{11} \Omega.
\]

It follows that

\[
R \gamma R^{-1} = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \times \left[ \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right] \times \left[ \begin{array}{cc} a'' & b'' \\ c'' & d'' \end{array} \right]
\]

with

\[
\begin{align*}
-a c + b = c' a^2 - d c, & \quad ad - bc = 1, \\
-a' c^2 + b' = c' a'^2 - d c^2, & \quad a'd' - b'c' = 1, \\
-a'' c^4 + b'' = c'' a''^2 - d'' c^4, & \quad a''d'' - b''c'' = 1.
\end{align*}
\]

The general solution of \( a, b, c, d \) is given by

\[
\begin{align*}
a &= \cos \theta - \cot \frac{2\pi}{9} \sin \theta, & \quad b = -\sec \frac{2\pi}{9} \sin \theta, \\
c &= \sec \frac{2\pi}{9} \sin \theta, & \quad d = \cos \theta + \cot \frac{2\pi}{9} \sin \theta,
\end{align*}
\]

\( \theta \in \mathbb{R} \).

The characteristic polynomial of \( \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \) is \( X^2 - 2 \cos \theta X + 1 \), hence \( 2 \cos \theta \) is an algebraic integer of degree 1 or 3. On the other hand, the fact that \( \gamma \) is an element of finite order implies \( e^{i \theta} \) is a root of unity. Now we have the following cases:

**Case I.** If \( 2 \cos \theta \) is an algebraic integer of degree 3, then the characteristic polynomial of \( \gamma \) is an irreducible polynomial of degree 6. Since \( M \) satisfies this case, the characteristic polynomial of \( \gamma \) is \( X^6 + X^3 + 1 \) or \( X^6 - X^3 + 1 \). This leads to the fact that \( \theta = 2\pi/9 \) or \( 4\pi/9 \) or \( 8\pi/9 \) and \( \gamma \) is one of the following elements: \( \pm M, \pm M^2, \pm M^4, \pm M^5, \pm M^7, \pm M^8 \).

**Case II.** If \( 2 \cos \theta = \pm 1 \), then \( \gamma \) is an element of order 3. Then \( \gamma = \pm M^3 \) or \( \pm M^6 \) by a direct calculation.

**Case III.** If \( 2 \cos \theta = \pm 2 \), then \( \gamma = \pm E_6 \).

**Case IV.** If \( 2 \cos \theta = 0 \), then

\[
\gamma = R^{-1}\left[ \begin{array}{cc} -\cot \eta & -\sec \eta \\ \sec \eta & \cot \eta \end{array} \right] \times \left[ \begin{array}{cc} -\cot 2\eta & -\sec 2\eta \\ \sec 2\eta & \cot 2\eta \end{array} \right] \times \left[ \begin{array}{cc} -\cot 4\eta & -\sec 4\eta \\ \sec 4\eta & \cot 4\eta \end{array} \right] R
\]

with \( \eta = 2\pi/9 \). Such a \( \gamma \) is not an integral matrix.

By the above discussion, we conclude \( C(M, \mathbb{Z}) \) is a group of order 9 generated by \( M \).
<table>
<thead>
<tr>
<th>No.</th>
<th>Representative in $U(3)$</th>
<th>Minimal polynomial</th>
<th>Order of centralizer</th>
<th>No. of conjugates in isotropy group</th>
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<tbody>
<tr>
<td>1</td>
<td>$e[1/2,1/2,1/2]$</td>
<td>$X^2 + 1$</td>
<td>192</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$e[1/2,1/4,5/4]$</td>
<td>$(X^2 + 1)(X^4 + 1)$</td>
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<td>12</td>
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<td>$e[1/2,3/4,7/4]$</td>
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<td>12</td>
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<td>$e[1/6,5/6,9/6]$</td>
<td>$X^6 + 1$</td>
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<td>32</td>
</tr>
<tr>
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<td>Minimal polynomial</td>
<td>Order of centralizer</td>
<td>No. of conjugates in isotropy group</td>
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With the same argument, we get the following result by simply replacing the role of \( e^{2\pi i/9} \) by \( e^{2\pi i/7} \).

**Lemma 5.** Let \( M \) be an element of \( \text{Sp}(3, \mathbb{Z}) \) of order 1. Then \( M \) is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to one of the following (\( v = e^{2\pi i/7} \)):

\[
[v, v^2, v^3], \ [v, v^2, v^4], \ [v, v^4, v^5], \ [v, v^3, v^5], \\
[v^2, v^3, v^6], \ [v^2, v^4, v^6], \ [v^4, v^5, v^6], \ [v^3, v^5, v^6].
\]

**Lemma 6.** The number of conjugacy classes of regular elliptic elements of order 7 in \( \text{Sp}(3, \mathbb{Z}) \) is 8.

**Theorem 3.** Suppose \( \alpha, \beta, \gamma \) are distinct roots of the equation \( X^3 + X^2 - 2X + 1 \) (or more precisely, \( \alpha = 2 \cos 2\pi/7, \beta = 2 \cos 4\pi/7, \gamma = 2 \cos 6\pi/7 \)),

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{bmatrix}, \quad \Omega' = \begin{bmatrix}
\alpha + \alpha^2 & \beta + \beta^2 & \gamma + \gamma^2 \\
1 + 3\alpha & 1 + 3\beta & 1 + 3\gamma \\
1 + 2\alpha & 1 + 2\beta & 1 + 2\gamma \\
1 + 3\alpha & 1 + 3\beta & 1 + 3\gamma \\
\alpha^2 & \beta^2 & \gamma^2 \\
1 + 3\alpha & 1 + 3\beta & 1 + 3\gamma
\end{bmatrix}
\]

and

\[
M = \begin{bmatrix}
B & E \\
-E & 0
\end{bmatrix}.
\]

Then

1. \( M \) is an element of order 7 in \( \text{Sp}(3, \mathbb{Z}) \) and has an isolated fixed point at

\[
Z_{12} = -\frac{1}{2}B + i\Omega'(E - \frac{1}{4}\Omega' B^2 \Omega')^{1/2} \Omega',
\]

2. \( M \) is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to \([v, v^2, v^3]\),
3. the centralizer of \( M \) in \( \text{Sp}(3, \mathbb{Z})/\{\pm 1\} \) is a group of order 7 generated by \( M \).

Note that \([v, v^2, v^4]\) and \([v^3, v^6, v^5]\) are exclusive in the set of all powers of \([v, v^2, v^3]\). To find all representatives for elliptic conjugacy classes of order 7, it suffices to get a representative which is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to \([v, v^2, v^4]\).

**Theorem 4.** Let \( B, \Omega' \) be matrices as in Theorem 3,

\[
U = \text{diag}[1, 1, -1] \quad \text{and} \quad M = \begin{bmatrix}
B & E + B \\
-(E + B)^{-1} & 0
\end{bmatrix}.
\]

Then

1. \( M \) is an element of order 7 in \( \text{Sp}(3, \mathbb{Z}) \) with isolated fixed point at

\[
Z_{13} = -\frac{1}{2}(B + E) + i\Omega' \left( (E - \frac{1}{4} \Omega' B^2 \Omega')^{1/2} \Omega' (B + E) \Omega U \right) \Omega',
\]

2. \( M \) is conjugate in \( \text{Sp}(3, \mathbb{R}) \) to \([v, v^2, v^4]\),
3. the centralizer of \( M \) in \( \text{Sp}(3, \mathbb{Z})/\{\pm 1\} \) is a finite group of order 7 generated by \( M \).
Proof. Since \(\det(E + B) = 1\) and
\[
\Omega'(E + B)\Omega' = \text{diag}\left[1 + 2\cos\frac{2\pi}{7}, 1 + 2\cos\frac{4\pi}{7}, 1 + 2\cos\frac{6\pi}{7}\right]
\]
has signature +, +, −, it follows that \(M \in \text{Sp}(3, \mathbb{Z})\) and \(M\) is conjugate in \(\text{Sp}(3, \mathbb{R})\) to
\[
M' = \begin{bmatrix}
2\cos\theta & 1 \\
-1 & 0
\end{bmatrix} \times \begin{bmatrix}
2\cos\frac{2\theta}{1} & 1 \\
-1 & 0
\end{bmatrix} \times \begin{bmatrix}
2\cos\frac{3\theta}{1} & -1 \\
1 & 0
\end{bmatrix}, \quad \theta = \frac{2\pi}{7}.
\]
Indeed, if we let \(R' = \Omega'\Lambda\) with
\[
\Lambda = \text{diag}\left[(1 + 2\cos\frac{2\pi}{7})^{-1/2}, (1 + 2\cos\frac{4\pi}{7})^{-1/2}, \left(-1 - 2\cos\frac{6\pi}{7}\right)^{-1/2}\right],
\]
then \((R')^{-1}MR' = M'\). Hence (1) and (2) follow as a direct calculation. By a similar argument as in (3) of Theorem 2, we get (2).

By Theorems 1, 2, 3 and 4, we obtain the following table for conjugacy classes of regular elliptic elements in \(\text{Sp}(3, \mathbb{Z})\).

4. Application. Contributions from conjugacy classes of regular elliptic elements in \(\text{Sp}(n, \mathbb{Z})\) to the dimension formula for Siegel cusp forms of degree \(n\) and weight \(k\) [4] are given by
\[
\sum |C(M, \mathbb{Z})|^{-1} \prod_{i=1}^{n} \prod_{i<j} \left(1 - \lambda_i\lambda_j\right)^{-1}.
\]
Here the summation over all conjugacy classes of regular elliptic elements in \(\text{Sp}(n, \mathbb{Z})\). \(M\) is conjugate in \(\text{Sp}(n, \mathbb{R})\) \([A, B] = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \lambda_i\lambda_j \neq 1\) for all \(i, j\) and \(C(M, \mathbb{Z})\) is the centralizer of \(M\) in \(\text{Sp}(3, \mathbb{Z})\). Applying this formula to the case \(n = 3\), we get all contributions from 86 regular elliptic conjugacy classes in \(\text{Sp}(3, \mathbb{Z})\).

For the case \(n = 1\) and \(n = 2\), the contribution from a particular regular elliptic conjugacy class appears to be a residue of a generating function at a simple pole. For example, the contribution from the conjugacy class of regular elliptic elements of order 5 in \(\text{Sp}(2, \mathbb{Z})\) is given by
\[
K = \frac{1}{5} \left[\omega^{-6k}(1 - \omega^{-2}) + \omega^{-2k}(1 - \omega^{-4}) + \omega^{-8k}(1 - \omega^{-6}) + \omega^{-4k}(1 - \omega^{-8})\right],
\]
which is precisely the negative of the sum of residues of the function
\[
\frac{1}{(1 - T^4)(1 - T^6)(1 - T^{10})(1 - T^{12})T^{k+1}}
\]
at \(T = e^{i\theta}\) with \(\theta = \pm \pi/5, \pm 2\pi/5, \pm 3\pi/5, \pm 4\pi/5\) when \(k\) is even.

It is easy to see that the total contribution from conjugacy classes of elements of order 2 or 3 in \(\text{SL}_2(\mathbb{Z})\) is the negative of the sum of residues of the function
\[
\frac{1}{(1 - T^4)(1 - T^6)T^{k+1}}
\]
at \(T = e^{i\theta}\) with \(\theta = \pm \pi/2, \pm \pi/3, \pm 2\pi/3\) when \(k\) is even.
Note that
\[
\frac{1}{(1 - T^4)(1 - T^6)} \quad \text{and} \quad \frac{1}{(1 - T^4)(1 - T^6)(1 - T^{10})(1 - T^{12})}
\]
are well known to be generating functions of dimension formulas for modular forms of degree 1 and degree 2, respectively. It is hopeful to find a generating function of a dimension formula for modular forms of degree 3 by computing contributions from conjugacy classes of regular elliptic elements in \( \text{Sp}(3, \mathbb{Z}) \). However, we can write down explicitly the conjugacy classes of \( \text{Sp}(3, \mathbb{Z}) \) simply by using our results in this paper and reduction theory in [2, 3]. Thus a dimension formula for Siegel cusps forms of degree 3 can be obtained by the Selberg trace formula and results of [5].

REFERENCES


INSTITUTE OF MATHEMATICS, ACADEMA SINICA, NANKANG, TAIPEI, TAIWAN, REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, NATIONAL TSING-HUA UNIVERSITY, HSINCHU, TAIWAN, REPUBLIC OF CHINA (Current address of Chuang-Yuan Lin)

Current address (Minking Eie): Sonderforschungsbereich 170, Mathematisches Institut der Georg-August-Universität, Göttingen, Bunsenstrasse 3–5, D-3400 Göttingen, Federal Republic of Germany

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