CONVOLUTION EQUATIONS IN SPACES OF
DISTRIBUTIONS WITH ONE-SIDED BOUNDED SUPPORT

BY

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ABSTRACT. Let $\mathcal{D}'(0, \infty)$ be the space of distributions on $R$ with support in $[0, \infty)$ and $\mathcal{D}''(0, \infty)$ its subspace consisting of tempered distributions. We characterize the distributions $S \in \mathcal{D}'(0, \infty)$ for which $S * \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, where $*$ is the convolution. We also characterize the distributions $S \in \mathcal{D}''(0, \infty)$ for which $S * \mathcal{D}''(0, \infty) = \mathcal{D}''(0, \infty)$.

The problem of existence of solutions of convolution equations in spaces of distributions on $R^n$ was thoroughly investigated by L. Ehrenpreis [1]. Later, L. Hörmander [2] gave a systematic exposition of the range of convolution operators in spaces of distributions on arbitrary open sets $\Omega \subset R^n$. In particular, consider the space $\mathcal{D}'$ of all distributions on $R^n$ and its subspace $\mathcal{D}'$ consisting of distributions of compact support. If $S \in \mathcal{D}'$, the convolution equation

\[ S * u = v \]

has a solution $u \in \mathcal{D}'$, for every $v \in \mathcal{D}'$, if and only if the Fourier transform $\hat{S}$ of $S$ has the following property:

There are constants $A_1, A_2$ and $A_3$ such that for every $\xi \in R^n$ there exists $\eta \in R^n$ satisfying the conditions

\[ |\xi - \eta| \leq A_1 \log(2 + |\xi|) \quad \text{and} \quad |\hat{S}(\eta)| \geq (A_2 + |\xi|)^{-A_3}. \]

In that case $S$ is said to be invertible and $\hat{S}$ slowly decreasing.

However, if we consider the space $\mathcal{D}'(0, \infty)$ of distributions of one variable with support in $[0, \infty)$, then the convolution $S * u$ is defined for any $S, u \in \mathcal{D}'(0, \infty)$ as a distribution in $\mathcal{D}'(0, \infty)$. The question now arises: Under what conditions on $S \in \mathcal{D}'(0, \infty)$ has equation (1) a solution $u \in \mathcal{D}'(0, \infty)$ for every $v \in \mathcal{D}'(0, \infty)$? Since the Fourier transform $\hat{S}$ of a distribution $\hat{S} \in \mathcal{D}'(0, \infty)$ is, in general, not even a distribution, conditions (2) are now meaningless.

In this paper we characterize the distributions $S \in \mathcal{D}'(0, \infty)$ for which $S * \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$. We also solve the problem of existence of solutions of equation (1) in the space $\mathcal{D}'(0, \infty)$ of tempered distributions with support in $[0, \infty)$. 

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1. Preliminaries. We denote by $\mathcal{D}_-$ the space of all $C^\infty$-functions with support bounded from the right and by $\mathcal{D}'_+$ its dual space. $\mathcal{D}'_+$ is the space of distributions with support bounded from the left. We also denote by $\mathcal{D}'(\alpha, \infty)$ the space of distributions with support in $[\alpha, \infty)$. For details about these spaces we refer to [4].

If $S, T \in \mathcal{D}'_+$, then the convolution $S \ast T$ is well defined as a distribution in $\mathcal{D}'_+$. We recall some of the properties of the convolution which we apply later. First we have

\begin{equation}
(3) \quad \text{supp}(S \ast T) \subset \text{supp } S + \text{supp } T.
\end{equation}

In particular, if $S \in \mathcal{D}'(\alpha, \infty)$ and $T \in \mathcal{D}'(\beta, \infty)$, then $S \ast T \in \mathcal{D}'(\alpha + \beta, \infty)$.

Furthermore, the following theorem of Titchmarsh (see [3 and 6]) is valid for distributions in $\mathcal{D}'(0, \infty)$:

If $S$ and $T$ are distributions in $\mathcal{D}'(0, \infty)$ and $S \ast T \in \mathcal{D}'(\gamma, \infty)$, $\gamma > 0$, then $S \in \mathcal{D}'(\alpha, \infty)$ and $T \in \mathcal{D}'(\beta, \infty)$, where $\alpha + \beta = \gamma$, $\alpha, \beta > 0$.

Finally, if $S \in \mathcal{E}'$, then the value of the convolution $S \ast T$ on an open set $\Omega$ depends only on the value of $T$ in $\Omega - \text{supp } S$.

We denote by $\mathcal{S}'(0, \infty)$ the space of tempered distributions with support in $[0, \infty)$. Distributions in $\mathcal{S}'(0, \infty)$ can be characterized in terms of their Fourier transforms (see [5]). Let $C_- = \{ \xi \in C : \text{Im } \xi < 0 \}$ and denote by $M_n$ the function defined on $( - \infty, 0)$ by

\begin{equation}
(4) \quad M_n(\eta) = \begin{cases} 
1 & \text{if } \eta \leq -1, \\
|\eta|^{-n} & \text{if } -1 < \eta < 0;
\end{cases}
\end{equation}

here $n$ is an integer. A tempered distribution $T$ is in $\mathcal{S}'(0, \infty)$ if and only if its Fourier transform $\hat{T}$ can be continued in $C_-$ to an analytic function $\hat{t}$ satisfying the estimate

\begin{equation}
(5) \quad |\hat{t}(\xi)| \leq C(1 + |\xi|)^m M_n(\eta),
\end{equation}

where $\xi = \xi + i\eta$, $m$ and $n$ are positive integers and $C$ is a constant.

If $S$ and $T$ are both in $\mathcal{S}'(0, \infty)$ and if $\hat{s}$ and $\hat{t}$ are the corresponding analytic continuations in $C_-$ of their Fourier transforms, then $\hat{st}$ is the analytic continuation of a tempered distribution $\hat{S} \ast \hat{T}$ such that

\[ \hat{S} \ast \hat{T} = \hat{s} \ast \hat{t}. \]

2. The case of convolution operators of compact support. We first consider convolution operators $S$ with compact support and assume that $\text{supp } S \subset [0, \infty)$; in that case $S \ast \mathcal{D}'(0, \infty) \subset \mathcal{D}'(0, \infty)$.

If $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, then $0 \in \text{sing supp } S$. In fact, if $\text{sing supp } S \subset [\alpha, \infty)$, $\alpha > 0$, then $\text{sing supp } S \ast u \subset [\alpha, \infty)$ for every $u \in \mathcal{D}'(0, \infty)$, since the value of $S \ast u$ in $( - \infty, \alpha)$ depends only on the value of $S$ in $( - \infty, \alpha)$. This shows that $S \ast \mathcal{D}'(0, \infty) \neq \mathcal{D}'(0, \infty)$.

In what follows we use the notion of invertibility of a convolution operator $S \in \mathcal{E}'$ in the same sense as in [1] (see the introduction). We also denote by $R^-_j$ and $R^+_j$, $j = 0, \pm 1, \pm 2, \ldots$, the intervals $( - \infty, j]$ and $[ j, \infty)$, respectively.
Theorem 1. If \( S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty) \), then \( S \) is invertible and satisfies the conditions

\[ (6) \quad u \in \mathcal{D}', \quad \text{sing supp}(S \ast u) \subset R^+_j \Rightarrow \text{sing supp } u \subset R^+_j, \quad j = 0, \pm 1, \pm 2, \ldots. \]

Proof. If \( \varphi \in \mathcal{D} \), we pick \( h \geq 0 \) such that \( \text{supp } \tau_h \varphi \subset [0, \infty) \). By assumption, there exists \( u \in \mathcal{D}'(0, \infty) \) such that \( S \ast u = \tau_h \varphi \). Hence \( S \ast \tau_{-h} u = \varphi \), and, obviously, \( \tau_{-h} u \in \mathcal{D}' \). In other words, we proved that \( S \ast \mathcal{D}' \supset \mathcal{D} \), which implies that \( S \) is invertible by Theorem 2.5 in [1].

Furthermore, it follows from the hypothesis that \( S \) has a fundamental solution in \( \mathcal{D}'(0, \infty) \), i.e. there exists \( E \in \mathcal{D}'(0, \infty) \) such that \( S \ast E = \delta \), where \( \delta \) is the Dirac measure. Consequently, every distribution \( u \in \mathcal{D}'(0, \infty) \) can be represented in the form \( u = (S \ast u) \ast E \). If now \( \text{sing supp}(S \ast u) \subset R^+_j \), then \( S \ast u \) is a \( C^\infty \)-function on the complement of \( R^+_j \), and therefore \( u = (S \ast u) \ast E \) is a \( C^\infty \)-function on \( (-\infty, j) \). Thus \( \text{sing supp } u \subset R^+_j \).

Remark 1. If \( S \) is any distribution in \( \mathcal{D}'(0, \infty) \) such that \( 0 \in \text{supp } S \), then it follows from the Titchmarsh theorem that

\[ (7) \quad u \in \mathcal{D}', \quad \text{supp}(S \ast u) \subset R^+_j \Rightarrow \text{supp } u \subset R^+_j, \]

\[ j = 0, \pm 1, \pm 2, \ldots. \] In fact, pick an integer \( k \geq 0 \) such that \( \tau_k u \in \mathcal{D}'(0, \infty) \). If \( \text{supp}(S \ast u) \subset R^+_j \), we have \( \text{supp}(S \ast \tau_k u) = \text{supp } \tau_k (S \ast u) \subset R^+_j \). But \( 0 \in \text{supp } S \), and so \( \tau_k u \subset R^+_j \), by Titchmarsh’s theorem. Hence \( \text{supp } u \subset R^+_j \).

Remark 2. Conditions (6) and (7) are equivalent to the conditions

\[ (8) \quad u \in \mathcal{D}'_-, \quad \text{sing supp}(\tilde{S} \ast u) \subset R^-_j \Rightarrow \text{sing supp } u \subset R^-_j, \]

\[ j = 0, \pm 1, \pm 2, \ldots, \] where \( \tilde{S} \) is the symmetric distribution to \( S \). This follows immediately from the relation \( (\tilde{S} \ast u)^* = S \ast \tilde{u} \).

We now prove the converse of Theorem 1.

Theorem 2. If \( S \) is invertible and satisfies conditions (6), then \( S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty) \).

Proof. It suffices to show that

\[ (10) \quad S \ast \mathcal{D}' = \mathcal{D}' \]

and then apply Titchmarsh’s theorem. The proof of (10) is basically the same as that of Theorem 4.5 in [2]. We therefore restrict ourselves to a brief sketch of the proof.

Let \( v \) be an arbitrary distribution in \( \mathcal{D}'_- \) and let \( p \) be a seminorm in \( \mathcal{D}_- \) such that

\[ (11) \quad |v(\varphi)| \leq p(\varphi), \quad \varphi \in \mathcal{D}_-. \]

The main part of the proof is the construction of a seminorm \( q \) in \( \mathcal{D}_- \) satisfying the condition

\[ (12) \quad p(\varphi) \leq q(\tilde{S} \ast \varphi), \quad \varphi \in \mathcal{D}_-. \]
If this is accomplished, conditions (11) and (12) yield \(|\varphi(\varphi)| \leq q(\tilde{S} \ast \varphi), \varphi \in \mathcal{D}_{-}\), and hence, by the Hahn-Banach theorem, the linear form \(\tilde{S} \ast \varphi \to \varphi(\varphi), \varphi \in \mathcal{D}_{-}\), can be extended to a linear form \(u\) on \(\mathcal{D}_{-}\) such that \(|u(\varphi)| \leq q(\varphi), \varphi \in \mathcal{D}_{-}\). Thus \(u \in \mathcal{D}_{+}\), and since \(u(\tilde{S} \ast \varphi) = \varphi(\varphi), \varphi \in \mathcal{D}_{-}\), we have \(\tilde{S} \ast u(\varphi) = \varphi(\varphi), \varphi \in \mathcal{D}_{-}\), or \(S \ast u = \varphi\).

The construction of \(q\) is based on the following lemma.

**Lemma.** Let \(q\) be a seminorm in \(\mathcal{D}_{-}\) such that

\[
q(\psi) \geq \sup_{\alpha \leq x} |\psi(x)|, \quad \psi \in \mathcal{D}_{-},
\]

for some \(\alpha \in R\), and assume that

\[
p(\varphi) \leq q(\tilde{S} \ast \varphi), \quad \varphi \in \mathcal{D}_{-}, \quad \text{supp} \varphi \subset R_{j},
\]

where \(j > \alpha\). Then, under the conditions of the theorem, for every \(\varepsilon > 0\) there exists another seminorm \(q'\) in \(\mathcal{D}_{-}\) such that

\[
p(\varphi) \leq q'(\tilde{S} \ast \varphi), \quad \varphi \in \mathcal{D}_{-}, \quad \text{supp} \varphi \subset R_{j+1},
\]

and

\[
q'(\psi) = (1 + \varepsilon)q(\psi), \quad \psi \in \mathcal{D}_{-}, \quad \text{supp} \varphi \subset R_{j-1}.
\]

We observe that \(q(\psi)\) depends only on the values of \(\psi\) in \(R_{\lambda}^{+}\) for some \(\lambda \in R\). Otherwise one could find a sequence of functions \(\psi_{n} \in \mathcal{D}_{-}\) such that \(\text{supp} \psi_{n} \subset R_{j}^{-}\), where \(\lambda_{n} \to -\infty\), and \(q(\psi_{n}) \geq 1\). Since \(\psi_{n} \to 0\) in \(\mathcal{D}_{-}\), this would be a contradiction.

In view of that, the conditions of the lemma need only to be satisfied by functions with compact support and the proof is then the same as in [2].

Given \(\varepsilon_{j} > 0\) such that \(\sum_{j=1}^{\infty} \varepsilon_{j} < \infty\), one can now apply the lemma to successively construct seminorms \(q_{j}\) in \(\mathcal{D}_{-}\) such that

\[
q_{j+1}(\psi) = (1 + \varepsilon_{j})q_{j}(\psi), \quad \psi \in \mathcal{D}_{-}, \quad \text{supp} \varphi \subset R_{j-1},
\]

and

\[
p(\varphi) \leq q_{j}(\tilde{S} \ast \varphi), \quad \varphi \in \mathcal{D}_{-}, \quad \text{supp} \varphi \subset R_{j+1}.
\]

If \(q(\psi) = \lim_{j \to \infty} q_{j}(\psi)\), then \(q\) is a continuous seminorm in \(\mathcal{D}_{-}\) and conditions (13) imply (12).

**Corollary 1.** For \(S \in \mathcal{E}'\) with \(\text{supp} S \subset [0, \infty)\), the following conditions are equivalent.

(i.1) \(S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)\).

(i.2) \(S \ast \mathcal{D}' = \mathcal{D}'\).

(i.3) There exists a fundamental solution for \(S\) in \(\mathcal{D}'(0, \infty)\).

(i.4) \(S\) is invertible and satisfies conditions (6).

**3. The general case.** We now consider equation (1), where \(S\) is an arbitrary distribution in \(\mathcal{D}'(0, \infty)\). We prove that the solvability of this equation in \(\mathcal{D}'(0, \infty)\) depends only on the values of \(S\) in an arbitrary small neighborhood of the origin.
THEOREM 3. If $S \in \mathcal{D}'(0, \infty)$ and $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$, then, for every $\alpha > 0$, $S$ admits a decomposition

$$S = S_1 + S_2,$$

where $S_1$ is a distribution in $\mathcal{E}'$ satisfying the equivalent conditions in Corollary 1, and $S_2 \in \mathcal{D}'(\alpha, \infty)$.

PROOF. Given $\alpha > 0$, we can always decompose $S$ as in (14), with $S_1 \in \mathcal{E}'$ and $S_2 \in \mathcal{D}'(\alpha, \infty)$. In fact, if $\psi$ is a function in $\mathcal{D}$ such that $\psi = 1$ on $[-\alpha, \alpha]$, we can define $S_1 = \psi S$ and $S_2 = S - S_1$.

By the hypothesis, there exists $E \in \mathcal{D}'(0, \infty)$ such that $S \ast E = \delta$. Hence

$$S_1 \ast E = \delta - S_2 \ast E$$

and $S_2 \ast E \in \mathcal{D}'(\alpha, \infty)$, by (3).

Consider now a solution $u_1 \in \mathcal{D}'(0, \infty)$ of the equation $S \ast u = S_2 \ast E$. By assumption, $u_1$ exists. Also, since $S_2 \ast E \in \mathcal{D}'(\alpha, \infty)$ and $0 \in \text{sing supp } S$, $u_1$ is in $\mathcal{D}'(\alpha, \infty)$ by Titchmarsh's theorem. It follows that

$$S_1 \ast u_1 = S_2 \ast E - S_2 \ast u_1$$

and $S_2 \ast u_1 \in \mathcal{D}'(2\alpha, \infty)$ by (3).

Suppose that $u_k \in \mathcal{D}'(k\alpha, \infty)$ has been defined for $k \geq 1$. We then define $u_{k+1}$ as a solution in $\mathcal{D}'(0, \infty)$ of the equation $S \ast u = S_2 \ast u_k$. Since $S_2 \ast u_k \in \mathcal{D}'((k + 1)\alpha, \infty)$, $u_{k+1}$ must be in $\mathcal{D}'((k + 1)\alpha, \infty)$ by Titchmarsh's theorem, and we have

$$S_1 \ast u_{k+1} = S_2 \ast u_k - S_2 \ast u_{k+1}.$$

In this way we have defined a sequence of distributions $u_k \in \mathcal{D}'(k\alpha, \infty)$ satisfying equations (16) and (17). Adding both sides of equations (15), (16), and (17) for $k = 1, 2, \ldots, n$, we obtain

$$S_1 \ast \left(E + \sum_{k=1}^{n+1} u_k\right) = \delta - S_2 \ast u_{n+1}.$$

Since $u_n \in \mathcal{D}'(n\alpha, \infty)$, the series $\sum_{k=1}^{\infty} u_k$ converges and $S_2 \ast u_{n+1} \rightarrow 0$ in $\mathcal{D}'(0, \infty)$, as $n \rightarrow \infty$. Thus, from (18) it follows that

$$S_1 \ast \left(E + \sum_{k=1}^{\infty} u_k\right) = \delta,$$

i.e. $E_1 = E + \sum_{k=1}^{\infty} u_k$ is a fundamental solution for $S_1$ in $\mathcal{D}'(0, \infty)$.

The converse of Theorem 3 is also true.

THEOREM 4. Let $S$ be a distribution in $\mathcal{D}'(0, \infty)$ and let $\alpha > 0$. If $S$ admits a decomposition (14), where $S_1 \in \mathcal{E}'$, $S_2 \in \mathcal{D}'(\alpha, \infty)$, and $S_1$ satisfies the equivalent conditions of Corollary 1, then $S \ast \mathcal{D}'(0, \infty) = \mathcal{D}'(0, \infty)$.

PROOF. It suffices to construct a fundamental solution $E \in \mathcal{D}'(0, \infty)$ for $S$.

By assumption, there exists $E_1 \in \mathcal{D}'(0, \infty)$ such that $S_1 \ast E_1 = \delta$. We now consider a solution $v_1 \in \mathcal{D}'(0, \infty)$ of the equation $S_1 \ast v = -S_2 \ast E_1$. Since $S_2$ is in
$\mathcal{D}'(\alpha, \infty)$, $S_2 \ast E_1$ is also in $\mathcal{D}'(\alpha, \infty)$ by (3). But $0 \in \text{sing supp } S_1$, and so $v_1 \in \mathcal{D}'(\alpha, \infty)$ by Titchmarsh’s theorem. We also have

$$S \ast (E_1 + v_1) = \delta + S_2 \ast v_1.$$

Suppose that $v_k$ is defined for $k = 1, 2, \ldots, n$, so that $v_k \in \mathcal{D}'(k\alpha, \infty)$ and

$$S \ast \left( E_1 + \sum_{k=1}^{n} v_k \right) = \delta + S_2 \ast v_n.$$

We define $v_{n+1}$ to be a solution in $\mathcal{D}'(0, \infty)$ of the equation $S_1 \ast v = -S_2 \ast v_n$. By the same argument as above, we can show that $v_{n+1} \in \mathcal{D}'((n + 1)\alpha, \infty)$ and

$$S \ast \left( E_1 + \sum_{k=1}^{n+1} v_k \right) = \delta + S_2 \ast v_{n+1}.$$

We have thus constructed a sequence of distributions $v_n \in \mathcal{D}'(n\alpha, \infty)$ which satisfy (19). But $v_n \to 0$, and therefore $S_2 \ast v_n \to 0$ as $n \to \infty$, and the series $\sum_{n=1}^{\infty} v_n$ converges in $\mathcal{D}'(0, \infty)$. Consequently, it follows from (19) that $E = E_1 + \sum_{n=1}^{\infty} v_n$ is a fundamental solution for $S$ in $\mathcal{D}'(0, \infty)$. 

**Corollary 2.** If $S \in \mathcal{D}'(0, \infty)$, the following conditions are equivalent.

1. $S \ast \mathcal{S}'(0, \infty) = \mathcal{S}'(0, \infty)$.

2. For some $\alpha > 0$ (or equivalently, for every $\alpha > 0$), $S$ admits a decomposition (14), where $S_1 \in \mathcal{S}'$, $S_2 \in \mathcal{D}'(\alpha, \infty)$, and $S_1$ satisfies the equivalent conditions in Corollary 1.

We now establish necessary and sufficient conditions on a distribution $S \in \mathcal{S}'(0, \infty)$ in order that $S \ast \mathcal{S}'(0, \infty) = \mathcal{S}'(0, \infty)$.

**Theorem 5.** Let $S \in \mathcal{S}'(0, \infty)$ and let $\hat{S}$ be the analytic continuation in $C_-$ of the Fourier transform $\hat{S}$ of $S$. Then the following conditions are equivalent.

1. $S \ast \mathcal{S}'(0, \infty) = \mathcal{S}'(0, \infty)$.

2. $\hat{S}(\xi) \neq 0$ for all $\xi \in C_-$, and there exist positive integers $m, n$ and a constant $c > 0$ such that

$$|\hat{S}(\xi)| \geq c(1 + |\xi|)^{-m} M_n(n),$$

where $\xi = \xi + i\eta$ and $M_n$ is the function defined by (4).

**Proof.** It follows from (1) that there is $E \in \mathcal{S}'(0, \infty)$ such that

$$S \ast E = \delta.$$

If $\hat{E}$ is the analytic continuation in $C_-$ of the Fourier transform $\hat{E}$ of $E$, then (21) implies that $\hat{S}(\xi)\hat{E}(\xi) = 1$, $\xi \in C_-$. Hence $\hat{S}(\xi) \neq 0$ for all $\xi \in C_-$, and there are positive integers $m, n$ and a constant $c$ such that

$$|1/\hat{S}(\xi)| = |\hat{E}(\xi)| \leq C(1 + |\xi|)^{m} M_n(n),$$

in view of (5). This is equivalent to (20) with $c = 1/C$.

Conversely, if the analytic continuation $\hat{S}$ of $\hat{S}$ satisfies conditions (12), then $\hat{E}(\xi) = 1/\hat{S}(\xi)$ satisfies the estimate (5) in $C_-$, and so $\hat{E}(\xi)$ is the analytic continuation of the Fourier transform $\hat{E}$ of a distribution $E \in \mathcal{S}'(0, \infty)$. Moreover, since $\hat{S}(\xi)\hat{E}(\xi) = 1$, we have $S \ast E = \delta$, i.e. $E$ is a fundamental solution in $\mathcal{S}'(0, \infty)$ for $S$. 

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