CONVOLUTION EQUATIONS IN SPACES OF DISTRIBUTIONS WITH ONE-SIDED BOUNDED SUPPORT

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ABSTRACT. Let $\mathcal{D}'(0, \infty)$ be the space of distributions on $\mathbb{R}$ with support in $[0, \infty)$ and $\mathcal{D}''(0, \infty)$ its subspace consisting of tempered distributions. We characterize the distributions $S \in \mathcal{D}'(0, \infty)$ for which $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, where $\ast$ is the convolution. We also characterize the distributions $S \in \mathcal{D}''(0, \infty)$ for which $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$.

The problem of existence of solutions of convolution equations in spaces of distributions on $\mathbb{R}^n$ was thoroughly investigated by L. Ehrenpreis [1]. Later, L. Hörmander [2] gave a systematic exposition of the range of convolution operators in spaces of distributions on arbitrary open sets $\Omega \subset \mathbb{R}^n$. In particular, consider the space $\mathcal{D}'$ of all distributions on $\mathbb{R}^n$ and its subspace $\mathcal{E}'$ consisting of distributions of compact support. If $S \in \mathcal{E}'$, the convolution equation

$$S \ast u = v$$

has a solution $u \in \mathcal{D}'$, for every $v \in \mathcal{D}'$, if and only if the Fourier transform $\hat{S}$ of $S$ has the following property:

There are constants $A_1, A_2$ and $A_3$ such that for every $\xi \in \mathbb{R}^n$ there exists $\eta \in \mathbb{R}^n$ satisfying the conditions

$$(2) \quad |\xi - \eta| \leq A_1 \log(2 + |\xi|) \quad \text{and} \quad |\hat{S}(\eta)| \geq (A_2 + |\xi|)^{-A_3}.$$  

In that case $S$ is said to be invertible and $\hat{S}$ slowly decreasing.

However, if we consider the space $\mathcal{D}'(0, \infty)$ of distributions of one variable with support in $[0, \infty)$, then the convolution $S \ast u$ is defined for any $S, u \in \mathcal{D}'(0, \infty)$ as a distribution in $\mathcal{D}'(0, \infty)$. The question now arises: Under what conditions on $S \in \mathcal{D}'(0, \infty)$ has equation (1) a solution $u \in \mathcal{D}'(0, \infty)$ for every $v \in \mathcal{D}'(0, \infty)$?

Since the Fourier transform $\hat{S}$ of a distribution $\hat{S} \in \mathcal{D}'(0, \infty)$ is, in general, not even a distribution, conditions (2) are now meaningless.

In this paper we characterize the distributions $S \in \mathcal{D}'(0, \infty)$ for which $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$. We also solve the problem of existence of solutions of equation (1) in the space $\mathcal{D}'(0, \infty)$ of tempered distributions with support in $[0, \infty)$. 

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1. Preliminaries. We denote by $\mathcal{S}_-$ the space of all $C^\infty$-functions with support bounded from the right and by $\mathcal{D}_+$ its dual space. $\mathcal{D}_+$ is the space of distributions with support bounded from the left. We also denote by $\mathcal{D}'(\alpha, \infty)$ the space of distributions with support in $[\alpha, \infty)$. For details about these spaces we refer to [4].

If $S, T \in \mathcal{D}_+$, then the convolution $S \ast T$ is well defined as a distribution in $\mathcal{D}_+$. We recall some of the properties of the convolution which we apply later. First we have

\[ \text{supp}(S \ast T) \subset \text{supp} S + \text{supp} T. \]

In particular, if $S \in \mathcal{D}'(\alpha, \infty)$ and $T \in \mathcal{D}'(\beta, \infty)$, then $S \ast T \in \mathcal{D}'(\alpha + \beta, \infty)$.

Furthermore, the following theorem of Titchmarsh (see [3 and 6]) is valid for distributions in $\mathcal{D}'(0, \infty)$:

If $S$ and $T$ are distributions in $\mathcal{D}'(0, \infty)$ and $S \ast T \in \mathcal{D}'(\gamma, \infty)$, $\gamma > 0$, then $S \in \mathcal{D}'(\alpha, \infty)$ and $T \in \mathcal{D}'(\beta, \infty)$, where $\alpha + \beta = \gamma$, $\alpha, \beta > 0$.

Finally, if $S \in \mathcal{D}'$, then the value of the convolution $S \ast T$ on an open set $\Omega$ depends only on the value of $T$ in $\Omega - \text{supp} S$.

We denote by $\mathcal{S}'(0, \infty)$ the space of tempered distributions with support in $[0, \infty)$. Distributions in $\mathcal{S}'(0, \infty)$ can be characterized in terms of their Fourier transforms (see [5]). Let $C_- = \{ \xi \in C : \text{Im} \xi < 0 \}$ and denote by $M_n$ the function defined on $(-\infty, 0)$ by

\[ M_n(\eta) = \begin{cases} 1 & \text{if } \eta \leq -1, \\ |\eta|^{-n} & \text{if } -1 < \eta < 0; \end{cases} \]

where $n$ is an integer. A tempered distribution $T$ is in $\mathcal{S}'(0, \infty)$ if and only if its Fourier transform $\hat{T}$ can be continued in $C_-$ to an analytic function $\hat{i}$ satisfying the estimate

\[ |\hat{i}(\xi)| \leq C (1 + |\xi|)^m M_n(\eta), \]

where $\xi = \xi + i\eta$, $m$ and $n$ are positive integers and $C$ is a constant.

If $S$ and $T$ are both in $\mathcal{S}'(0, \infty)$ and if $\hat{s}$ and $\hat{t}$ are the corresponding analytic continuations in $C_-$ of their Fourier transforms, then $\hat{s} \hat{t}$ is the analytic continuation of a tempered distribution $\hat{S} \ast \hat{T}$ such that

\[ S \ast T = \hat{S} \ast \hat{T}. \]

2. The case of convolution operators of compact support. We first consider convolution operators $S$ with compact support and assume that $\text{supp} S \subset [0, \infty)$; in that case $S \ast \mathcal{D}'(0, \infty) \subset \mathcal{D}'(0, \infty)$.

If $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, then $0 \in \text{sing supp} S$. In fact, if $\text{sing supp} S \subset [\alpha, \infty)$, $\alpha > 0$, then $\text{sing supp} S \ast u \subset [\alpha, \infty)$ for every $u \in \mathcal{D}'(0, \infty)$, since the value of $S \ast u$ in $(-\infty, \alpha)$ depends only on the value of $S$ in $(-\infty, \alpha)$. This shows that $S \ast \mathcal{D}'(0, \infty) \neq \mathcal{D}'(0, \infty)$.

In what follows we use the notion of invertibility of a convolution operator $S \in \mathcal{D}'$ in the same sense as in [1] (see the introduction). We also denote by $R_j^-$ and $R_j^+$, $j = 0, \pm 1, \pm 2, \ldots$, the intervals $(-\infty, j]$ and $[j, \infty)$, respectively.
THEOREM 1. If $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, then $S$ is invertible and satisfies the conditions

(6) $u \in \mathcal{D}'_+$, $\text{sing supp}(S \ast u) \subset R^+_j \Rightarrow \text{sing supp } u \subset R^+_j$, $j = 0, \pm 1, \pm 2, \ldots$.

PROOF. If $\varphi \in \mathcal{D}$, we pick $h > 0$ such that $\text{supp } \tau_h \varphi \subset [0, \infty)$. By assumption, there exists $u \in \mathcal{D}'(0, \infty)$ such that $S \ast u = \tau_h \varphi$. Hence $S \ast \tau_{-h} u = \varphi$, and, obviously, $\tau_{-h} u \in \mathcal{D}'$. In other words, we proved that $S \ast \mathcal{D}' \supset \mathcal{D}$, which implies that $S$ is invertible by Theorem 2.5 in [1].

Furthermore, it follows from the hypothesis that $S$ has a fundamental solution in $\mathcal{D}'(0, \infty)$, i.e. there exists $E \in \mathcal{D}'(0, \infty)$ such that $S \ast E = \delta$, where $\delta$ is the Dirac measure. Consequently, every distribution $u \in \mathcal{D}'(0, \infty)$ can be represented in the form $u = (S \ast u) \ast E$. If now $\text{sing supp}(S \ast u) \subset R^+_j$, then $S \ast u$ is a $C^\infty$-function on the complement of $R^+_j$, and therefore $u = (S \ast u) \ast E$ is a $C^\infty$-function on $(-\infty, j)$. Thus $\text{sing supp } u \subset R^+_j$.

REMARK 1. If $S$ is any distribution in $\mathcal{D}'(0, \infty)$ such that $0 \in \text{supp } S$, then it follows from the Titchmarsh theorem that

(7) $u \in \mathcal{D}'_+$, $\text{supp}(S \ast u) \subset R^+_j \Rightarrow \text{supp } u \subset R^+_j$, $j = 0, \pm 1, \pm 2, \ldots$.

In fact, pick an integer $k > 0$ such that $\tau_k u \in \mathcal{D}'(0, \infty)$. If $\text{supp}(S \ast u) \subset R^+_j$, we have $\text{supp}(S \ast \tau_k u) = \text{supp } \tau_k (S \ast u) \subset R^+_j \subset R^+_j \subset R^+_j$, and thus $\text{supp } u \subset R^+_j$.

REMARK 2. Conditions (6) and (7) are equivalent to the conditions

(8) $u \in \mathcal{D}'_-$, $\text{sing supp}(\tilde{S} \ast u) \subset R^-_j \Rightarrow \text{sing supp } u \subset R^-_j$, $j = 0, \pm 1, \pm 2, \ldots$, where $\tilde{S}$ is the symmetric distribution to $S$. This follows immediately from the relation $(\tilde{S} \ast u)^* = S \ast u$.

We now prove the converse of Theorem 1.

THEOREM 2. If $S$ is invertible and satisfies conditions (6), then $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$.

PROOF. It suffices to show that

(10) $S \ast \mathcal{D}'_+ = \mathcal{D}'_+$

and then apply Titchmarsh's theorem. The proof of (10) is basically the same as that of Theorem 4.5 in [2]. We therefore restrict ourselves to a brief sketch of the proof.

Let $\nu$ be an arbitrary distribution in $\mathcal{D}'_+$ and let $p$ be a seminorm in $\mathcal{D}_-$ such that

(11) $|\nu(\varphi)| \leq p(\varphi)$, $\varphi \in \mathcal{D}_-$.

The main part of the proof is the construction of a seminorm $q$ in $\mathcal{D}_-$ satisfying the condition

(12) $p(\varphi) \leq q(\tilde{S} \ast \varphi)$, $\varphi \in \mathcal{D}_-$.
If this is accomplished, conditions (11) and (12) yield $|v(\varphi)| \leq q(\tilde{S} \star \varphi)$, $\varphi \in \mathcal{D}_-$, and hence, by the Hahn-Banach theorem, the linear form $\tilde{S} \star \varphi \to v(\varphi)$, $\varphi \in \mathcal{D}_-$, can be extended to a linear form $u$ on $\mathcal{D}_-$ such that $|u(\psi)| \leq q(\psi)$, $\psi \in \mathcal{D}_-$. Thus $u \in \mathcal{D}'_-$, and since $u(\tilde{S} \star \varphi) = v(\varphi)$, $\varphi \in \mathcal{D}_-$, we have $\tilde{S} \star u(\varphi) = v(\varphi)$, $\varphi \in \mathcal{D}_-$, or $S \star u = v$.

The construction of $q$ is based on the following lemma.

**Lemma.** Let $q$ be a seminorm in $\mathcal{D}_-$ such that

$$ q(\psi) \geq \sup_{\alpha \leq x} |\psi(\alpha)|, \quad \psi \in \mathcal{D}_-, $$

for some $\alpha \in \mathbb{R}$, and assume that

$$ p(\varphi) \leq q(\tilde{S} \star \varphi), \quad \varphi \in \mathcal{D}_-, \supp \varphi \subset R_j^-,$$

where $j > \alpha$. Then, under the conditions of the theorem, for every $\varepsilon > 0$ there exists another seminorm $q'$ in $\mathcal{D}_-$ such that

$$ p(\varphi) \leq q'(\tilde{S} \star \varphi), \quad \varphi \in \mathcal{D}_-, \supp \varphi \subset R_{j+1}^-,$$

and

$$ q'(\psi) = (1 + \varepsilon)q(\psi), \quad \psi \in \mathcal{D}_-, \supp \psi \subset R_j^-.$$

We observe that $q(\psi)$ depends only on the values of $\psi$ in $R^+_{\lambda}$ for some $\lambda \in \mathbb{R}$. Otherwise one could find a sequence of functions $\psi_n \in \mathcal{D}_-$ such that $\supp \psi_n \subset R^-_{\lambda_n}$, where $\lambda_n \to -\infty$, and $q(\psi_n) \to 1$. Since $\psi_n \to 0$ in $\mathcal{D}_-$, this would be a contradiction.

In view of that, the conditions of the lemma need only to be satisfied by functions with compact support and the proof is then the same as in [2].

Given $\varepsilon_j > 0$ such that $\sum_{j=1}^{\infty} \varepsilon_j < \infty$, one can now apply the lemma to successively construct seminorms $q_j$ in $\mathcal{D}_-$ such that

$$ q_{j+1}(\psi) = (1 + \varepsilon_j)q_j(\psi), \quad \psi \in \mathcal{D}_-, \supp \psi \subset R^-_{j+1}, $$

and

$$ p(\varphi) \leq q_j(\tilde{S} \star \varphi), \quad \varphi \in \mathcal{D}_-, \supp \varphi \subset R^-_{j+1}. $$

If $q(\psi) = \lim_{j \to \infty} q_j(\psi)$, then $q$ is a continuous seminorm in $\mathcal{D}_-$ and conditions (13) imply (12).

**Corollary 1.** For $S \in \mathcal{E}'$ with $\supp S \subset [0, \infty)$, the following conditions are equivalent.

(i) $S \star \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$.

(ii) $S \star \mathcal{D}'_+ = \mathcal{D}'_+$.

(iii) There exists a fundamental solution for $S$ in $\mathcal{D}'(0, \infty)$.

(iv) $S$ is invertible and satisfies conditions (6).

3. The general case. We now consider equation (1), where $S$ is an arbitrary distribution in $\mathcal{D}'(0, \infty)$. We prove that the solvability of this equation in $\mathcal{D}'(0, \infty)$ depends only on the values of $S$ in an arbitrary small neighborhood of the origin.
Theorem 3. If \( S \in \mathcal{D}'(0, \infty) \) and \( S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty) \), then, for every \( \alpha > 0 \), \( S \) admits a decomposition

\[
S = S_1 + S_2,
\]

where \( S_1 \) is a distribution in \( \mathcal{D}' \) satisfying the equivalent conditions in Corollary 1, and \( S_2 \in \mathcal{D}'(\alpha, \infty) \).

Proof. Given \( \alpha > 0 \), we can always decompose \( S \) as in (14), with \( S_1 \in \mathcal{D}' \) and \( S_2 \in \mathcal{D}'(\alpha, \infty) \). In fact, if \( \psi \) is a function in \( \mathcal{D} \) such that \( \psi = 1 \) on \([-\alpha, \alpha]\), we can define \( S_1 = \psi S \) and \( S_2 = S - S_1 \).

By the hypothesis, there exists \( E \in \mathcal{D}'(0, \infty) \) such that \( S \ast E = \delta \). Hence

\[
S_1 \ast E = \delta - S_2 \ast E
\]

and \( S_2 \ast E \in \mathcal{D}'(\alpha, \infty) \), by (3).

Consider now a solution \( u_1 \in \mathcal{D}'(0, \infty) \) of the equation \( S \ast u = S_2 \ast E \). By assumption, \( u_1 \) exists. Also, since \( S_2 \ast E \in \mathcal{D}'(\alpha, \infty) \) and \( 0 \in \text{sing supp } S \), \( u_1 \) is in \( \mathcal{D}'(\alpha, \infty) \) by Titchmarsh’s theorem. It follows that

\[
S_1 \ast u_1 = S_2 \ast E - S_2 \ast u_1
\]

and \( S_2 \ast u_1 \in \mathcal{D}'(2\alpha, \infty) \) by (3).

Suppose that \( u_k \in \mathcal{D}'(k\alpha, \infty) \) has been defined for \( k \geq 1 \). We then define \( u_{k+1} \) as a solution in \( \mathcal{D}'(0, \infty) \) of the equation \( S \ast u = S_2 \ast u_k \). Since \( S_2 \ast u_k \in \mathcal{D}'((k+1)\alpha, \infty) \), \( u_{k+1} \) must be in \( \mathcal{D}'((k+1)\alpha, \infty) \) by Titchmarsh’s theorem, and we have

\[
S_1 \ast u_{k+1} = S_2 \ast u_k - S_2 \ast u_{k+1}.
\]

In this way we have defined a sequence of distributions \( u_k \in \mathcal{D}'(k\alpha, \infty) \) satisfying equations (16) and (17). Adding both sides of equations (15), (16), and (17) for \( k = 1, 2, \ldots, n \), we obtain

\[
S_1 \ast \left( E + \sum_{k=1}^{n+1} u_k \right) = \delta - S_2 \ast u_{n+1}.
\]

Since \( u_n \in \mathcal{D}'(n\alpha, \infty) \), the series \( \sum_{k=1}^{\infty} u_k \) converges and \( S_2 \ast u_{n+1} \to 0 \) in \( \mathcal{D}'(0, \infty) \), as \( n \to \infty \). Thus, from (18) it follows that

\[
S_1 \ast \left( E + \sum_{k=1}^{\infty} u_k \right) = \delta,
\]

i.e. \( E_1 = E + \sum_{k=1}^{\infty} u_k \) is a fundamental solution for \( S_1 \) in \( \mathcal{D}'(0, \infty) \).

The converse of Theorem 3 is also true.

Theorem 4. Let \( S \) be a distribution in \( \mathcal{D}'(0, \infty) \) and let \( \alpha > 0 \). If \( S \) admits a decomposition (14), where \( S_1 \in \mathcal{D}' \), \( S_2 \in \mathcal{D}'(\alpha, \infty) \), and \( S_1 \) satisfies the equivalent conditions of Corollary 1, then \( S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty) \).

Proof. It suffices to construct a fundamental solution \( E \in \mathcal{D}'(0, \infty) \) for \( S \).

By assumption, there exists \( E_1 \in \mathcal{D}'(0, \infty) \) such that \( S_1 \ast E_1 = \delta \). We now consider a solution \( v_1 \in \mathcal{D}'(0, \infty) \) of the equation \( S_1 \ast v = -S_2 \ast E_1 \). Since \( S_2 \) is in
\( \mathcal{D}'(\alpha, \infty) \), \( S_2 \ast E_1 \) is also in \( \mathcal{D}'(\alpha, \infty) \) by (3). But \( 0 \in \text{sing supp} S_1 \), and so \( v_1 \in \mathcal{D}'(\alpha, \infty) \) by Titchmarsh's theorem. We also have
\[
S \ast (E_1 + v_1) = \delta + S_2 \ast v_1.
\]

Suppose that \( v_k \) is defined for \( k = 1, 2, \ldots, n \), so that \( v_k \in \mathcal{D}'(k\alpha, \infty) \) and
\[
S \ast \left( E_1 + \sum_{k=1}^{n} v_k \right) = \delta + S_2 \ast v_n.
\]
We define \( v_{n+1} \) to be a solution in \( \mathcal{D}'(0, \infty) \) of the equation \( S_1 \ast v = -S_2 \ast v_n \). By the same argument as above, we can show that \( v_{n+1} \in \mathcal{D}'((n+1)\alpha, \infty) \) and
\[
S \ast \left( E_1 + \sum_{k=1}^{n+1} v_k \right) = \delta + S_2 \ast v_{n+1}.
\]

We have thus constructed a sequence of distributions \( v_n \in \mathcal{D}'(n\alpha, \infty) \) which satisfy (19). But \( v_n \to 0 \), and therefore \( S_2 \ast v_n \to 0 \) as \( n \to \infty \), and the series \( \sum_{n=1}^{\infty} v_n \) converges in \( \mathcal{D}'(0, \infty) \). Consequently, it follows from (19) that \( F = E_1 + \sum_{n=1}^{\infty} v_n \) is a fundamental solution for \( S \) in \( \mathcal{D}'(0, \infty) \).

**Corollary 2.** If \( S \in \mathcal{D}'(0, \infty) \), the following conditions are equivalent.

1. \( S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty) \).
2. For some \( \alpha > 0 \) (or equivalently, for every \( \alpha > 0 \)), \( S \) admits a decomposition (14), where \( S_1 \in \mathcal{S}' \), \( S_2 \in \mathcal{D}'(\alpha, \infty) \), and \( S_1 \) satisfies the equivalent conditions in Corollary 1.

We now establish necessary and sufficient conditions on a distribution \( S \in \mathcal{S}'(0, \infty) \) in order that \( S \ast \mathcal{S}'(0, \infty) = \mathcal{S}'(0, \infty) \).

**Theorem 5.** Let \( S \in \mathcal{S}'(0, \infty) \) and let \( \hat{S} \) be the analytic continuation in \( C_- \) of the Fourier transform \( \hat{S} \) of \( S \). Then the following conditions are equivalent.

1. \( S \ast \mathcal{S}'(0, \infty) = \mathcal{S}'(0, \infty) \).
2. \( \hat{S}(\xi) \neq 0 \) for all \( \xi \in C_- \), and there exist positive integers \( m, n \) and a constant \( c > 0 \) such that
\[
|\hat{S}(\xi)| > c(1 + |\xi|)^{-m} M_{-n}(\eta),
\]
where \( \xi = \xi + i\eta \) and \( M_n \) is the function defined by (4).

**Proof.** It follows from (1) that there is \( E \in \mathcal{S}'(0, \infty) \) such that
\[
S \ast E = \delta.
\]
If \( \hat{\delta} \) is the analytic continuation in \( C_- \) of the Fourier transform \( \hat{E} \) of \( E \), then (21) implies that \( \hat{S}(\xi)\hat{\delta}(\xi) = 1, \xi \in C_- \). Hence \( \hat{S}(\xi) \neq 0 \) for all \( \xi \in C_- \), and there are positive integers \( m, n \) and a constant \( C \) such that
\[
|1/\hat{S}(\xi)| = |\hat{\delta}(\xi)| < C(1 + |\xi|)^m M_n(\eta),
\]
in view of (5). This is equivalent to (20) with \( c = 1/C \).

Conversely, if the analytic continuation \( \hat{S} \) of \( \hat{S} \) satisfies conditions (1), then \( \hat{\delta}(\xi) = 1/\hat{S}(\xi) \) satisfies the estimate (5) in \( C_- \), and so \( \hat{\delta}(\xi) \) is the analytic continuation of the Fourier transform \( \hat{E} \) of a distribution \( E \in \mathcal{S}'(0, \infty) \). Moreover, since \( \hat{S}(\xi)\hat{\delta}(\xi) = 1 \), we have \( S \ast E = \delta \), i.e. \( E \) is a fundamental solution in \( \mathcal{S}'(0, \infty) \) for \( S \).
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