HARMONIC FUNCTIONS ON SEMIDIRECT EXTENSIONS OF TYPE H NILPOTENT GROUPS

BY
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ABSTRACT. Let $S = NA$ be a semidirect extension of a Heisenberg type nilpotent group N by the one-parameter group of dilations, equipped with the Riemannian structure, which generalizes this of the symmetric space. Let $\{P_a(y)\}_{a>0}$ be a Poisson kernel on N with respect to the Laplace-Beltrami operator. Then every bounded harmonic function F on S is a Poisson integral $F(yb) = f * P_b(y)$ of a function $f \in L^\infty(N)$. Moreover the harmonic measures μ_a^b defined by $P_b = P_a * \mu_a^b$, $b > a$, are radial and have smooth densities. This seems to be of interest also in the case of a symmetric space of rank 1.

Introduction. We continue to investigate the harmonic functions on the Riemannian spaces S , studied in [2, 3], with respect to the Laplace-Beltrami operator Δ . S is a semidirect product of a type H nilpotent group N and the one-parameter group of dilations A , equipped with the Riemannian structure modeled on one of the symmetric spaces of rank 1. S includes those spaces as well as many more nonsymmetric ones [3].

The formula for what should be the Poisson kernel P_a , $a \in A$, has been written down by J. Cygan. In [2] the author has proved that the function $P(ya) = P_a(y)$, $y \in N$, is harmonic, as is the function $f * P_a(y)$ for every $f \in L^p(N)$, $1 \leq p \leq \infty$. Also it has been shown that $\lim_{a \rightarrow 0} f * P_a(y) = f(y)$ a.e.

The aim of this paper is to show that every bounded harmonic function on S is a Poisson integral of a L^∞ function on N . For the symmetric space it is, of course, well known [6, 9]. However all the proofs we know are based on the fact that S admits a large group K of isometries, which leaves a point in S invariant. By [3, 12] we know that such a group is in fact quite small for general S .

Thus our proof is based on a different idea, which seems to be new in the classical case also. The idea is based on a maximum principle and certain properties of reproducing measures μ_a^b on N , which are uniquely defined as solutions of the equation $P_b = P_a * \mu_a^b$, $b > a$. We show that these measures are radial and have smooth densities, though the explicit formulas for them, even in the case of the Siegel domain, seems to be hopeless. To investigate μ_a^b we apply a method, which began with [7] and was further developed in [1, 11].

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1. Preliminaries. Let $\mathcal{N} = V \oplus Z$ be a nonabelian Lie algebra of type H [8] with centre Z . V is the orthogonal complement to Z . Let $N = \exp \mathcal{N}$. We denote the element $\exp_N(v, z)$ by (v, z) so that

$$(v, z)(v', z') = (v + v', z + z' + \tfrac{1}{2}[v, v']).$$

As in [2] let $S = NA$, where A is the multiplicative group of R^+ , be a semidirect product of N and A , A acting on N as dilations $\delta_a(v, z) = (av, a^2z)$. We identify S with $V \times Z \times R^+$ so that $(v, z, a)(v', z', a') = (v + av', z + a^2z' + \tfrac{1}{2}a[v, v'], aa')$ and in the Lie algebra $\mathcal{S} = \mathcal{N} \oplus R$ of S we define the inner product

$$\langle (v, z, \log a), (v', z', \log a') \rangle_S = \langle v, v' \rangle + \langle z, z' \rangle + 4(\log a)(\log a')$$

[2, 3]. Let $\{e_i\}_{i=1, \dots, 2m}$, $\{e_r\}_{r=2m+1, \dots, 2m+l}$, $\{e_0\}$, where $2m = \dim V$, $l = \dim Z$, be an orthonormal basis of $\mathcal{S}$ corresponding to the decomposition $\mathcal{S} = V \oplus Z \oplus R$. Denote by E_β the left-invariant vector field on S determined by e_β , $\beta = 0, 1, \dots, 2m+l$. It has been shown in [2] that the Laplace-Beltrami operator associated to the left-invariant metric $\langle \cdot, \cdot \rangle_S$ has the form

$$\Delta = \sum_{\beta} E_{\beta}^2 - \frac{Q}{2} E_0,$$

where $Q = 2l + 2m$.

Let $\partial_i, \partial_r, \partial_0$ be the partial derivatives for the system of coordinates (v^i, z^r, a) corresponding to $\{e_i, e_r, e_0\}$. Since

$$E_0 = \tfrac{1}{2}a\partial_0, \quad E_r = a^2\partial_r,$$

$$E_i = a\partial_i + \tfrac{1}{2}a \sum_{r=2m+1}^{2m+l} \langle [v, e_i], e_r \rangle \partial_r,$$

a straightforward calculation yields

$$\begin{aligned} \Delta &= \tfrac{1}{4}(1 - Q)a\partial_0 + \tfrac{1}{4}a^2\partial_0^2 + a^2 \left(a^2 + \tfrac{1}{4}|v|^2 \right) \sum_{r=2m+1}^{2m+l} \partial_r^2 \\ (1.1) \quad &+ a^2 \sum_{i=1}^{2m} \partial_i^2 + a^2 \sum_{r=2m+1}^{2m+l} \sum_{i=1}^{2m} \langle [v, e_i], e_r \rangle \partial_r \partial_i. \end{aligned}$$

If f is a function on S depending only on $|v|, z$ and a , then

$$(1.2) \quad \sum_{r=2m+1}^{2m+l} \sum_{i=1}^{2m} \langle [v, e_i], e_r \rangle \partial_r \partial_i f = 0.$$

Finally the formula for the Poisson kernel (cf. [2]) is

$$(1.3) \quad P_a(y) = \frac{ca^Q}{((a^2 + |v|^2/4)^2 + |z|^2)^{Q/2}},$$

where $y = (v, z)$ and c is such that $\int_N P_a(y) dy = 1$.

2. A maximum principle. Let

$$N_a = \{ya : y \in N\}, \quad S_a = \{yb : y \in N, b > a\}.$$

We shall prove the following form of the maximum principle.

THEOREM 2.1. *For every $\varepsilon > 0$, $M > 0$ and $x = y_1 b \in S_{a_0}$ there is a ball $B \subset N_{a_0}$ with centre in $y_1 a_0$ such that, if F is harmonic in S_{a_0} (i.e. $\Delta F = 0$ on S_{a_0}), continuous in $\bar{S}_{a_0}$ and $|F| \leq M$, then*

$$F(x) \leq \sup_{y \in B} F(y) + \varepsilon.$$

PROOF. Since Δ commutes with left translations, it is sufficient to prove the theorem for $y_1 = e = (0, 0)$. Obviously, we can assume that $F \leq 0$ on B and $M/\varepsilon > 1$. We consider the function $G_\varepsilon = G_\varepsilon^1 + \eta G_\varepsilon^2$, where

$$G_\varepsilon^1(v, z, a) = -\varepsilon \frac{a - a_0}{b - a_0},$$

and

$$G_\varepsilon^2(v, z, a) = -\frac{\varepsilon}{M} \log(1 + |v|^2 + |z|^2)$$

in the domain

$$D = \left\{ (v, z, a) : a_0 < a < \frac{b - a_0}{\varepsilon} M + a_0, |v|^2 + |z|^2 < R^2 \right\},$$

where η is sufficiently small (see below) and R is such that $\varepsilon \eta \log(1 + R^2) \geq M^2$. We put

$$B = \{(v, z, a_0) : |v|^2 + |z|^2 \leq R^2\}.$$

Since by (1.1), (1.2)

$$\Delta G_\varepsilon^1 \geq \frac{1}{4}(Q - 1) \frac{\varepsilon a_0}{b - a_0} > 0$$

and ΔG_ε^2 is bounded in $\{(v, z, a) : a_0 \leq a \leq (b - a_0)M/\varepsilon + a_0\}$, taking η sufficiently small we obtain $\Delta G_\varepsilon \geq 0$. Moreover $G_\varepsilon + F \leq 0$ on ∂D , because $F \leq 0$ on B and $F \leq M$ on the rest of the boundary. Applying the classical maximum principle for elliptic operators to Δ on D we obtain $F(b) \leq \varepsilon$.

As an immediate consequence of Theorem 2.1 we have

COROLLARY 2.2. *For every $\varepsilon > 0$, $M > 0$ and $x = y_1 b \in S_{a_0}$ there is a ball $B \subset N_{a_0}$ with centre in $y_1 a_0$ such that if F is harmonic in S_{a_0} , continuous in $\bar{S}_{a_0}$ and $|F| \leq M$, then*

$$|F(x)| \leq \sup_{y \in B} |F(y)| + \varepsilon.$$

3. The representation theorem. First we are going to prove some properties of radial functions and measures on N . By $O(N)$ we denote the set of transformations $B: N \rightarrow N$ such that $B|_V$ is orthogonal and $B|_Z$ is the identity.

DEFINITION 3.1. We say that a Borel measure μ is radial if for every $B \in O(N)$ and every Borel set $X \subset N$, $\mu(BX) = \mu(X)$, and similarly a function f is radial if $f(By) = f(y)$, $y \in N$ [11].

By $M_v(N)$ and $L_v^1(N)$ we denote the set of radial measures and the set of radial integrable functions, respectively. In [11] F. Ricci proved that $L_v^1(N)$ is a commutative algebra and consequently since $L_v^1(N)$ is $*$ -weakly dense in $M_v(N)$, so is $M_v(N)$.

Let $C_{0,v}(N)$ ($C_{c,v}(N)$) denote the set of radial continuous functions vanishing at infinity (with compact support). Since

$$Ef(y) = \int_{O(N)} f(By) dB$$

is a projection on the radial functions and $\langle E\mu, f \rangle = \langle \mu, Ef \rangle$ is a projection on radial measures, the dual of $C_{0,v}(N)$ is $M_v(N)$. Of course, by (1.3), P_a , $a > 0$, are radial.

LEMMA 3.2. *The set $R^a = \{f * P_a : f \in C_{c,v}(N)\}$ is dense in $C_{0,v}(N)$.*

PROOF. Our proof is a modification of that of Przebinda [10]. Let $\mu \in M_v(N)$ be a functional vanishing on R^a , i.e. $0 = \langle f * P_a, \mu \rangle = \langle f, \mu * P_a \rangle$. Thus

$$(3.1) \quad \mu * P_a = 0.$$

Now we consider the harmonic function $F(yb) = \mu * P_b(y)$, $b > 0$, on S . By (3.1) and the maximum principle $F(yb) = 0$ for $b > a$. Since F is harmonic, it is real analytic and so $F(yb) = 0$ for all $b > 0$. Thus $\mu = 0$, because P_a is an approximate identity.

Similarly we have

LEMMA 3.3. *The set $T^a = \{f * P_a : f \in C_c(N)\}$ is dense in $C_0(N)$.*

Now we return to our main theme. Let

$$H^a = \{F \in C^2(S_a) \cap C(\bar{S}_a) \cap L^\infty(\bar{S}_a) : \Delta F = 0\}, \quad U^a = \{F|_{N_a} : F \in H^a\}.$$

In view of the maximum principle the mapping $F \rightarrow F|_{N_a}$ is injective. Since T^a is dense in $C_0(N)$, the maximum principle implies that for every f in $C_0(N)$ there is an F in H^a such that $f = F|_{N_a}$. For every $x \in S$ we can define the functional

$$\langle \varphi_a^x, f \rangle = F(x), \quad f \in U^a \text{ and } f = F|_{N_a}.$$

By Theorem 2.1 $\|\varphi_a^x\| \leq 1$. φ_a^x defines a bounded measure μ_a^x such that

$$(3.2) \quad F(x) = f * \mu_a^x(e), \quad f \in C_0(N) \text{ and } f = F|_{N_a}.$$

Let us list a few elementary properties of μ_a^x . We have

$$(3.3) \quad \mu_a^{yx} = \mu_a^x * \delta_{y^{-1}}$$

because for g in the dense set T^a

$$(3.4) \quad g * \mu_a^{yx}(e) = g * \mu_a^x * \delta_{y^{-1}}(e).$$

Putting $x = b$ and $g = F_a = F|_{N_a}$ in (3.4) we obtain

$$(3.5) \quad F(yb) = F_a * \mu_a^b(y) \quad \text{for } F \in H^a \cap C_0(\bar{S}_a).$$

In particular

$$(3.6) \quad P_b = P_a * \mu_a^b.$$

(3.6) immediately gives

$$(3.7) \quad \|\mu_a^b\| = 1 \quad \text{and} \quad \mu_a^b \geq 0,$$

and (3.6) combined with Lemma 3.3 yields

$$(3.8) \quad \mu_c^a * \mu_a^b = \mu_c^b.$$

LEMMA 3.4. *The measures μ_a^b are uniquely defined by (3.6).*

PROOF. If $P_a * \nu = 0$, then by Lemma 3.3 ν defines the zero functional on C_0 ; hence $\nu = 0$.

LEMMA 3.5. *If $f \in C_{0,v}(N)$, then $f * \mu_a^b \in C_{0,v}(N)$.*

PROOF. If $f \in R^a$, that is $f = g * P_a$ for a $g \in C_{c,v}(N)$, then by (3.6) $f * \mu_a^b = g * P_b$, which is radial. The assertion follows now by Lemma 3.2.

LEMMA 3.6. *The measures μ_a^b are radial.*

PROOF. Let $f \in C_0(N)$ and $B \in O(N)$. By the previous lemma $P_\varepsilon * \mu_a^b \in L_v^1(N)$ for $\varepsilon > 0$. Then we have

$$\langle P_\varepsilon * f, \mu_a^b \rangle = \langle f, P_\varepsilon * \mu_a^b \rangle = \langle f \circ B, P_\varepsilon * \mu_a^b \rangle = \langle P_\varepsilon * (f \circ B), \mu_a^b \rangle.$$

Hence, if $\varepsilon \rightarrow 0$, we obtain $\langle f, \mu_a^b \rangle = \langle f \circ B, \mu_a^b \rangle$, which completes the proof.

Now we look at the relation of these measures to bounded harmonic functions. First of all we notice that (3.2) is true for all $f \in U^a$, because the functional φ_a^x is defined on the space $U^a \subset C(N) \cap L^\infty(N)$ and in view of (3.7) attains its norm on $C_0(N)$. As before, (3.2) combined with (3.3) gives

$$(3.9) \quad F(yb) = F_a * \mu_a^b(y) \quad \text{for } F \in H^a.$$

Now we are in a position to prove the main theorem of this section.

THEOREM 3.7. *If F is a bounded harmonic function on S , then there is an $f \in L^\infty(N)$ such that $F(yb) = f * P_b(y)$.*

PROOF. Since the family $\{F_a\}_{a>0}$ is uniformly bounded there is a sequence F_{a_n} which is convergent $*$ -weakly to a function $f \in L^\infty(N)$ when $a_n \rightarrow 0$. In particular, $F_{a_n} * P_b \rightarrow f * P_b$. On the other hand, Lemma 3.6 and (3.9) imply

$$F_{a_n} * P_b = F_{a_n} * (P_{a_n} * \mu_{a_n}^b) = (F_{a_n} * \mu_{a_n}^b) * P_{a_n} = F_b * P_{a_n}.$$

Since P_{a_n} is an approximate identity, $F_b * P_{a_n} \rightarrow F_b$ a.e. when $a_n \rightarrow 0$ [2] and the theorem follows.

4. Smoothness of μ_a^b . In this section we investigate more precisely the measures μ_a^b by means of the Gelfand transform of $L_v^2(N)$. The Gelfand transform of $L_v^1(N)$ is described in [11]. There are two families of multiplicative functionals. For a real nonnegative ρ we write

$$\hat{f}(\rho) = \int_N f(v, z) e^{i\rho\langle v, v_0 \rangle} dv dz,$$

where v_0 is a fixed unit vector in V . For $w \in Z \setminus \{0\}$ and a nonnegative integer n we have

$$(4.1) \quad \hat{f}(w, n) = \left(\binom{n+m-1}{n} \right)^{-1} \int_N f(v, z) e^{i\langle w, z \rangle} \mathcal{L}_n^{m-1} \left(\frac{1}{2} |w| |v|^2 \right) dz dv,$$

where $\mathcal{L}_n^{m-1}(r) = e^{-r/2} L_n^{m-1}(r)$ and $L_n^{m-1}(r)$ is the Laguerre polynomial of degree n and order $m-1$ [4]. Formula (4.1) defines a unitary operator

$$\mathcal{F}: L_v^2(N) \rightarrow \mathcal{M} = \left\{ \chi: \|\chi\|_{\mathcal{M}}^2 = \int_Z \sum_{n=0}^{\infty} |\chi(w, n)|^2 \binom{n+m-1}{n} |w|^m dw < \infty \right\};$$

that is a Plancherel theorem holds [11]:

$$\|f\|_{L^2}^2 = (2\pi)^{-m-l} \|\hat{f}\|_{\mathcal{M}}^2.$$

Thus, of course, we have

$$(4.2) \quad \mathcal{F}(P_a * f) = \mathcal{F}P_a \cdot \mathcal{F}f.$$

Let

$$(4.3) \quad \mathcal{F}^{-1}(\chi) = (2\pi)^{-m-l} \sum_{n=0}^{\infty} \int_Z \chi(w, n) e^{-i\langle w, z \rangle} \mathcal{L}_n^{m-1} \left(\frac{1}{2} |w| |v|^2 \right) |w|^m dw.$$

Applying the ordinary inversion formula for the Fourier transform and the orthogonality relation

$$\frac{n!}{(n+m-1)!} \int_0^{\infty} L_n^{m-1}(r) L_{n_1}^{m-1}(r) r^{m-1} e^{-r} dr = \delta_{n, n_1}$$

we obtain by routine calculation that

$$\mathcal{F}^{-1} \circ \mathcal{F} = I \quad \text{and} \quad \mathcal{F} \circ \mathcal{F}^{-1} = I.$$

Using (4.3) we can write the formula for what should be the density of μ_a^b , $b > a$. First we show that $\hat{P}_b(w, n)/\hat{P}_a(w, n) \in \mathcal{X}$, and second that $\psi_a^b = \mathcal{F}^{-1}(\hat{P}_b/\hat{P}_a)$ satisfies the equation $P_b = P_a * \psi_a^b$. If we also prove that $\psi_a^b \in L_v^1(N)$, then we shall have $\mu_a^b = \psi_a^b$.

LEMMA 4.1.

$$(4.4) \quad \begin{aligned} \hat{P}_a(w, n) &= c_1 |w|^{-m} e^{-|w|a^2} \\ &\times \int_0^{\infty} \left(\frac{t}{1+t} \right)^n e^{-2|w|a^2 t} t^{(2m+l-1)/2} (1+t)^{(l-1)/2} dt, \end{aligned}$$

where

$$c_1 = 2^{m+1} \pi^{(2m+l+1)/2} c a^Q \left(\Gamma \left(\frac{Q}{2} \right) \right)^{-1} \left(\Gamma \left(\frac{2m+l+1}{2} \right) \right)^{-1}$$

and c is the constant in (1.3).

PROOF. By (5.1) of [1] and (1.3) we have

$$(4.5) \quad \begin{aligned} \int_Z P_a(v, z) e^{i\langle w, z \rangle} dz &= c_2 e^{-|w|(a^2 + |v|^2/4)} \\ &\times \int_0^{\infty} e^{-|w|(a^2 + |v|^2/4)t} (t^2 + 2t)^{(2m+l-1)/2} dt, \end{aligned}$$

where

$$c_2 = c a^Q 2^{1-2m-l} \pi^{(l+1)/2} \left(\Gamma \left(\frac{Q}{2} \right) \cdot \Gamma \left(\frac{2m+l+1}{2} \right) \right)^{-1}.$$

Hence

$$(4.6) \quad \begin{aligned} \hat{P}_a(w, n) &= \binom{n+m-1}{n}^{-1} c_2 e^{-|w|a^2} \int_V \int_0^{\infty} e^{-|w|(a^2 + |v|^2/4)t} \\ &\times (t^2 + 2t)^{(2m+l-1)/2} e^{-|w||v|^2/2} L_n^{m-1} \left(\frac{1}{2} |w| |v|^2 \right) dt dv. \end{aligned}$$

Integrating first over v , by

$$L_n^{m-1}(r) = \frac{1}{n!} r^{-m+1} e^r D^n(r^{n+m-1} e^{-r})$$

we obtain

$$\begin{aligned} & \int_V e^{-|w||v|^2 t/4} e^{-|w||v|^2/2} L_n^{m-1}\left(\frac{1}{2}|w||v|^2\right) dv \\ &= \left(\frac{2}{|w|}\right)^m \frac{\pi^m}{n!(m-1)!} \int_0^\infty e^{-rt/2} D^n(r^{n+m-1} e^{-r}) dr, \end{aligned}$$

which by parts is equal to

$$(4.7) \quad \left(\frac{2}{|w|}\right)^m \frac{\pi^m}{n!(m-1)!} \left(\frac{t}{2}\right)^n \frac{(n+m-1)!}{(1+t/2)^{n+m}}.$$

Finally putting (4.7) into (4.6) we get (4.4).

LEMMA 4.2.

$$(4.8) \quad \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} \leq \begin{cases} c_3 e^{-c_4 \sqrt{|w|n}} e^{-|w|(b^2-a^2)} & \text{if } 0 < |w| \leq 1, \\ c_3 e^{-c_4 \sqrt{n}} e^{-|w|(b^2-a^2)} |w|^{(l-1)/2} & \text{if } |w| > 1, \end{cases}$$

where $c_3 = (b/a)^{l-1} + (b^2+1)^{(l-1)/2} c_5$, $c_4 = (b^2-a^2)/(c_6+b^2)$, $c_6 = \frac{3}{2}a+1$, $c_5 > 0$ depends only on $m+l-1$ and a .

PROOF. By (4.4)

$$\frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} = e^{-|w|(b^2-a^2)} \frac{\int_0^\infty e^{-2t} (t/(t+|w|b^2))^n t^{(2m+l-1)/2} (|w|b^2+t)^{(l-1)/2} dt}{\int_0^\infty e^{-2t} (t/(t+|w|a^2))^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2} dt}.$$

Let

$$I_n(a, A) = \int_0^A e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2} dt$$

and

$$J_n(a, A) = \int_A^\infty e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2} dt.$$

Analogously we define $I_n(b, A)$ and $J_n(b, A)$. We estimate separately

$$(i) \quad \frac{I_n(b, c_6 \sqrt{n|w|})}{I_n(a, \infty)} \quad \text{and} \quad (ii) \quad \frac{J_n(b, c_6 \sqrt{n|w|})}{I_n(a, \infty)}.$$

Since $(t+|w|a^2)/(t+|w|b^2)$ increases,

$$\begin{aligned} I_n(b, c_6 \sqrt{n|w|}) &= \int_0^{c_6 \sqrt{n|w|}} e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n \left(\frac{t+|w|a^2}{t+|w|b^2}\right)^n t^{(2m+l-1)/2} \\ &\quad \times (|w|a^2+t)^{(l-1)/2} \left(\frac{|w|b^2+t}{|w|a^2+t}\right)^{(l-1)/2} dt \end{aligned}$$

$$d \leq \left(\frac{b}{a}\right)^{l-1} \left(\frac{c_6 \sqrt{n} + \sqrt{|w|a^2}}{c_6 \sqrt{n} + \sqrt{|w|b^2}}\right)^n I_n(a, c_6 \sqrt{n|w|}).$$

Hence for (i) we have

$$(4.9) \quad \frac{I_n(b, c_6 \sqrt{n|w|})}{I_n(a, \infty)} \leq \left(\frac{b}{a}\right)^{l-1} \exp\left(-\frac{b^2 - a^2}{c_6 + \sqrt{|w|}b^2} \sqrt{|w|}n\right).$$

To estimate (ii), we notice that if $t \geq c_6 \sqrt{n|w|}$, then

$$(|w|b^2 + t) \leq \begin{cases} (b^2 + 1)t & \text{if } |w| \leq 1, \\ (b^2 + 1)|w|t & \text{if } |w| > 1 \end{cases}$$

and

$$(4.10) \quad J_n(b, c_6 \sqrt{n|w|}) \leq c_7 \int_{c_6 \sqrt{n|w|}}^{\infty} e^{-2t} t^{m+l-1} dt,$$

where $c_7 = (b^2 + 1)^{(l-1)/2} \max(1, |w|^{(l-1)/2})$. Since

$$\left(\frac{t}{t + |w|a^2}\right)^n \geq \left(\frac{t}{t + |w|a^2}\right)^{t^2/a^2|w|} \geq e^{-t}$$

for $t \geq a\sqrt{n|w|}$, we have

$$(4.11) \quad I_n(a, \infty) \geq \int_{a\sqrt{n|w|}}^{\infty} e^{-3t} t^{m+l-1} dt.$$

There is a constant c_5 depending only on $m + l - 1$, a and c_6 such that

$$(4.12) \quad \frac{\int_{c_6 \sqrt{n|w|}}^{\infty} e^{-2t} t^{m+l-1} dt}{\int_{a\sqrt{n|w|}}^{\infty} e^{-3t} t^{m+l-1} dt} \leq c_5 \exp((-2c_6 + 3a)\sqrt{n|w|}) \\ = c_5 e^{-2\sqrt{n|w|}}.$$

Finally putting (4.10)–(4.12) together we obtain

$$\frac{J_n(b, c_6 \sqrt{n|w|})}{I_n(a, \infty)} \leq c_7 c_5 e^{-2\sqrt{n|w|}}$$

which with (4.9) implies (4.8).

LEMMA 4.3. *For every nonnegative integer p*

$$\sum_{n=0}^{\infty} \int_Z \left| \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} \right|^2 (2n + m)^p n^{m-1} |w|^{m+p} dw < \infty.$$

PROOF. Let $h_n(w) = |\hat{P}_b(w, n)/\hat{P}_a(w, n)|^2 |w|^{m+p}$. Since

$$\int_{|w| \geq 1} h_n(w) dw \leq c_3^2 e^{-2c_4 \sqrt{n}} \int_{|w| \geq 1} \exp(-2|w|(b^2 - a^2)) |w|^{m+p+l-1} dw$$

and

$$\sum_{n=0}^{\infty} e^{-2c_4 \sqrt{n}} (2n + m)^p n^{m-1} < \infty$$

by (4.8), we have

$$\sum_{n=0}^{\infty} (2n + m)^p n^{m-1} \int_{|w| \geq 1} h_n(w) dw < \infty.$$

On the other hand,

$$\begin{aligned} \int_{|w| \leq 1} h_n(w) dw &\leq c_3^2 \int_{|w| \leq 1} \exp(-2c_4 \sqrt{n|w|}) \exp(-2|w|(b^2 - a^2)) |w|^{m+p} dw \\ &\leq c_3^2 \int_Z e^{-2c_4 \sqrt{n|w|}} |w|^{m+p} dw \\ &= c_3^2 \frac{2l\pi^{1/2}}{\Gamma(l/2 + 1)} \frac{(2(m+p+l) - 1)!}{(2c_4)^{2(m+p+l)}} \frac{1}{n^{m+p+l}} \end{aligned}$$

and $\sum_{n=0}^{\infty} (2n+m)^p / n^{1+p+l} < \infty$ for $l > 0$. Hence also

$$\sum_{n=0}^{\infty} (2n+m)^p n^{m-1} \int_{|w| \leq 1} h_n(w) dw < \infty$$

which concludes the proof.

THEOREM 4.4. *The measures μ_a^b are absolutely continuous and their densities ψ_a^b are given by the formula*

$$(4.13) \quad \psi_a^b(v, z) = (2\pi)^{-l-m} \sum_{n=0}^{\infty} \int_Z \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} \mathcal{L}_n^{m-1}(\tfrac{1}{2}|w||v|^2) e^{-i\langle w, z \rangle} |w|^m dw.$$

PROOF. The previous lemma and the inequality

$$\binom{n+m-1}{n} \leq \frac{\text{const}}{(m-1)!} n^{m-1}$$

yield $\hat{P}_b/\hat{P}_a \in \mathcal{H}$. Hence $\psi_a^b \in L_v^2(N)$, $\mathcal{F}(\psi_a^b) = \hat{P}_b/\hat{P}_a$ and by (4.2) $\mathcal{F}(P_b) = \mathcal{F}(P_a * \psi_a^b)$. Thus we have

$$(4.14) \quad P_b = P_a * \psi_a^b.$$

Let f be a continuous function with compact support. Then by (4.14) $(f * P_a) * \psi_a^b(e) = (f * P_a) * \mu_a^b(e)$, which shows that ψ_a^b defines a continuous functional on $C_0(N)$. Hence $\psi_a^b \in L^1(N)$ and in view of the uniqueness of μ_a^b as the solution of (3.6) ψ_a^b is the density of μ_a^b .

THEOREM 4.5. *The functions ψ_a^b are smooth and, for every left-invariant operator ∂ on N , $\partial(\psi_a^b) \in L^2(N)$.*

PROOF. Let $L = -\sum_{i=1}^{2m} E_i^2$. Here E_i is the left-invariant field on N corresponding to e_i . Analogously to the case of Δ we can easily check that

$$L = \sum_{i=1}^{2m} \partial_i^2 + \sum_{s=2m+1}^{2m+l} \sum_{i=1}^{2m} \langle [v, e_i], e_s \rangle \partial_s \partial_i + \frac{1}{4} |v|^2 \sum_{s=2m+1}^{2m+l} \partial_s^2.$$

If f is a radial function, then by (1.2)

$$(4.15) \quad Lf = \left(\frac{\partial^2}{\partial r^2} + \frac{2m-1}{r} \frac{\partial}{\partial r} + \frac{1}{4} r^2 \sum_{s=2m+1}^{2m+l} \partial_s^2 \right) f,$$

where $r = |v|$. Applying (4.15) and the equality

$$r(L_n^{m-1})''(r) + (m-r)(L_n^{m-1})'(r) + nL_n^{m-1}(r) = 0$$

[4, vol. 2, p. 188] we obtain that $\mathcal{L}_n^{m-1}(\frac{1}{2}|w||v|^2)e^{-i\langle w,z \rangle}$ are eigenfunctions of L with eigenvalues $(2n+m)|w|$. Let

$$g_n(v, z) = (2\pi)^{-l-m} \int_Z \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} \mathcal{L}_n^{m-1} \left(\frac{1}{2}|w||v|^2 \right) e^{-i\langle w,z \rangle} |w|^m dw.$$

$g_n \in C^\infty(N)$ and by (4.13) $\sum_{n=0}^\infty g_n$ is convergent in $L^2(N)$ to ψ_a^b . We have

$$L^k g_n(v, z) = (2\pi)^{-l-m} \int_Z \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} (2n+m)^k \mathcal{L}_n^{m-1} \left(\frac{1}{2}|w||v|^2 \right) e^{-i\langle w,z \rangle} |w|^{m+k} dw$$

and

$$\|L^k g_n\|_{L^2}^2 = (2\pi)^{-l-m} \binom{n+m-1}{n} (2n+m)^{2k} \int_Z \left| \frac{\hat{P}_b(w, n)}{\hat{P}_a(w, n)} \right|^2 |w|^{m+2k} dw.$$

Hence by Lemma 4.3

$$\sum_{n=0}^\infty \|L^k g_n\|_{L^2}^2 < \infty.$$

This shows that ψ_a^b belongs to the domain of the closure of L^k and so ψ_a^b is smooth. Since, by [5], for every left-invariant differential operator ∂ on N there is a k and d such that

$$\|\partial f\|_{L^2} \leq d(\|L^k f\|_{L^2} + \|f\|_{L^2}),$$

the theorem follows.

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