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DEGREE THEORY ON ORIENTED INFINITE DIMENSIONAL VARIETIES AND THE MORSE NUMBER OF MINIMAL SURFACES SPANNING A CURVE IN $\mathbf{R}^n$. PART I: $n \geq 4$

BY

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Dedicated to Stefan Hildebrandt and Steven Smale

ABSTRACT. A degree theory applicable to Plateau's problem is developed and the Morse equality for minimal surfaces spanning a contour in $\mathbf{R}^n$, $n \geq 4$, is proved.

In [**10**] the author and David Elworthy developed a theory of degree for Fredholm maps on oriented Banach manifolds, a theory which generalized the now classical Leray-Schauder degree theory.

The purpose of this paper is to show how one can use a slight extension of Elworthy-Tromba theory to develop a "Morse-theory" for minimal surfaces of disc type spanning a wire in $\mathbf{R}^n$. We prove, via this theory, the first and last of Morse's "inequalities", and in so doing obtain an existence theorem to the classical problem of Plateau, and simultaneously a generalization of the famous mountain pass result for minimal surfaces due to Morse, Shiffman and Tompkins. Before proceeding with a detailed statement of the results we would like to present a bit of historical background.

The problem of finding a surface of least area spanning a wire Γ in Euclidian three space has interested mathematicians since the last century, primarily due to the pioneering soap film experiments of the Belgian physicist Plateau (1801–1883) and the mathematical work of Riemann, Weierstrass and H. A. Schwarz all dating from about 1860. However, the mathematical proof of the existence of such a soap film of the topological type of the disc, and more generally the existence of a solution to the corresponding "normalized" Euler equation, eluded researchers until the major breakthrough by Jesse Douglas and Tibor Rado in 1931. Douglas proved the existence of a solution to "Plateau's problem" not by minimizing the area function considered as a function on the space of all surfaces spanning Γ, but by showing that the critical points of the classical Dirichlet's[1] integral on this space of surfaces were "minimal surfaces" as defined by Plateau. Consequently the absolute minimum of this integral (which he proved existed) yielded a solution.

However, Douglas did not know either how smooth his solution was nor whether it was immersed or embedded. The regularity question was eventually settled in

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[1]Douglas actually used another functional, called the Douglas functional, which for harmonic surfaces agrees with Dirichlet's functional.

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1968 by S. Hildebrandt [**16**] (with later improvements by others). The Douglas solution was found to be an immersion on the interior of the disc by Osserman [**23**], Gulliver [**13**] and Alt [**1**] (1968), and in the case of analytic wires Γ this solution is an immersion on the closed disc (Gulliver and Lesley [**14**]).

In the thirties and forties, as a result of Douglas' work on disc type minimal surfaces, many mathematicians became interested in extending the work of Marston Morse on the Calculus of Variations in the large to several variables and in particular to Plateau's problem. These included Morse (for example, see Raoul Bott's recent AMS Bulletin survey article on Morse's work [**6**]), Shiffman, Tompkins and Courant. A full "Morse theory" relating the topology of the level sets of the associated Dirichlet's functional with the Morse indexes of the critical points was never achieved and only a Morse type inequality was proved independently by Morse and Tompkins [**18**] and Shiffman [**25**]. This result states that if there exist two isolated (in the C^0 topology) disc type minimal surfaces spanning a wire Γ which are also strict minima in the C^0 topology and with respect to Dirichlet's integral, then there exists a third unstable minimal surface. Morse and Tompkins did, however, develop [**18**] some kind of Morse theory for this problem. Morse had already developed an abstract Morse theory for lower semicontinuous real valued functions on a metric space [**17**] and Morse and Tompkins showed that this theory "might" apply to Plateau's problem for disc surfaces. Since their functions were not C^1 (they were not even continuous) they introduced the notion of a homotopy critical point σ and the notion of the "kth cap type number" of σ. They showed that homotopy critical points are minimal surfaces but not necessarily vice-versa. Assuming that the number of minimal surfaces spanning a given curve was finite in number (a hypothesis which was not known at this time to hold) they were able to relate k-cap type numbers to the Betti numbers of the level sets of Dirichlet's functional. It is not clear to this author how these inequalities are related to the standard Morse inequalities one learns today.

The method of Morse-Tompkins and Shiffman did not permit an extension to the cases where the minima were known to be isolated in a finer topology like a Sobolev H^s, or Hölder $C^{k,\alpha}$ topology, i.e. it was unknown whether a third surface would exist. In addition the methods of these authors would not allow one to say anything about whether the third surface was immersed or exactly what singularity type it had.

We have already mentioned the regularity result of Hildebrandt in 1968. This suggested the possibility of developing a degree theory of minimal surfaces of disc type along the lines of Elworthy-Tromba. However, the full development of this theory had to await the development of a general global index and stability theory of minimal surfaces of disc type due to the author and Reinhold Böhme [**5**]. It was here that generic nondegeneracy and generic finiteness results for minimal surfaces of disc type were proved, the first concrete information on the number and isolated nature of solutions to Plateau's problem to be obtained. However, the degree theory of Elworthy-Tromba requires some extension to fit this situation; namely an extension to infinite dimensional oriented varieties.

It was one of our objectives to develop such an extension, to show how oriented infinite dimensional manifolds and varieties arise naturally in variational problems, and then finally to apply these ideas to obtain a "degree theory" for Plateau's

problem. This theory can be viewed as a limited Morse theory for minimal surfaces of disc type.

Up to this time the only application of this new degree theory, in which Leray-Schauder theory did not in some form also apply, has been to give a proof of a generalized asymptotic fixed point formula [**32**].

Statement of main result. Let D be the unit disc in $\mathbf{R}^2$, $\partial D = S^1$ and $\alpha: S^1 \to \mathbf{R}^n$, $n \geq 4$, an embedding of Sobolev class H^r, with $\Gamma^\alpha = \alpha(S^1)$ the image wire. Let $\mathcal{N}(\alpha)$ be the space of all H^s maps, $s \geq 5/2$, $r \geq 2s+4$, $u: D \to \mathbf{R}^n$, $u = (u^1, \ldots, u^n)$ such that for all i, $\Delta u^i = 0$ (each u^i is harmonic) and $u: S^1 \to \Gamma^\alpha$ is homotopic to α.

A minimal surface of disc type is an element $u \in \mathcal{N}(\alpha)$ satisfying

$$\langle u_x, u_y \rangle = 0 = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle, \tag{1}$$

$$\|u_x\| = \|u_y\|, \tag{2}$$

$$u: S^1 \to \Gamma^\alpha \quad \text{is a homomorphism,} \tag{3}$$

where $\langle\ ,\ \rangle$ and $\|\ \|$ denote the $\mathbf{R}^n$ scalar product and norm respectively.

A map $u \in \mathcal{N}(\alpha)$ satisfying (1)–(3) is a critical point of Dirichlet's functional $E_\alpha: \mathcal{N}(\alpha) \to R$ defined by

$$E_\alpha(u) = \frac{1}{2} \sum_{i=1}^{n} \int_D \nabla u^i \cdot \nabla u^i.$$

Let $\mathcal{G}$ be the three dimensional conformal group of the disc. Each $g \in \mathcal{G}$ is of the form $g(z) = c \cdot (z - a)/(1 - \bar{a}z)$, $|c| = 1$, $|a| < 1$. $\mathcal{G}$ acts on $\mathcal{N}(\alpha)$ via $g_\#(u)(z) = u(g(z))$, and E_α is equivariant with respect to this action. Consequently if u is any critical point of E_α, then $\mathcal{O}_u(\mathcal{G})$, the orbit of $\mathcal{G}$ through u, consists of critical points. For such a critical point let $D^2E(u): T_u\mathcal{N}(\alpha) \times T_u\mathcal{N}(\alpha) \to R$ denote the Hessian or second derivative, where $T_u\mathcal{N}(\alpha)$ is the tangent space to $\mathcal{N}(\alpha)$ at u. By equivariance, it follows that $T_u\mathcal{O}_u(\mathcal{G})$, the tangent space to $\mathcal{O}_u(\mathcal{G})$ at u, will always be in the kernel of the quadratic form $D^2E_\alpha(u)$. The surface u is said to be "nondegenerate" if this is the only kernel. If u is nondegenerate there is (modulo this kernel) a maximal subspace on which the Hessian is negative, and this dimension is called the *Morse index* of u.

By the Fredholm index theory for minimal surfaces of Böhme-Tromba, almost all (open-dense) wires α (in $\mathbf{R}^n$, $n \geq 4$) enjoy the property that there are only a finite number of nondegenerate minimal surfaces spanning Γ^α. Suppose that for such an α they are $u_1, \ldots, u_m$ of Morse indexes $\lambda_1, \ldots, \lambda_m$ respectively. Then our *main* result is

THEOREM. *For any such* α, $\sum_{i=1}^m (-1)^{\lambda_i} = 1$.

REMARK. The number $\sum_{i=1}(-1)^{\lambda_i}$ is called the *Morse number* of minimal surfaces spanning Γ^α. So the above theorem says that the Morse number of minimal surfaces spanning "any" wire is one.

In the work at hand we have restricted our attention only to minimal surfaces in $\mathbf{R}^n$, $n > 3$; why did we omit $n = 3$? The fact is that the structure of the space of all

minimal surfaces $\Sigma \subset \mathcal{N}$ is quite a bit more complicated when $n = 3$. It is possible [**5**] that there may be minimal surfaces u which are formally degenerate in that the derivative of the vector field X_α at u is *not* an isomorphism from $T_u\mathcal{O}_u(\mathcal{G})^\perp$ to itself, yet stable under perturbations of the boundary wire α.

Thus in attempting to measure a Morse number of minimal surfaces one is forced generically to deal with branched (nonimmersed) minimal surfaces. This is not the case in $\mathbf{R}^n$, $n \geq 4$. The following theorem is again a brief summary of what can happen in $\mathbf{R}^3$, and is part of the Index Theorem [**5**].

THEOREM. *An open dense set $\hat{\mathcal{A}}$ in the space of all wires $\mathcal{A} = \{\alpha \in H^r(S^1, \mathbf{R}^3) |$ α an embedding, $\kappa(\Gamma^\alpha) \leq M$, $\kappa(\Gamma^\alpha)$ total curvature of $\Gamma^\alpha\}$ enjoys the property that for $\alpha \in \hat{\mathcal{A}}$ all minimal surfaces spanning Γ^α will be*

(i) *immersed,*

(ii) *have simple interior branch points,*

where by simple interior branch points for a minimal surface $u: D \to \mathbf{R}^3$ we mean that the holomorphic map $F: D \to \mathbf{C}^3$ defined by $F(z) = u_x - iu_y$ has only simple interior zeros. Moreover

(iii) *those with simple interior branch points can be stable under perturbations of α.*

In a further work [**34**] we have developed a degree theory for minimal surfaces in $\mathbf{R}^3$, defined a Morse number and shown that, as expected, this Morse number of minimal surfaces is one.

Furthermore, under certain circumstances one can define the Morse number of minimal immersion, embeddings and simple branched surfaces. We have the following result [**33**]:

THEOREM. *Let α be a curve on the boundary of a convex body in $\mathbf{R}^3$ (such an α is called an extreme curve). Then one can define:*

(i) *The Morse number $\mathcal{M}(\mathcal{E}, \alpha)$ of embeddings spanning Γ^α.*

(ii) *The Morse number $\mathcal{M}(\mathcal{I}_0, \alpha)$ of immersions spanning Γ^α which are not embeddings.*

(iii) *For each p, the Morse number (modulo 2) $\mathcal{M}(\mathcal{I}_p, \alpha)$ of branched minimal surfaces spanning Γ^α with p simple interior branch points.*

Then $\mathcal{M}(\mathcal{E}, \alpha) = 1$, $\mathcal{M}(\mathcal{I}_0, \alpha) = 0$ and $\mathcal{M}(\mathcal{I}_p, \alpha) = 0$, $p \geq 1$.

This work was completed in August 1981.

Recently, aware of the author's proofs of the first and last Morse inequalities, M. Struwe was able to prove all of the Morse inequalities in the case $n \geq 4$, and in fact showed that a complete Morse theory holds in this situation. Such a theory does not hold in $\mathbf{R}^3$ where a degree theoretic approach seems to be necessary.

1. A review of Elworthy-Tromba theory. Let M be an infinite dimensional Banach manifold without boundary modelled on a Banach space E. Let $L_c(\mathbf{E})$ denote those linear operators of the form identity plus compact, $\mathrm{GL}(E)$ the invertible linear operators on E, and $\mathrm{GL}_c(\mathbf{E}) = \mathrm{GL}(\mathbf{E}) \cap L_c(\mathbf{E})$. From [**10**] we have the following result.

THEOREM 1.1. *The path component space $\pi_0(\mathrm{GL}_c(\mathbf{E}))$ of the Banach Lie group $\mathrm{GL}_c(\mathbf{E})$ is isomorphic to Z_2 and consequently has two components.*

Let $\mathrm{GL}_c^+(\mathbf{E})$ *denote the component of the identity in* $\mathrm{GL}_c(E)$ *and* $\mathrm{GL}_c^-(\mathbf{E})$ *the other component.*

DEFINITION 1.2. A smooth C^r Banach manifold M is a *Fredholm* manifold or admits a C^r Fredholm structure Φ if there is a collection of coordinate charts (φ_i, U_i) covering M so that the derivative of the transition maps $\varphi_i \circ \varphi_j^{-1}$ (when $U_i \cap U_j \neq \emptyset$) is in $\mathrm{GL}_c(\mathbf{E})$. M is *oriented* with respect to Φ if a subcollection of charts of Φ can be found to have the property that the derivative of the transition maps is always in $\mathrm{GL}_c^+(\mathbf{E})$.

REMARK 1.3. Every open subset of a Banach space has a natural orientable Fredholm structure induced by $\mathrm{GL}_c^+(\mathbf{E})$ itself. It is possible [**10**] that an infinite dimensional manifold can be orientable with respect to one Fredholm structure and not another.

From [**10**] we have results on the existence of Fredholm and orientable Fredholm structures. Before we describe these results we need the notion of a Fredholm map.

DEFINITION 1.4. A differentiable map between two Banach manifolds M and N is Fredholm if the derivative $df(x): T_xM \to T_{f(x)}N$ is linear Fredholm for each $x \in M$. This means that $\dim \operatorname{Ker} df(x) < \infty$, $\dim \operatorname{Coker} df(x) < \infty$ and the range of $df(x)$ is closed. The index of $df(x)$ at x is defined to be $\dim \operatorname{Ker} df(x) - \dim \operatorname{Coker} df(x)$. One can show that this index is constant on the components of M and thus if M is connected one can speak of the index of f. If M is not connected one requires the index to be the same on all components. Thus one speaks of Fredholm maps of index m.

We then have the following results from [**10**].

THEOREM 1.5. *Let M be a Banach manifold modelled on $\mathbf{E}$ which admits C^r partitions of unity. Then M admits a C^r Fredholm structure if and only if there exists a Fredholm map $f: M \to \mathbf{E}$ of index zero.*

THEOREM 1.6. *Let M be a C^r Banach manifold admitting C^r partitions of unity and such that* $\mathrm{GL}(\mathbf{E})$ *is contractible. Then M admits a Fredholm structure.*

THEOREM 1.7. *A C^r Banach manifold M is orientable with respect to every possible Fredholm structure if $H^1(M; Z_2) = 0$; the first singular cohomology of M with Z_2 coefficients is zero. Conversely if* $\mathrm{GL}(E)$ *is contractible and M admits C^1 partitions of unity, then the orientability of M with respect to every Fredholm structure implies $H^1(M; Z_2) = 0$.*

The existing application of this theorem which has arisen in analysis [**11**] is where M is an open subset of a ball B in a Banach space E and the Fredholm structure on M is a restriction of the Fredholm structure on B. Thus this Fredholm structure is orientable.

We shall describe the Elworthy-Tromba degree theory for the case of Fredholm maps of index zero from a Banach manifold M to an open subset $\mathcal{A}$ of a Banach space $\mathbf{F}$. The case where $\mathcal{A}$ is a Banach manifold and the index is not necessarily zero is treated in [**10**]. In the case of nonzero index p the degree is no longer an integer but an element of a stable p-stem of the homotopy group of spheres.

Suppose $\pi: M \to \mathcal{A}$ is Fredholm of index zero and proper (by this we mean that the inverse image of compact sets is compact). We would like to assign an oriented degree to the map π. The two principal tools needed to accomplish this will be the

generalization of the Sard theorem due to Smale and the pull back Theorem 1.9 which follows.

DEFINITION 1.8. Let M be a C^r Banach manifold which admits a C^r-Fredholm structure Φ. A Fredholm map $\pi: M \to \mathcal{A}$ of index zero is said to be *admissible* with respect to Φ if for $(\varphi_i, U_i) \in \Phi$ the Fréchet derivative $D(\pi \circ \varphi_i^{-1})(x) \in L_c(\mathbf{E})$; i.e. as a linear map is of the form identity plus compact.

THEOREM 1.9 (PULL BACK THEOREM, ELWORTHY-TROMBA). *Let $\pi: M \to \mathcal{A}$ be as above with M a Banach manifold, $\mathcal{A} \subset \mathbf{F}$ an open subset and π Fredholm of index zero. The Fredholm structure Φ on $\mathcal{A}$ that $\mathcal{A}$ inherits from $\mathbf{F}$ (cf. Remark* 1.3) *induces via π a unique Fredholm structure $\pi^*\Phi$ on M with respect to which π is admissible. We call $\pi^*\Phi$ the pull back of Φ under π.*

In order to define a degree for such a map π we will need to know that π is proper and that $\pi^*\Phi$ is orientable.

The question of properness is naturally a question of estimates, e.g. a priori elliptic estimates for solutions to nonlinear elliptic systems. The question of orientability is in a strong sense the heart of this paper. One could ask whether $H^1(M; Z_2) = 0$ and if so invoke Theorem 1.7 and this certainly works for the class of problems considered in [**11**]. However one may not be able to say anything whatsoever about $H^k(M; Z_2)$ for any k. This is the case with Plateau's problem. So we need some other condition to insure that $\pi^*\Phi$ is orientable and this will be given in §3. We shall therefore now assume outright that $\pi^*\Phi$ is orientable. Recall that this means that there is a subcollection of charts (φ_i, U_i) in $\pi^*\Phi$ covering M such that the derivative $D(\varphi_i \circ \varphi_j^{-1})$ when defined is in $\mathrm{GL}_c^+(\mathbf{F})$.

In this case we define the oriented degree of π, $\deg \pi$, as follows. Let $y \in \mathcal{A}$ be a regular value of π (i.e. $x \in \pi^{-1}(y)$ implies that $D\pi(x): T_xM \to F$ is surjective). Since π is index zero this means that for each $x \in \pi^{-1}(y)$, $D\pi(x)$ is an isomorphism. The inverse function theorem guarantees that $\pi^{-1}(y)$ is a discrete set and properness implies that $\pi^{-1}(y)$ is a finite set of points $x_1, \ldots, x_m$. For each x_i let (φ_i, U_i) be a coordinate chart in an orientation of M relative to $\pi^*\Phi$. Define

$$\operatorname{sgn} D\pi(x_i) = \begin{cases} +1 & \text{if } D(\pi \circ \varphi_i^{-1})(\varphi_i(x)) \in \mathrm{GL}_c^+(\mathbf{F}), \\ -1 & \text{if not} \end{cases} \tag{1.10}$$

and $\deg \pi = \sum_i \operatorname{sgn} D\pi(x_i)$.

If $\mathcal{A}$ is connected it can be shown (cf. [**10**]) that $\deg \pi$ is independent of the choice of regular value y and is also independent, under certain types of homotopies, of the choice of element in a restricted homotopy class of π. But most importantly if $\deg \pi \neq 0$, π is surjective.

2. Oriented varieties and degree theory. We would now like to extend the notion of degree to maps $\pi: M \to \mathcal{A}$, where M is no longer a manifold, but some infinite dimensional oriented subvariety of some Banach space $\mathbf{E}$. This is certainly the case for Plateau's problem [**5**].

We shall suppose as before that $\mathcal{A}$ is connected and open in a Banach space $\mathbf{F}$. Moreover we shall suppose M to be made up of pieces $\mathcal{M}_d$, $M = \bigcup_{d \in \mathcal{D}} \mathcal{M}_d$, these pieces being each C^2 Banach manifolds indexed by some linearly ordered set $\mathcal{D}$, with order relation $>$ in such a way that the following assumptions on stratification hold (cf. [**28**]):

(2.1) (i) $\partial \mathcal{M}_{d_i} \subset \bigcup_{d>d_i} \overline{\mathcal{M}}_d$,

(ii) $\pi: M \to \mathcal{A}$ is proper and C^2 Fredholm of some nonpositive index on each $\mathcal{M}_d$,

(iii) that there is one "strata" or piece $\mathcal{M}_0$ such that π on $\mathcal{M}_0$ is C^2 Fredholm of index zero, and $d > 0$ for all d,

(iv) on all other $\mathcal{M}_d$, π is C^2 Fredholm of index less than or equal to negative 2, and finally

(v) $\mathcal{M}_0$ is orientable with respect to the pull back structure $\pi^*\Phi$ of the canonical Fredholm structure Φ on $\mathcal{A}$.

Then we claim that there is an oriented Elworthy-Tromba integer degree for π. In order to see this we need a result due to Tomi and Tromba [**27**].

THEOREM 2.2. *Under the stratification, index and properness assumptions on π, the image $\pi(\bigcup_{d>0} \mathcal{M}_d)$ has a closed nowhere dense image in $\mathcal{A}$ and does not disconnect $\mathcal{A}$.*

REMARK 2.3. It will be very important to note that Theorem 2.2 holds under a weakening of the stratification hypotheses (2.1), namely we may modify (ii) to assume that each $\mathcal{M}_d$, $d \neq 0$, need not be a manifold but is *locally* contained in some manifold $\mathcal{N}_d$ such that $\pi|\mathcal{N}_d$ is Fredholm of index less than negative two (again see [**28**]).

Using 2.2 we can construct an oriented degree on $\mathcal{M}$ as follows. Let $\alpha \in \mathcal{A}$ be a regular value for π. By 2.2 we may assume that $\alpha \notin \pi(\bigcup_{d>0} \mathcal{M}_d)$. From the fact that π on $\mathcal{M}$ is proper and π is index zero on $\mathcal{M}_0$ we know that $\pi^{-1}(\alpha)$ consists of a finite number of points $x_1, \ldots, x_m$. Since $\mathcal{M}_0$ is orientable with respect to $\pi^*\Phi$ we can define

$$\deg(\pi, \alpha) = \sum_{x_i} \operatorname{sgn} D\pi(x_i),$$

where $\operatorname{sgn} D\pi(x_i)$ is defined as in (1.10).

The main difficulty now, of course, is to show that $\deg(\pi, \alpha)$ is independent of the choice of $\alpha \in \mathcal{A}$. Therefore let us suppose that $\beta \in \mathcal{A}$ is another regular value of $\pi, \beta \notin \pi(\bigcup_{d>0} \mathcal{M}_d)$. By 2.2 we can find a path $\sigma: I \to \mathcal{A}$, I the unit interval such that $\sigma(0) = \alpha$, $\sigma(1) = \beta$, and with $\sigma(I) \subset \mathcal{A} - \pi(\bigcup_{d>0} \mathcal{M}_d)$. Since $\pi(\bigcup_{d>0} \mathcal{M}_d)$ is closed we may apply Smale's transversality theorem to perturb σ to a map $\tilde{\sigma}$ such that $\tilde{\sigma}(0) = \alpha$, $\tilde{\sigma}(1) = \beta$, π is transversal to $\tilde{\sigma}$, and $\tilde{\sigma}(I) \subset \mathcal{A} - \pi(\bigcup_{d>0} \mathcal{M}_d)$. Thus $\pi^{-1}(\sigma(I))$ effects a cobordism between $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$ and then the argument of Elworthy-Tromba [**10**] show that $\deg(\pi, \beta) = \deg(\pi, \alpha)$ and we take this to be the degree of π.

3. The existence of natural orientations on Banach manifolds. In §1 we described two results (Theorems 1.6 and 1.7) which give conditions when a Banach manifold has a Fredholm structure which is orientable. As was already remarked there has only been one application [**32**] of this result to analysis. In this section we develop several results on how oriented Banach manifolds may naturally arise in

problems in nonlinear analysis. Applications to Plateau's problem will be given in §5. We begin with the first basic result on oriented infinite dimensional manifolds.

THEOREM 3.1. *Let $\mathcal{A} \subset \mathbf{F}$ be open in a Banach space $\mathbf{F}$, $\mathcal{O} \subset \mathcal{A} \times \mathbf{E}$ open, $\mathbf{E}$ a Banach space, and let $f\colon \mathcal{O} \to E$ be a C^r, $r \geq 2$, map such that whenever $f(a,e) = 0$ the partial derivative of f in the E direction $D_2 f(a,e)\colon \mathbf{E} \to \mathbf{E}$ is a linear Fredholm map of the form identity plus compact, i.e. $D_2 f(a,e) \in L_c(\mathbf{E})$. Suppose further than whenever $x \in f^{-1}(0)$ the total derivative $Df(x)\colon \mathbf{F} \times \mathbf{E} \to \mathbf{E}$ is surjective.*

Then $\Sigma = f^{-1}(0)$ is a C^2 Banach manifold which inherits from $\mathbf{E}$ a natural "orientable" Fredholm structure $\tilde{\Phi}$. Moreover the projection map $\pi\colon \mathcal{A} \times \mathbf{E} \to \mathcal{A}$ which is projection onto the first factor restricts to a map $\pi_\Sigma\colon \Sigma \to \mathcal{A}$ which is Fredholm of index zero.

If Φ denotes the Fredholm structure which $\mathcal{A}$ inherits as an open subset of $\mathbf{F}$, then $\pi_\Sigma^* \Phi = \tilde{\Phi}$. Finally and perhaps most importantly, $\alpha \in \mathcal{A}$ is a regular value for π iff $D_2 f(\alpha, e)\colon \mathbf{E} \to \mathbf{E}$ is an isomorphism for any e for which $f(\alpha, e) = 0$. Moreover, with respect to the natural orientation which f induces on Σ, if $x \in \pi_\Sigma^{-1}(\alpha)$, then $\operatorname{sgn} D\pi_\Sigma(x) = \operatorname{sgn} D_2 f(\alpha, e)$, *where $x = (\alpha, e)$ and where we recall that*

$$\operatorname{sgn} D_2 f(\alpha, e) = \begin{cases} +1 & \text{if } D_2 f(\alpha, e) \in \mathrm{GL}_c^+(\mathbf{E}), \\ -1 & \text{otherwise.} \end{cases}$$

This last fact is a link between degree theory and Calculus of Variations in the Large.

PROOF. The fact that Σ is a manifold is, of course, a direct application of the implicit function theorem. We shall, however, be forced to examine the proof of this fact at greater depth in order to obtain the additional information we seek. We begin by proving a small lemma.

LEMMA 3.2. *Let $\mathcal{A}$, $\mathbf{E}$ and Y be Banach spaces with $L\colon \mathcal{A} \times \mathbf{E} \to Y$ a linear and surjective operator such that $L_2 = L(0, \cdot)$ as a map from $\mathbf{E}$ to Y is Fredholm. Let π denote the projection of $\mathcal{A} \times \mathbf{E} \to \mathcal{A}$ and $M = \operatorname{Ker} L$. Then we have* $\dim \operatorname{Ker}(\pi|M) = \dim \operatorname{Ker}(L_2)$ *and* $\dim \operatorname{Coker}(\pi|M) = \dim \operatorname{Coker}(L_2)$. *Thus if L_2 is Fredholm of index zero so is $\pi|M$.*

PROOF OF 3.2. Setting $L_1 = L(\cdot, 0)$ we have $L(x_1, x_2) = L_1(x_1) + L_2(x_2)$. Obviously

$$\begin{aligned} \operatorname{Ker} \pi|M &= \{(x_1, x_2) | L_1(x_1) + L_2(x_2) = 0, x_1 = 0\} \\ &= \{0\} \times \operatorname{Ker} L_2 \end{aligned}$$

which proves the statement on the kernels.

Since $Y = \operatorname{range} L_1 + \operatorname{range} L_2$ we can find a subspace $\mathcal{A}_0$ of $\mathcal{A}$ such that $\mathcal{A}_0 \cap \operatorname{Ker} L_1 = \{0\}$ and $Y = L_1(\mathcal{A}_0) \oplus L_2(\mathbf{E})$. Choosing a complimentary subspace $\mathbf{E}_1$ of $\operatorname{Ker} L_2$ in E it is easily seen that $L\colon (\mathcal{A}_0 \times 0) + (0 \times \mathbf{E}_1) \to Y$ is a bijection. We have therefore, the splitting

$$\mathcal{A} \times E = M \oplus (\mathcal{A}_0 \times 0) \oplus (0 \times \mathbf{E}_1)$$

from which we obtain $\mathcal{A} = \pi(M) \oplus \mathcal{A}_0$. This proves the lemma since $\dim \mathcal{A}_0 = \dim \operatorname{Coker} L_2$.

COROLLARY 3.3. *If $\mathcal{A}_1$ denotes any complement to $\mathcal{A}_0$ in $\mathcal{A}$ and $\mathbf{E}_0 = \operatorname{Ker} L_2$, then the kernel M of L is naturally isomorphic to $\mathcal{A}_1 \times \mathbf{E}_0$.*

PROOF. From above we see that M is a complement to $\mathcal{A}_0 \times \mathbf{E}_1$ in $\mathcal{A} \times \mathbf{E}$, but then so is $\mathcal{A}_1 \times \mathbf{E}_0$ and this gives the result. More specifically let $\tilde{P}$ be the projection of $\mathcal{A} \times \mathbf{E}$ onto $\mathcal{A}_0 \times \mathbf{E}_1$ and $P = I - \tilde{P}$. Then $P: M \to \mathcal{A}_1 \times \mathbf{E}_0$ isomorphically.

PROOF OF THEOREM 3.1. The fact that Σ is a manifold follows immediately from the implicit function theorem. In addition $T_{(\alpha,e)}\Sigma = \operatorname{Ker} Df(\alpha, e)$. That π_Σ is Fredholm of index zero and the statement that α is a regular value iff $D_2 f(\alpha, e): \mathbf{E} \to \mathbf{E}$ is an isomorphism follows immediately from Lemma 3.2 and Corollary 3.3 by setting $D\pi_\Sigma = \pi$ (the π now of 3.2) and $L = Df(\alpha, e)$ and noting in 3.3 that $D_2 f(\alpha, e)$ is an isomorphism iff $\mathbf{E}_0 = \operatorname{Ker} L_2 = \operatorname{Ker} D_2 f(\alpha, e) = \{0\}$, and that since $D_2 f \in L_c(E)$, $\dim \mathcal{A}_0 = \dim \mathbf{E}_0$ (all elements of $L_c(\mathbf{E})$ are of index zero).

To place an orientation on Σ is a bit trickier. Let $(\tilde{a}, \tilde{e})$ be a point where $f(\tilde{a}, \tilde{e}) = 0$. We may assume without loss of generality that $(\tilde{a}, \tilde{e}) = (0, 0)$. As in 3.2 let $\mathbf{E}_0 = \operatorname{Ker} D_2 f(\tilde{a}, \tilde{e}) = \operatorname{Ker} L_2$ with E_1 a complement to E_0 in E. Also let $\mathcal{A}_0$ and $\mathcal{A}_1$ be chosen as in 3.2 with $L = Df(\tilde{a}, \tilde{e})$, $M = \operatorname{Ker} L$ and *identify* (cf. 3.3) M *with* $\mathcal{A}_1 \times E_0$. Finally let $T: E_0 \to \mathcal{A}_0$ be any linear isomorphism. Define $\mathcal{F}: \mathcal{A} \times E \to \mathcal{A} \times E$ as follows: write

$$(a, e) = (a_1, e_0) + (a_0, e_1) \tag{3.4}$$

and set

$$\mathcal{F}(a, e) = (a, 0) + (Te_0, 0) + (0, f(a, e)). \tag{3.5}$$

We claim that $D\mathcal{F} \in \mathrm{GL}_c(\mathbf{F} \times E)$, where

$$D\mathcal{F}(\tilde{a}, \tilde{e})[a, e] = (a_1, 0) + (Te_0, 0) + (0, Df(\tilde{a}, \tilde{e})[a, e]).$$

To see that $D\mathcal{F}(\tilde{a}, \tilde{e})$ is injective we express each (a, e) as $(a, e) = (a_1, e_0) + (a_0, e_1)$ with $Df(\tilde{a}, \tilde{e})[a_1, e_0] = 0$ $\forall a_1, e_0$ and where $Df(\tilde{a}, \tilde{e})|\mathcal{A}_0 \times \mathbf{E}_1$ is an isomorphism onto $\mathbf{E}$. Thus if $D\mathcal{F}(\tilde{a}, \tilde{e})[a, e] = 0$, then necessarily $Te_0 = 0$ and so $e_0 = 0$ and $a_1 = 0$. Then $Df(\tilde{a}, \tilde{e})[a_0, e_1] = 0$ which implies that $a_0 = 0$ and $e_1 = 0$, which shows injectivity.

To see that $D\mathcal{F}(\tilde{a}, \tilde{e})$ is surjective note that if $(a^*, e^*) \in \mathbf{F} \times \mathbf{E}$, then

$$(a^*, e^*) = (a_1^*, 0) + (a_0^*, 0) + (0, e_1^* + e_0^*).$$

Choose a, e to map onto (a^*, e^*) by choosing $a_1 = a_1^*$, e_0 so that $Te_0 = a_0^*$ and (a_0, e_1) so that $Df(\tilde{a}, \tilde{e})[a_0, e_1] = e_1^* + e_0^*$.

We must now verify that $D\mathcal{F}$ is of the form identity plus compact linear. So to do this it is necessary to show that $D\mathcal{F} - I$ is compact linear. Again using the fact that every (a, e) can be decomposed uniquely as in (3.5) we see that for any $(\hat{a}, \hat{e})$

$$\begin{aligned}
&D\mathcal{F}(\hat{a}, \hat{e})[a, e] - [a, e] \\
&\quad = (a_1, 0) + (Te_0, 0) + (0, Df(\hat{a}, \hat{e})[a, e]) - \{(a_1, e_0) + (a_0, e_1)\} \\
&\quad = (Te_0 - a_0, -e_0) + (0, Df(\hat{a}, \hat{e})[a, e] - e_1) \\
&\quad = (Te_0 - a_0, -e_0) + (0, Df(\hat{a}, \hat{e})[(a_1, e_0) + (a_0, e_1)] - e_1) \\
&\quad = (Te_0 - a_0, -e_0) + (0, D_1 f(\hat{a}, \hat{e})[a_0] + D_2 f(a, e)[e_1] - e_1).
\end{aligned}$$

Since $\dim \mathcal{A}_0 < \infty$, $\dim \mathbf{E}_0 < \infty$ and $D_2 f(\hat{a}, \hat{e})[e_1] = e_1 + Ke_1$, K compact linear, the result follows. Thus $D\mathcal{F}(\hat{a}, \hat{e}) \in \mathrm{GL}_c(\mathbf{F} \times \mathbf{E})$ and so belongs to $\mathrm{GL}_c^+(\mathbf{F} \times \mathbf{E})$ or $\mathrm{GL}_c^-(\mathbf{F} \times \mathbf{E})$. If $D\mathcal{F}(\tilde{a}, \tilde{e}) \in \mathrm{GL}_c^-[\mathbf{F} \times \mathbf{E}]$ we modify it as follows. Let $J \in \mathrm{GL}_c^-(\mathcal{A}_1)$ and define a new $\mathcal{F}$, say $\mathcal{F}^*$, by

$$\mathcal{F}^*(a, e) = (Ja_1, 0) + (Te_0, 0) + (0, f(a, e)). \tag{3.6}$$

Then $D\mathcal{F}^*(\tilde{a}, \tilde{e}) \in \mathrm{GL}_c^+(\mathbf{F} \times E)$. Therefore without loss of generality we shall assume that $D\mathcal{F}(\tilde{a}, \tilde{e}) \in \mathrm{GL}_c^+(\mathbf{F} \times \mathbf{E})$.

REMARK 3.7. In the case that $\dim \mathbf{F} < \infty$ one might have to modify the construction of $\mathcal{F}^*$ ($\mathcal{A}_1$ could be 0) by using the J on the "$\mathbf{E}_0$" term.

By the inverse function theorem $\mathcal{F}$ has a local inverse G. It follows that $f \circ G(a, e) = e$.

Thus *locally* G maps a neighborhood U of 0 in $\mathbf{F} \times \mathbf{E}$ diffeomorphically onto a neighborhood of $(\tilde{a}, \tilde{e}) \in \Sigma = f^{-1}(0)$, and thus G restricted to a neighborhood of the origin in $\mathcal{A}$, say $\overline{G}$, gives a coordinate mapping (actually the inverse of a coordinate mapping) for Σ about $(\tilde{a}, \tilde{e}), \overline{G}(0) = (\tilde{a}, \tilde{e})$.

To finish the theorem we must check four facts:

(i) If $\overline{G}_1$ and $\overline{G}_2$ are any two such coordinate mappings, then $D(\overline{G}_1^{-1} \circ \overline{G}_2) \in \mathrm{GL}_c(\mathbf{F})$.

(ii) That, in fact, $D(\overline{G}_1^{-1}, \overline{G}_2) \in \mathrm{GL}_c^+(\mathbf{F})$.

(iii) $D(\pi \circ \overline{G}) \in L_c(\mathbf{F})$ and for $a \in U$.

(iv) $D(\pi \circ \overline{G})(a) \in \mathrm{GL}_c^+(\mathbf{F})$ iff $D_2 f(\overline{G}(a)) \in \mathrm{GL}_c^+(E)$.

PROOF. (i), (ii). From the construction of the G_i's it is clear that $D(G_1^{-1} \circ G_2) \in \mathrm{GL}_c^+(\mathbf{F} \times E)$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote G_1^{-1} and G_2^{-1} respectively. Now $D(G_1^{-1} \circ G_2) = D(\mathcal{F}_1 \circ G_2)$ and $\mathcal{F}_1 \circ G_2(a, e) = (-, f \circ G_2(a, e)) = (-, e)$. From this it follows that $D(G_1^{-1} \circ G_2)$ can be represented by the matrix

$$\begin{pmatrix} D(\overline{G}_1^{-1} \circ \overline{G}_2) & \sim \\ 0 & I \end{pmatrix},$$

where I is the identity mapping. It follows that the derivative of $D(\overline{G}_1^{-1} \circ \overline{G}_2)$ is in $\mathrm{GL}_c(\mathbf{F})$ and moreover must be in $\mathrm{GL}_c^+(\mathbf{F})$. Thus the selection of these G_i's not only give us a Fredholm structure for Σ but an orientation as well.

(iii) A simple computation shows that for $h \in \mathcal{A}$

$$D(\pi \circ \overline{G})_{(a)}[h] = D(\pi_\Sigma \circ \overline{G})_{(a)}[h] = \begin{cases} h_1 + K(h) & \text{or} \\ Jh_1 + K(h), \end{cases}$$

where K is a finite dimensional linear map and $J \in \mathrm{GL}_c^-(\mathcal{A}_1)$ (cf. (3.6)). Thus, since $\mathcal{A}_1$ is finite codimensional it follows that $D(\pi_\Sigma \circ \overline{G}) \in L_c(\mathbf{F})$, and so by the uniqueness Theorem 2.9 the pull back under π_Σ of the Fredholm structure of $\mathcal{A}$ must be the same as the Fredholm structure defined by the G's above.

To check point (iv) suppose that $D_2 f(\tilde{a}, \tilde{e}) \colon E \to E$ is an isomorphism in $\mathrm{GL}_c^+(E)$. Then $\mathcal{F}$ as defined in (3.5) will be given by $\mathcal{F}(a, e) = (a, 0) + (0, f(a, e))$ and clearly $D(\pi_\Sigma \circ \overline{G})_0[h] = h$.

If $D_2 f(\tilde{a}, \tilde{e}) \in \mathrm{GL}_c^-(E)$, then in defining the orientation for Σ we have defined $\mathcal{F}(a, e) = (Ja, 0) + (0, f(a, e))$ and $D(\pi_\Sigma \circ \overline{G})_{(0)}[h] = Jh$. Thus

$$D(\pi_\Sigma \circ \overline{G})_{(0)} \in \mathrm{GL}_c^-(\mathbf{F}).$$

If $a \in U$ is any other point with $D_2 f(\overline{G}(a)) \in \mathrm{GL}_c^+(\mathbf{E})$, then construct another $\overline{G}_1$ with $\overline{G}_1(0) = \overline{G}(a)$ as in (3.5) which is a coordinate chart for $f^{-1}(0) = \Sigma$.

Then from what we have just done $D(\pi_\Sigma \circ \overline{G}_1)_{\overline{G}_1(0)} \in \mathrm{GL}_c^+(\mathbf{E})$ iff $D_2 f(\overline{G}_1(0)) = D_2 f(\overline{G}(a)) \in \mathrm{GL}_c^+(\mathbf{E})$. However our construction guarantees that $D(\overline{G}_1^{-1} \circ \overline{G}) \in \mathrm{GL}_c^+(\mathbf{F})$ from which it follows that $D(\pi_\Sigma \circ \overline{G})_{(a)} \in \mathrm{GL}_c^+(\mathbf{F})$ iff $D_2 f(\overline{G}(a)) \in \mathrm{GL}_c^+(\mathbf{E})$ and the proof of 3.1 is complete.

The situation in Theorem 3.1 is not the actual situation which arises in several nonlinear problems in analysis, as for example Plateau's problem. We shall now, in steps, lead up to a result which does apply to Plateau's problem.

Let $\mathcal{A} \subset \mathbf{F}$ be open and suppose that for each $a \in \mathcal{A}$ we have a Banach manifold $\mathcal{M}_a$ modelled on a Banach space $\mathbf{E}$ in such a way that $\bigcup_{a \in \mathcal{A}} \mathcal{M}_a = \mathcal{M}$ has the structure of a smooth fibre bundle with bundle projection map $\pi\colon \mathcal{M} \to \mathcal{A}$ (cf. [**29**]).

For each $a \in \mathcal{A}$ suppose that $X_a\colon \mathcal{M}_a \to T\mathcal{M}_a$ is a vector field on $\mathcal{M}_a$ such that the induced vector field X on $\mathcal{M}$ defined by $X|\mathcal{M}_a = X_a$ is C^r, $r \geq 2$.

DEFINITION 3.8. A vector field with the property that whenever $X_a(x) = 0$, the derivative $DX_a(x)\colon T_x\mathcal{M}_a \hookleftarrow$ is in $L_c(T_x\mathcal{M}_a)$ is said to be *Palais-Smale*. For $a \in \mathcal{A}$ fixed, a zero x of X_a is nondegenerate whenever $DX_a(x) \in \mathrm{GL}_c(T_x\mathcal{M}_a)$. Thus the sign of a zero is well defined depending on whether or not $DX_a(x)$ belongs to GL_c^+ or GL_c^-. Formally

$$\operatorname{sgn} DX_a(x) = \begin{cases} +1 & \text{if } DX_a(x) \in \mathrm{GL}_c^+(T_x\mathcal{M}_a), \\ -1 & \text{if } DX_a(x) \in \mathrm{GL}_c^-(T_x\mathcal{M}_a). \end{cases}$$

Warning. The derivative of a vector field is *not* a well-defined object as a morphism from a tangent space to itself away from a zero. Thus it makes no sense to speak of $DX_a(x) \in L_c(T_x\mathcal{M}_a)$ away from a zero x.

By restricting ourselves to a coordinate neighborhood $\tilde{\mathcal{O}} \subset \mathcal{M}$ about a point $m \in \mathcal{M}$ we can, using a coordinate mapping $\varphi\colon \tilde{\mathcal{O}} \to \mathcal{O}$ pass to an open subset $\mathcal{O} \subset \mathcal{A} \times E$ and represent $X|\tilde{\mathcal{O}}$ as a C^r map $f_\varphi\colon \mathcal{O} \to E$ such that whenever $f_\varphi(\tilde{a}, \tilde{e}) = 0$, $D_2 f_\varphi(\tilde{a}, \tilde{e}) \in L_c(\mathbf{E})$. If for a fixed $\tilde{a}$ the point $\tilde{e}$ represents a "nondegenerate" zero of X_a, then $D_2 f_\varphi(\tilde{a}, \tilde{e}) \in \mathrm{GL}_c^+$ or GL_c^- and it easily follows that whether it is in GL_c^+ or GL_c^- is independent of the choice of coordinate representation. The $\operatorname{sgn} D_2 f_\varphi(\tilde{a}, \tilde{e})$ agrees with that of $\operatorname{sgn} DX_{\tilde{a}}(x)$ above. The intrinsic nature of $\operatorname{sgn} D_2 f_\varphi(\tilde{a}, \tilde{e})$ permits the following important extension of Theorem 3.1.

THEOREM 3.9. *Let $X\colon \mathcal{M} \to T\mathcal{M}$ be a C^r vector field ($r \geq 2$) on a trivial Banach fibre bundle $\mathcal{M} = \mathcal{A} \times M$ over an open subset $\mathcal{A} \subset \mathbf{F}$ such that for each $a \in \mathcal{A}$, $X_a = X|\mathcal{M}_a$ is a Palais-Smale field on $\mathcal{M}_a$. Assume that M admits some* ***orientable*** *Fredholm structure ¶. Suppose further that whenever $X(m) = 0$ the total derivative $DX(m)\colon T_m\mathcal{M} \to T_m\mathcal{M}_a$ is surjective. Then the zero set Σ of X is a smooth submanifold of $\mathcal{M}$ which has a natural orientable Fredholm structure $\tilde{\Phi}$. The projection map $\pi\colon \mathcal{M} \to \mathcal{A}$ restricts to a map $\pi_\Sigma\colon \Sigma \to \mathcal{A}$ which is Fredholm of index zero. If Φ denotes the Fredholm structure which $\mathcal{A}$ inherits as an open subset of $\mathbf{F}$, then the pull back Fredholm structure $\pi_\Sigma^*\Phi = \tilde{\Phi}$. Moreover $a \in \mathcal{A}$ is a regular value for π_Σ iff $DX_a(x)\colon T_x\mathcal{M}_a \to T_x\mathcal{M}_a$ is an isomorphism for any x for which $X_a(x) = 0$.*

Finally with respect to the natural orientation which X induces on Σ, if $x \in \pi_\Sigma^{-1}(a)$ then

$$\operatorname{sgn} D\pi_\Sigma(x) = \operatorname{sgn} DX_a(x). \tag{3.10}$$

We repeat that (3.10) *is a principal link between degree theory and the Calculus of Variations in the Large.*

PROOF. That Σ is a manifold and $\pi_\Sigma\colon \Sigma \to \mathcal{A}$ Fredholm of index zero follows immediately from 3.1. We must show that the differentiable structure induced on Σ by 3.1 is an orientable Fredholm structure.

Let $\{\overline{\varphi}_i, \tilde{\mathcal{O}}_i\}_{i\in I}$ denote the coordinate charts in ¶ with $D(\overline{\varphi}_i \circ \overline{\varphi}_j^{-1}) \in \mathrm{GL}_c^+(\mathbf{E})$, where M is modelled on $\mathbf{E}$. This coordinate system induces a coordinate system on the product bundle $\mathcal{A} \times M$ as follows: $\overline{\varphi}_i\colon \mathcal{A} \times \tilde{\mathcal{O}}_i \to \mathcal{A} \times \mathbf{E}$ is simply $\varphi_i(a,m) = (a, \overline{\varphi}_i(m))$.

For each such $(\overline{\varphi}, \tilde{\mathcal{O}})$ we may represent X locally as $f_\varphi\colon \mathcal{A} \times \mathcal{O} \to \mathbf{E}$ as we just discussed. Each choice of $\overline{\varphi}$ leads by 3.1 to a collection of coordinate charts $\{\varphi^{-1} \circ \overline{G}_\varphi\}$ for $\Sigma \cap (\mathcal{A} \times \tilde{\mathcal{O}})$ which for each fixed $\overline{\varphi}$ yields an orientable Fredholm structure for $\Sigma \cap (\mathcal{A} \times \tilde{\mathcal{O}})$.

By construction, π_Σ is admissible (cf. 1.8) with respect to all the $\varphi^{-1} \circ \overline{G}_\varphi$'s and so this structure must by 1.9 be a Fredholm structure. We only need to show that for any choice of φ, ψ (as long as $\psi \circ \varphi^{-1}$ is defined)

$$D(G_\psi^{-1} \circ \psi\varphi^{-1} \circ \overline{G}_\varphi) \in \mathrm{GL}_c^+(\mathbf{F}).$$

Recall from 3.1 that each $\overline{G}_\psi = G_\psi | V \times \{0\}$, V open in $\mathbf{F}$, and $G_\psi = \mathcal{F}_\psi^{-1}$ where $\mathcal{F}_\psi(a,e) = (-, f_\psi(a,e))$ and $D\mathcal{F}_\psi, D\mathcal{F}_\varphi \in \mathrm{GL}_c^+(\mathbf{F} \times E)$. Now

$$G_\psi^{-1}\psi \circ \varphi^{-1} G_\varphi = (-, f_\psi \circ \psi \circ \varphi^{-1} G_\varphi) = (-, D\overline{\psi} \circ f \circ \varphi^{-1} \circ G_\varphi),$$

where we are writing $X_a(x) = (a, f(a,x)) \in \mathcal{A} \times T_x M$. Continuing the equality this is clearly equal to $(-, D(\overline{\psi} \circ \overline{\varphi}^{-1}) f_\varphi \circ G_\varphi)$. By hypothesis $D(\overline{\psi} \circ \overline{\varphi}^{-1}) \in \mathrm{GL}_c^+(\mathbf{E})$ and by construction $f_\varphi G_\varphi(a,e) = e$. From this it immediately follows that at a zero of X_a, $D(G_\psi^{-1}\psi\varphi^{-1}G_\varphi)$ is a matrix of the form

$$\begin{pmatrix} D(\overline{G}_\psi^{-1} \circ \psi \circ \varphi^{-1} \circ \overline{G}_\varphi) & \sim \\ 0 & D(\overline{\psi} \circ \overline{\varphi}^{-1}) \end{pmatrix} \in \mathrm{GL}_c^+(\mathbf{F} \times \mathbf{E}).$$

Hence $D(\overline{G}_\psi^{-1} \circ \psi \circ \varphi^{-1} \circ \overline{G}_\varphi) \in \mathrm{GL}_c^+(\mathbf{F})$. The remainder of the theorem follows as in 3.1.

REMARK 3.11. Theorem 3.9 has an obvious extension to the case of nontrivial bundles, namely one requires that each fibre $\mathcal{M}_a$ be modelled on M, that each $\mathcal{M}_a$ has an orientable Fredholm structure and finally that a collection of bundle trivialization can be found which fibrewise preserve these Fredholm structures.

4. The index theorem for classical minimal surfaces. Our main reference for this section is the Index Theorem of Böhme and Tromba [**5**]. Let Γ be a smooth wire in $\mathbf{R}^n$ which is the image of a differentiable embedding $\alpha\colon S^1 \to \mathbf{R}^n$, $\alpha(S^1) = \Gamma^\alpha$. A solution to the classical Plateau problem for Γ is a mapping $x\colon \overline{D} \to \mathbf{R}^n$, $\overline{D}$ the closed unit disc in $\mathbf{R}^2$, $\partial D = S^1$ such that:

(1) $\Delta x = 0$ (each component of x is harmonic),

(2) $x_u \cdot x_v = 0$ (the coordinates of $\mathbf{R}^2$ being labeled by u, v),
(3) $\|x_u\|^2 = \|x_v\|^2$,
(4) $x: S^1 \to \Gamma^\alpha$ is a homeomorphism (or monotonic).

Conditions (2) and (3) imply that the surface is conformally parametrized. Moreover these are the Euler equations for critical points of Dirichlet's integral on the space of all surfaces spanning Γ.

A point $p \in D$, where $F(z) = x_u + ix_v$, $i = \sqrt{-1}$, vanishes, is called an interior branch point of x. Since F is holomorphic p has a finite order λ ($F(z) = (z-p)^\lambda G(z)$, $G(p) \neq 0$). A point $q \in S^1$ where F vanishes is called a boundary branch point and by results of Heinz and Tomi [**15**] q has a well-defined order. Equations (2) and (3) also imply that all points where x fails to be an immersion are branch points. Moreover the monotonicity assumption (4) implies that boundary branch points will all be of even order.

For integers r and s, $r \geq 2s+4$, define

$$\mathcal{D} = \mathcal{D}^s = \{u: S^1 \to S^1 | \deg u = 1 \text{ and } u \in H^s(S^1, \mathbf{C})\},$$

where H^s denotes the Sobolev space of s-times differentiable (in the distribution sense) functions with values in the complexes $\mathbf{C}$, set

$$\mathcal{A} = \{\alpha: S^1 \to \mathbf{R}^n | \alpha \in H^r\{S^1, \mathbf{R}^n) = \mathbf{F},\ \alpha \text{ an embedding}\} \tag{4.0}$$

(i.e. α is one-to-one and $\alpha^1(p) \neq 0$ for all $p \in S^1$) and the total curvature of Γ^α is bounded by $\pi(s-2)$.

Let $\pi: \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ denote the projection map onto the first factor. A minimal surface $x: \overline{D} \to \mathbf{R}^n$ spanning $\alpha \in \mathcal{A}$ can be viewed as an element of $\mathcal{A} \times \mathcal{D}$, since x is harmonic and therefore determined by its boundary values

$$\alpha | \partial D = x | S^1 = \alpha \circ u, \quad \text{where } (\alpha, u) \in \mathcal{A} \times \mathcal{D}.$$

The classical approach to minimal surfaces was to understand the set of minimal surfaces spanning a given fixed wire α; that is the set of minimal surfaces in $\pi^{-1}(\alpha)$. Our approach is to first understand the structure of the set of minimal surfaces as a subset of the bundle $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ as fibre bundle over $\mathcal{A}$ and then to approach the question of the set of minimal surfaces in the fibre $\pi^{-1}(\alpha)$ in terms of the singularities of the projection map π restricted to a suitable subvariety of $\mathcal{N}$. This is in the spirit of Thom's original approach to unfoldings of singularities.

Let us say that a minimal surface $x \in \mathcal{A} \times \mathcal{D}$ has branching type (λ, ν), $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbf{Z}^p$, $\nu = (\nu_1, \ldots, \nu_q) \in \mathbf{Z}^q$, each $\lambda_i, \nu_i \geq 0$ if x has p distinct but arbitrarily located interior branch points $z_1, \ldots, z_p$ in D of integer orders $\lambda_1, \ldots, \lambda_p$ and q distinct boundary branch points $\xi_1, \ldots, \xi_q$ in S^1 of (even) integer orders $\nu_1, \ldots, \nu_q$. In a formal sense the subset $\mathcal{M}$ of minimal surfaces in $\mathcal{N}$ is an algebraic subvariety of $\mathcal{N}$ [**5**] and is a stratified set, stratified by branching types. To be more precise let $\mathcal{M}_\nu^\lambda$ denote the minimal surfaces of branching type (λ, ν). We can now state the index result of Böhme and Tromba [**5**].

THEOREM 4.1 (INDEX THEOREM FOR DISC SURFACES). *The set $\mathcal{M}_0^\lambda$ is a C^{r-s-1} submanifold of $\mathcal{N}$ and the restriction π^λ of π to $\mathcal{M}_0^\lambda$ is C^{r-s-1}, Fredholm of index $I(\lambda) + 3 = 2(2-n)|\lambda| + 2p + 3$, where $|\lambda| = \Sigma\lambda_i$.*

Moreover, locally, for $\nu \neq 0$, $\mathcal{M}_\nu^\lambda \subset W$ such that W_ν^λ is a submanifold[2] *of $\mathcal{N}$ and where the restriction π_ν^λ of π to W_ν^λ is Fredholm of index $I(\lambda,\nu)+3 = 2(2-n)|\lambda|+(2-n)|\nu|+2p+q+3$. The number 3 comes from the equivariance of the problem under the action of the three dimensional conformal group of the disc.*

It is easy to see that if $n \geq 3$, then $I(\lambda,\nu) \leq 0$. This index measures (in some sense) the stability of minimal surfaces of branching type (λ,ν) in $\mathbf{R}^n$ and the likelihood of finding such surfaces; the more negative the index of π_ν^λ the less likely it is to find a wire admitting minimal surfaces of branching type (λ,ν) which span it.

These stratification and index results are the basis to prove the generic (open-dense) finiteness and stability of minimal surfaces of the type of the disc as discussed in [**5**]; i.e. there exists an open dense subset $\hat{\mathcal{A}} \subset \mathcal{A}$ such that if $\alpha \in \hat{\mathcal{A}}$, then there exists only a finite number of minimal surfaces bounded by α, and these minimal surfaces are stable under perturbations of α. This open dense set will be the set of regular values of the map π.

Moreover if $n > 3$ the minimal surfaces spanning $\alpha \in \hat{\mathcal{A}}$ are all immersed up to the boundary, and if $n = 3$ they are simply branched. It should be emphasized that in this theory we are considering not only area minimizing minimal surfaces bounded by $\alpha \in \mathcal{A}$, but *all* critical points of the area functional defined on the space of surfaces spanning α which satisfy the classical monotonicity condition along the boundary.

Finally there are some other surprising consequences of this index formula. For example minimal surfaces in $\mathbf{R}^3$ are free of interior branch points if they minimize area [**1, 14, 23**], whereas most minimal surfaces with simple interior branch points are stable with respect to perturbations of the boundary [**4, 5**]. Second, for $n \geq 4$, minimal surfaces in $\mathbf{R}^n$ may have branch points even if they are area minimizing, but for such n *no* such minimal surface in $\mathbf{R}^n$ is stable as a branched surface under perturbation of the boundary.

We are now ready to proceed with the development of degree theory for minimal surfaces.

5. Degree theory for minimal surfaces of disc type in $\mathbf{R}^n$, $n \geq 4$. In this section we show how the degree theory developed in §2 applies to the Plateau problem discussed in the last section.

Let $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ be the bundle over $\mathcal{A}$ introduced in the last section. Let $\alpha \in \mathcal{A}$ and let Γ^α be the image of such an embedding. Consider the manifold of maps $H^s(S^1,\Gamma^\alpha)$. In [**29**] it is shown that $H^s(S^1,\Gamma^\alpha)$ is a C^{r-s} submanifold of $H^s(S^1,\mathbf{R}^n)$. Let $\mathcal{N}(\alpha)$ denote the component of $H^s(S^1,\Gamma^\alpha)$ determined by α. (In [**29**], the notation $\mathcal{N}_\alpha$ was used for $\mathcal{N}(\alpha)$.) Recall that the tangent space to $\mathcal{N}(\alpha)$ at the point $x \in \mathcal{N}(\alpha)$ can be identified with the H^s maps $h\colon S^1 \to \mathbf{R}^n$ with $h(\theta) \in T_{x(\theta)}\Gamma^\alpha$ (the tangent space to Γ^α at $x(\theta)$). By harmonic extension we can identify elements of $\mathcal{N}(\alpha)$ with harmonic surfaces spanning Γ^α. *We shall always assume this identification.*

[2]Recently Ursula Thiel, in her Saarbrücken Ph.D. thesis, has shown that M_ν^λ is in fact a manifold with the index of π_ν^λ equal to $3 + I_\nu^\lambda$.

In [**28**] it was shown[3] that there exists a smooth C^{r-s-1} vector field X_α on $\mathcal{N}(\alpha)$ whose zeros are precisely the minimal surfaces spanning Γ^α. We should note here that each of these zeros are minimal surfaces in a more general sense than classically defined, since a zero x of X_α viewed as a harmonic map $x: D \to \mathbf{R}^n$ need not induce a homeomorphism of S^1 onto Γ^α. We have the following theorem which is of great importance to us.

THEOREM 5.1. *If $X_\alpha(x) = 0$, then the Fréchet derivative of X_α at x maps $T_x\mathcal{N}(\alpha)$ into itself and is of the form identity plus a compact linear operator. Thus each X_α is a Palais-Smale field.*

We can give a definition of the vector field X_α as follows. Let $\mathcal{T}_i: S^1 \to \mathbf{R}^n$, $i = 1, \ldots, n$, be a smooth framing of Γ^α; i.e. for each $p \in \Gamma^\alpha$, $\{\mathcal{T}_i(p)\}_{i=1}^n$ forms an orthonormal basis for $\mathbf{R}^n$. We shall assume that $\mathcal{T}_1$ is always tangential so that $\mathcal{T}_1(p) \in T_p\Gamma^\alpha$. Then the vector field is characterized by the following conditions. For each $x \in \mathcal{N}(\alpha)$, $X_\alpha(x): D \to \mathbf{R}^n$ is harmonic so

$$\text{(i)} \quad \Delta X_\alpha(x) \equiv 0$$

satisfying the mixed Neumann-Dirichlet boundary conditions

$$\begin{aligned} &\text{(ii)} \quad \frac{\partial X_\alpha}{\partial r}(x) \cdot \mathcal{T}_1(x) = \frac{\partial x}{\partial r} \cdot \mathcal{T}_1(x), \\ &\text{(iii)} \quad X_\alpha(x) \cdot \mathcal{T}_j(x) = 0, \qquad j = 2, \ldots, n, \end{aligned} \tag{5.2}$$

where $\mathcal{T}_j(x)$ denotes the composition $\mathcal{T}_j(x(\theta))$ and $\partial/\partial r$ denotes the normal or radial derivative along S^1. We can paraphrase these boundary conditions as follows: Let $\Omega: \Gamma^\alpha \to OP(\mathbf{R}^n)$, the orthogonal projections on $\mathbf{R}^n$, be the C^{r-2} map such that $\Omega(p)$ is the projection of $\mathbf{R}^n$ onto $T_p\Gamma^\alpha$. Then (5.2) can be rewritten as

$$\Omega(x)\frac{\partial X^\alpha}{\partial r}(x) = \Omega(x)\frac{\partial x}{\partial r}, \tag{5.2$'$}$$

where $\Omega(x)$ again denotes the map $\theta \to \Omega(x(\theta))$.

Let $\mathcal{N}^* = \bigcup_\alpha \mathcal{N}(\alpha)$ and $\pi: \mathcal{N}^* \to \mathcal{A}$ be the natural projection map $\pi(\mathcal{N}(\alpha)) = \alpha$. The space $\mathcal{N}^*$ (in [**28**] we used the notation $\mathcal{N}$ for $\mathcal{N}^*$) has the structure of a smooth fibre bundle over $\mathcal{A}$ which is bundle equivalent to the product bundle $\mathcal{N} = \mathcal{A} \times \mathcal{D}$ via the map $w: (\alpha, u) \to \alpha \circ u$ and hence $\mathcal{N}^*$ is globally trivial. We shall often identify $\mathcal{N}$ with $\mathcal{N}^*$ via this trivialization and hopefully no confusion should arise from this.

The family of vector fields X_α induces a C^{r-s-1} vector field X on $\mathcal{N}$ by the rule $X(x) = X_\alpha(x)$ if $x \in \mathcal{N}(\alpha)$. This vector field will be vertical in the sense that $X_\alpha(x) \in T_x\mathcal{N}(\alpha)$. If $x \in \mathcal{N}(\alpha)$ is a zero of X (and hence a zero of X_α) the Fréchet derivative of X can be viewed as a map $DX(x): T_x\mathcal{N} \to T_x\mathcal{N}(\alpha)$. If $x \in \mathcal{M}_\nu^\lambda$ we shall be interested in the corank of $DX(x)$; i.e. $\dim T_x\mathcal{N}/DX[T_x\mathcal{N}(\alpha)]$.

Before stating a theorem which gives the answer, we would like to discuss the action of the conformal group $\mathcal{G}$ of the disc on the space $\mathcal{D}$ (and hence on $\mathcal{N}(\alpha)$ and the bundle $\mathcal{N}$). So let $\mathcal{G}$ be the group of conformal transformations of the disc onto itself. It is well known that every $g \in \mathcal{G}$ is of the form

$$g(z) = c \cdot \frac{z - a}{1 - \bar{a}z}, \qquad |z| \le 1,$$

[3]In this paper the author took $s = 2$; however there is virtually no difference in the proof for $s > 2$.

where $|c| = 1$ and $|a| < 1$. $\mathcal{G}$ is a three dimensional Lie group which is not compact. $\mathcal{G}$ acts on $\mathcal{D}$ in the following manner. Let $u: S^1 \to \mathbf{C}$. For $g \in \mathcal{G}$ define $g_\#(u) \in H^s(S^1, \mathbf{C})$ by $g_\#(u)(z) = u(g(z))$. For fixed g, $g_\#$ is a linear isomorphism of H^s to itself which fixes $\mathcal{D}$. Thus $g_\#$ induces a diffeomorphism of $\mathcal{D}$ to itself and hence of $\mathcal{N}(\alpha)$ to itself. Again denote this diffeomorphism by g. The correspondence $\mathcal{G} \to \mathrm{Diff}(\mathcal{D})$ given by $g \to g_\#$ defines the action of $\mathcal{G}$ on $\mathcal{D}$.

This $\mathcal{G}$ action is not smooth (cf. [**29**, p. 39]), and this of course creates many technical difficulties for the theory. However

REMARK 5.3. If $u \in \mathcal{D}$ and $u \in H^{s+1}(S^1, \mathbf{C})$, then $g \to g_\#(u)$ is a smooth map into $\mathcal{D}$. Since the zeros of our vector field X_α are more than H^s smooth (in fact by Böhme's version [**3**] of Hildebrandt's regularity result, H^r smooth) the orbits of $\mathcal{G}$ through zeros will be smooth submanifolds.

If $x = \alpha(u)$, then the $\mathcal{G}$ action on $\mathcal{D}$ induces a $\mathcal{G}$ action on $\mathcal{N}(\alpha)$ via $g_\#(x) = \alpha(g_\#(u))$.

We have the following result from [**29**, p. 44].

THEOREM 5.4. *The vector field X_α (fixing α) is equivariant with respect to the $\mathcal{G}$-action on $\mathcal{N}(\alpha)$. This means that*

$$[Dg_\#(x)]^{-1} X_\alpha(g_\#(x)) = X_\alpha(x),$$

where D denotes the Fréchet derivative of the map $x \to g_\#(x)$ (this map being the restriction of a linear map is smooth).

The fact that each X_α is equivariant implies that the corank of the total derivative $DX(x): T_x\mathcal{N} \to T_x\mathcal{N}(\alpha)$ must be at least three and hence the assumptions of Theorem 3.9 cannot be satisfied. One can ask if this is all that can happen and the negative answer is given in

THEOREM 5.5. *If $x \in \mathcal{M}_\nu^\lambda$, then the corank of $DX(x)$ is equal to $2|\lambda| + |\nu| + 3$. The number three comes from the action of the conformal group, as was expected.*

For a proof see [**28**, p. 92] or the appendix of [**5**]. Result 5.5 tells us that only at a nonbranched minimal surface x will the total derivative of X at x be of minimal rank. We would like to apply "indirectly" the degree theory on varieties introduced in §2, and we begin as follows.

Let $\mathcal{I}(\alpha) = \{x \in \mathcal{N}(\alpha) | x(Q_j) = \alpha(Q_j)\}$, where Q_1, Q_2, Q_3 are three prescribed points on S^1. The codimension three subbundle

$$\mathcal{I} = \bigcup_\alpha \mathcal{I}(\alpha) \tag{5.6}$$

intersects each of the submanifolds $\mathcal{M}_0^\lambda$ (cf. Theorem 4.1) transversely and hence is also a submanifold which we denote by Σ_0^λ. The restriction of the projection map π, $\overline{\pi}_0^\lambda$, to Σ_0^λ is Fredholm of index $I(\lambda, 0) = 2(2-n)|\lambda| + 2p$. Moreover if $\Sigma_\nu^\lambda = \mathcal{I} \cap \mathcal{M}_\nu^\lambda$ it follows again from the index theorem that locally Σ_ν^λ is contained in a C^{r-s-1} submanifold W_ν^λ such that $\overline{\pi}_\nu^\lambda = \pi | W_\nu^\lambda$ has index $I(\lambda, \nu) = 2(2-n)|\lambda| + (2-n)|\nu| + 2p + q$.

The following calculation is elementary.

THEOREM 5.7. *If $n \geq 4$ and λ or ν is not zero, $I(\lambda, \nu) \leq -2$. The index of $\overline{\pi}_0^0$ is zero.*

From the results of Tomi and Tromba [**28**] it follows that there is a partial ordering on these strata Σ_ν^λ and that the complement in $\mathcal{A}$ of the union of the images $\bigcup_{\lambda \neq 0, \nu \neq 0} \pi(\Sigma_\nu^\lambda)$ does not disconnect any component of $\mathcal{A}$.

Consequently, *if* $\mathcal{A}$ were connected and if the strata Σ_0^0 carries a natural orientation which is related to the vector field X^α as in 3.9 and such that π restricted to $\Sigma = \bigcup \Sigma_\nu^\lambda$, say, is proper we would have achieved a degree theory for minimal surfaces, by employing the degree theory for π_Σ developed in §2. The properness of π_Σ is a direct consequence of Hildebrandt's regularity theorem. The proof of this result can be found in [**5**] and we state it formally as

THEOREM 5.8. *The map $\pi_\Sigma : \Sigma \to \mathcal{A}$ is a proper map.*

However in the beginning of §4 (cf. (4.0)) we imposed a curvature condition on the set $\mathcal{A}$ and so the set $\mathcal{A}$ is not "formally connected" even though the set of all H^r embeddings is. This is only a minor difficulty in constructing a degree theory for π_Σ which we take up again in 5.35.

Our next goal will be to put an orientation on Σ_0^0 related to the derivatives of the vector field X_α. To start we must introduce the weak Riemannian structure on the manifold $\mathcal{N}(\alpha)$.

DEFINITION 5.9. Let $x \in \mathcal{N}(\alpha)$ and $h, k \in T_x \mathcal{N}(\alpha)$. Define $\langle\langle h, k \rangle\rangle_x$, the weak inner product of h and k over x, by the formula

$$\langle\langle h, k \rangle\rangle_x = \int_{S^1} \left\langle \frac{\partial h}{\partial r}, k \right\rangle d\theta = \sum_i \int_D \langle \nabla h^i, \nabla k^i \rangle,$$

where again h and k are identified with their harmonic extensions.

THEOREM 5.10. *The weak inner product is $\mathcal{G}$-invariant, and consequently the Dirichlet functional $E_\alpha : \mathcal{N}(\alpha) \to R$ defined by*

$$E_\alpha(x) = \sum_{i=1}^n \int_D \nabla x^i \cdot \nabla x^i$$

is also $\mathcal{G}$-invariant.

Moreover the vector field X_α is the gradient of E_α with respect to $\langle\langle\ ,\ \rangle\rangle$.

PROOF. See [**29**].

Let $\mathcal{O}_x(\mathcal{G})$ be the orbit of the conformal group through x. By 5.3 for x a zero $\mathcal{O}_x(\mathcal{G})$ will be a smooth manifold. Let $T_x \mathcal{O}_x(\mathcal{G})$ denote the tangent space to this orbit at x. Then $T_x[\mathcal{O}_x(\mathcal{G})]$ has an orthogonal complement $T_x[\mathcal{O}_x(\mathcal{G})]^\perp$ w.r.t. the weak inner product $\langle\langle\ ,\ \rangle\rangle$. The equivariance of X_α, E_α and $\langle\langle\ ,\ \rangle\rangle$ under $\mathcal{G}$ implies the following

THEOREM 5.11. *At a zero x*

$$DX_\alpha(x) : T_x[\mathcal{O}_0(\mathcal{G})]^\perp \to T_x[\mathcal{O}_x(\mathcal{G})]^\perp.$$

DEFINITION 5.12. A minimal surface $x : D \to \mathbf{R}^n$, $n \geq 4$, is said to be nondegenerate if $DX_\alpha(x)$ restricted to $T_x[\mathcal{O}_x(\mathcal{G})]^\perp$ is an isomorphism.

REMARK 5.13. It is a consequence of the index formula of Böhme-Tromba that an open and dense set of wires $\hat{\mathcal{A}} \subset \mathcal{A}$ enjoy the property that if $\alpha \in \hat{\mathcal{A}}$, then there are only a finite number of *nondegenerate* minimal surfaces which span them. This nondegeneracy result is not true for surfaces in $\mathbf{R}^3$ and in this case a new notation of nondegeneracy is required. This result allows us to define the $\operatorname{sgn} DX_\alpha(x)$ if x is nondegenerate. Since X_α is Palais-Smale, $DX_\alpha(x) \in \mathrm{GL}_c[T_x(\mathcal{O}_x(\mathcal{G}))]^\perp$ and we define

$$\operatorname{sgn} DX_\alpha(x) = \begin{cases} +1 & \text{if } DX_\alpha(x) \in \mathrm{GL}_c^+[T_x(\mathcal{O}_x(\mathcal{G}))]^\perp, \\ -1 & \text{if } DX_\alpha(x) \in \mathrm{GL}_c^- . \end{cases} \tag{5.14}$$

This will be useful at a later point.

We know that $H^s(S^1, \Gamma^\alpha) \subset H^s(S^1, \mathbf{R}^n) \subset H^{1/2}(S^1, \mathbf{R}^n)$. The weak inner product is defined naturally as a strong (or Hilbert space) inner product on $H^{1/2}(S^1, \mathbf{R}^n)$ which by a trace theorem can be identified with H^1 harmonic mappings on the disc. For $u \in \mathcal{N}(\alpha)$, $T_u\mathcal{N}(\alpha) \subset H^{1/2}(S^1, \mathbf{R}^n)$ and there is a weak (w.r.t. $\langle\langle\ ,\ \rangle\rangle$) orthogonal projection $\mathbf{P}_u$ from $H^{1/2}(S^1, \mathbf{R}^n)$ onto the closure $\overline{T_u\mathcal{N}(\alpha)}$ in $H^{1/2}$.

THEOREM 5.15. *The projection $\mathbf{P}_u$ extends to a continuous map of $H^s(S^1, \mathbf{R}^n) \to T_u\mathcal{N}(\alpha)$ with $u \to \mathbf{P}_u$ C^{r-s-2} smooth.*

PROOF. Again see [**29**, p. 82].

As was already remarked the fibre bundle $\pi\colon \mathcal{N}^* \to \mathcal{A}$ is trivial. For fixed ρ consider the trivialization $\tau\colon \mathcal{A} \times \mathcal{N}(\rho) \to \mathcal{N}^*$ by $\tau(\alpha, x) = \tau_\alpha(x) = \alpha(\rho^{-1}(x))$. Using τ we can consider X_α, $\alpha \in \mathcal{A}$, as a parametrized family of vector fields $\tilde{X}_\alpha$ on $\mathcal{N}(\rho)$ given by

$$\tilde{X}_\alpha(x) = D\tau_\alpha^{-1} X_\alpha(\tau_\alpha(x)) \tag{5.16}$$

and $x \to \tau_\alpha(x)$ is for fixed α a diffeomorphism of $\mathcal{N}(\rho)$ with $\mathcal{N}(\alpha)$ which when $\alpha = \rho$ is the identity map of $\mathcal{N}(\rho)$ with itself, and thus $\tilde{X}_\rho = X_\rho$.

Denote the image curves of α and ρ by Γ^α and Γ^ρ. The map $\alpha \circ \rho^{-1}\colon \Gamma^\rho \to \Gamma^\alpha$ is a diffeomorphism which we shall denote by ψ. By composition $\mathbf{\Psi}$ induces the diffeomorphism τ_α of $\mathcal{N}(\rho)$ to $\mathcal{N}(\alpha)$.

THEOREM 5.17. *The map $D\tau_\alpha(x)\colon T_x\mathcal{N}(\rho) \to T_{\tau_\alpha(x)}\mathcal{N}(\alpha)$ has an adjoint $D\tau_\alpha^*(x)\colon T_{\tau_\alpha(x)}\mathcal{N}(\alpha) \to T_x\mathcal{N}(\rho)$ with respect to the weak Riemannian structures on $\mathcal{N}(\alpha)$ and $\mathcal{N}(\rho)$. Thus for all $g \in T_x\mathcal{N}(\alpha)$, $k \in T_{\tau_\alpha(x)}\mathcal{N}(\alpha)$*

$$\langle\langle D\tau_\alpha(x)h, k\rangle\rangle = \langle\langle h, D\tau_\alpha^*(x)k\rangle\rangle.$$

In addition the dependence $x \to D\tau_\alpha^*(x)$ is smooth and $D\tau_\alpha$, $D\tau_\alpha^*$ extend to isomorphisms in the $H^{\bar{s}}$ topology, $1 \le \bar{s} \le s$.

PROOF. [**29**, p. 99].

COROLLARY 5.18. *The family $\tilde{X}_\alpha$ is orthogonal at $x \in \mathcal{N}(\rho)$ (always w.r.t. $\langle\langle\ ,\ \rangle\rangle$) to the image $D\tau_\alpha^*(T_w\mathcal{O}_w(\mathcal{G}))$, $w = \tau_\alpha(x)$. In addition, at a zero of $\tilde{X}_\alpha$, the image $D\tilde{X}_\alpha[T_x\mathcal{N}(\rho)]$ is always orthogonal to $D\tau_\alpha^*(T_w\mathcal{O}_w(\mathcal{G}))$. Note that if $w \notin H^{s+1}$, $T_w\mathcal{O}_w(\mathcal{G}) \subset H^{s-1}(S^1, \mathbf{R}^n)$, but $D\tau_\alpha$ and $D\tau_\alpha^*$ extend to continuous maps in these topologies.*

REMARK 5.19. It is a consequence of the formula (5.16) defining $\tilde{X}_\alpha$ that at a zero x of $\tilde{X}_\alpha$ the kernel of $D\tilde{X}_\alpha(x)$ always contains $D\tau_\alpha^{-1}(T_w\mathcal{O}_w(\mathcal{G}))$, $w = \tau_\alpha(x)$.

LEMMA 5.20. *Let $x \in \mathcal{N}(\alpha)$, $x \in H^{s+1}$ and let $\Pi_x: T_x\mathcal{N}(\alpha) \to D\tau_\alpha^*(T_w\mathcal{O}_w(\mathcal{G}))$ be the weak orthogonal projection onto this three dimensional subspace of $T_x\mathcal{N}(\alpha)$. Then, if $\overline{x} \in H^{s+1}$ is H^{s+1} close to x, $\Pi_{\overline{x}}$ is close to Π_x as linear endomorphisms of their respective spaces.*

PROOF. This follows from the formula for Π_x, using the fact that $T_w\mathcal{O}_w(\mathcal{G})$ is spanned by w_θ, $\cos\theta \cdot w_\theta$, $\sin\theta \cdot w_\theta$.

As a variant of this we have the following

LEMMA 5.21. *Let $x \in \mathcal{N}(\alpha)$ and let $\overline{T_x\mathcal{N}(\alpha)}$ denote the closure of $T_x\mathcal{N}(\alpha)$ as a subspace of $H^{s-1}(S^1, \mathbf{R}^n)$. Again let $\Pi_x: \overline{T_x\mathcal{N}(\alpha)} \to D\tau_\alpha(T_w\mathcal{O}_w(\mathcal{G}))$ be the weak orthogonal projection onto this three dimensional subspace of $\overline{T_x\mathcal{N}(\alpha)}$. Then, if $\overline{x} \in \mathcal{N}(\alpha)$ is close to x, $\Pi_{\overline{x}}$ is close to Π_x as linear endomorphisms of their respective spaces.*

We will need yet another variant of these lemmas, namely

LEMMA 5.22. *Let α and $\overline{\alpha}$ be curves in $\mathcal{A}$ and x and $\overline{x}$ be minimal surfaces which span α and $\overline{\alpha}$ respectively. Thinking of α as fixed, then for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that whenever $\|\alpha - \overline{\alpha}\|_{H^r} < \delta(\varepsilon)$ and $\|x - \overline{x}\|_{H^s} < \delta(\varepsilon)$, then $\|x - \overline{x}\|_{H^{s+1}} < \varepsilon$.*

PROOF. Assume not. Then there exists an $\varepsilon > 0$ such that for all n we can find sequences $\overline{x}_n$ spanning $\overline{\alpha}_n$ such that $\|\overline{\alpha}_n - \alpha\|_{H^r} < 1/n$, $\|x - \overline{x}_n\|_{H^s} < 1/n$ and $\|x - \overline{x}_n\|_{H^{s+1}} \geq \varepsilon$. Since $r \gg s$, x_n has (by Hildebrandt's regularity result) a convergent subsequence H^{s+1}. Thus $x_n \to x$ in H^{s+1}, a clear contradiction.

REMARK 5.23. The implication of 5.22 is that for *minimal surfaces* the $H^r \times H^s$ topology on $\mathcal{N}$ (or $\mathcal{N}^*$) is equivalent to the $H^r \times H^{s+1}$ topology.

REMARK 5.24. It is a trivial observation to see that the bundle trivialization $\tau_\alpha: \mathcal{N}(\alpha) \to \mathcal{N}(\rho)$ carries $\mathcal{I}_\alpha$ diffeomorphically onto $\mathcal{I}_\rho$.

We are now ready to show how to construct an orientation on the manfold Σ_0^0. We, however, should note at this point that many of the technical difficulties and subsequent methods employed to overcome them could have been avoided if there were a weak orthogonal projection of $T_x\mathcal{N}(\alpha)$ onto $T_x\mathcal{I}_\alpha$.

Let $x_0 \in \Sigma_0^0 \cap \mathcal{N}(\rho)$ be a zero of $\tilde{X}_\rho = X_\rho$. Since we cannot project the family $\tilde{X}_\alpha$ onto a family of vector fields on $\mathcal{I}_\rho$ preserving at the same time its zero structure we shall extend $\tilde{X}_\alpha$ to a family of vector fields in a neighborhood of x_0 in $\mathcal{I}_\rho$ in such a way that the zeros of the extended field lie in $\mathcal{I}_\rho$.

First we have

THEOREM 5.25. *The manifolds $\mathcal{I}_\alpha$ are for each α contractible. Thus by Theorems 1.6 and 1.7 $\mathcal{I}_\rho$ admits an orientable Fredholm structure, and hence so does a tubular neighborhood.*

PROOF. Obvious.

REMARK 5.26. In order to show that $\mathcal{I}_\rho$ has an orientable Fredholm structure one could have appealed instead to the theorem of Eells and Elworthy [**9**], which shows that every Hilbert manifold can be embedded as an open subset of its model space and hence inherits an orientable Fredholm structure by pulling back the natural orientable Fredholm structure on any open subset of a Hilbert space via this embedding.

REMARK 5.27. We shall henceforth always assume that we are using a chart in some fixed orientable Fredholm structure for a tubular neighborhood of $\mathcal{I}_\rho$. Using the diffeomorphisms $\tau_\alpha\colon \mathcal{N}(\rho) \to \mathcal{N}(\alpha)$ we shall give a tubular neighborhood of each $\mathcal{I}_\alpha$ the Fredholm structure it inherits from $\mathcal{I}_\rho$ via τ_α.

Let $\mathbf{E}_1 \subset \mathbf{E}$ be a codimension three subspace of $\mathbf{E}$, the model space for $\mathcal{N}(\rho)$, with $\mathbf{E}_1$ serving as the model space for $\mathcal{I}_\rho$. We may locally flatten $\mathcal{I}_\rho$ about x_0 and represent the family $\tilde{X}_\alpha$ locally by a family of maps $f_\alpha\colon E \to E$ such that, say, x_0 corresponds to $0 \in E$ and $\mathcal{I}_\rho \cap \mathbf{E}_1$ is an open neighborhood of $0 \in \mathbf{E}_1$. Let $\mathbf{E}_0 \subset \mathbf{E}$ be the three dimensional subspace defined by $T_{x_0}[\mathcal{O}_{x_0}(\mathcal{G})]$. Since $\mathbf{E}_0$ and $\mathbf{E}_1$ span E we may write $\mathbf{E} = \mathbf{E}_1 \times \mathbf{E}_0$. We also know that $D\tilde{X}_\rho(x_0)$ is orthogonal to E_0.

Define a new family of vector fields $\tilde{Y}_\alpha$ on a neighborhood of x_0 by

$$Y_\alpha(e) = \tilde{X}_\alpha(e) + \Pi_{x_0} e = f(\alpha, e) + \Pi_{x_0} e, \tag{5.28}$$

where Π_{x_0} denotes the weak orthogonal complement (in the pull back metric) of $\mathbf{E}$ onto $\mathbf{E}_0$.

REMARK 5.29. If $\alpha \neq \rho$ we would take E_0 to be $D\tau_\alpha^* T_w(\mathcal{O}_{x_0}(\mathcal{G}))$, $w = \tau_\alpha(x_0)$.

THEOREM 5.30. *The zeros of the family Y_α on a neighborhood of $0 \in \mathbf{E}$ and for α sufficiently close to ρ are precisely the zeros of the family $\tilde{X}_\alpha$ on $\mathcal{I}_\rho$. The family Y_α is Palais-Smale and if $Y_\alpha(e) = 0$*

$$\operatorname{sgn} DY_\alpha(e) = \operatorname{sgn} D\tilde{X}_\alpha(e) = \operatorname{sgn} DX_\alpha(e), \tag{5.31}$$

where the later was defined in (5.14). *Moreover if our neighborhoods* (*of ρ and x_0*) *are sufficiently small, the total derivative of Y_α at a zero will be surjective.*

PROOF. Suppose $Y_\alpha(e) = 0$.

Let Π_e denote the projection onto $D\tau_\alpha^*[T_w(\mathcal{O}_w(\mathcal{G}))]$, $w = \tau_\alpha(e)$. Since $\tilde{X}_\alpha(e)$ is orthogonal to the range of Π_e it follows that $\Pi_e \tilde{X}_\alpha(e) = 0$. In addition, from 5.21, for e close to zero $\Pi_e \Pi_{x_0} e \neq 0$ unless $\Pi_{x_0} e = 0$. Thus if $Y_\alpha(e) = \tilde{X}_\alpha(e) + \Pi_{x_0} e = 0$ we get that $\Pi_e Y_\alpha(e) = \Pi_e \Pi_{x_0} e = 0$. Thus $\Pi_{x_0} e = 0$ and hence e must lie in $\mathcal{I}_\rho$ and $\tilde{X}_\alpha(e) = 0$. That $DY_\alpha(e)$ is surjective at any zero follows at once from the following facts:

(i) The weak orthogonal complement of $D\tilde{X}_\rho(0)$ is precisely $\mathbf{E}_0$.

(ii) $D(e \to \Pi_{x_0} e) \cdot h = \Pi_{x_0} h$.

From (i) and (ii) we see that

(iii) $DY_\rho(0)$ is surjective

from which the local surjectivity of DY_α follows. The remainder of 5.30 follows by construction.

Now we are in a position to apply the results of 3.9. We have a family of Palais-Smale fields Y_α defined on a neighborhood of ρ and x_0 whose zeros will be on $\mathcal{I}_\rho$ and such that the total derivative at any zero is onto. We may, therefore, apply the methods of 3.9 to produce a coordinate chart $\overline{G}_{x_0}$ for the zero set of $\tilde{X}_\alpha$ in a neighborhood of $0 \in \mathbf{E}$.

If φ was the coordinate chart for $\mathcal{N}(\rho)$ in a neighborhood of x_0 which gave use to the local representation $f(\alpha, \cdot)$ of $\tilde{X}_\alpha$, then $\varphi^{-1}\overline{G}_{x_0}$ gives a coordinate chart for the zero set of $\tilde{X}_\alpha$ in a neighborhood of x_0. That this is independent of the choice of φ in our given fixed Fredholm structure now follows essentially as in 3.9, as does the independence under bundle trivialization (cf. Remarks 5.27 and 3.11).

However suppose $x_1 \in \mathcal{I}_\rho$ is another minimal surface and hence a zero of $\tilde{X}_\alpha$ in a neighborhood of $0 \in \mathbf{E}$ (using now a fixed coordinate mapping φ) we must show that the choice of coordinate mappings $\overline{G}_{x_0}$ and $\overline{G}_{x_1}$ for the zero set in $\mathcal{I}_\rho$ are positively related.

In (5.28) the choice of Y_α yields a coordinate chart (as in 3.9)

$$G_{x_0}: V \times U \to \mathcal{W} \times \mathbf{E},$$

$V \times U$ a neighborhood of $(0,0)$ in $H^r(S^1, \mathbf{R}^n) \times \mathbf{E}$ and $\mathcal{W}$ a neighborhood of ρ.

Resrict Y_α to a sufficiently small neighborhood of (ρ, x_0) so that $(\overline{G}_{x_0})^{-1}$ is defined and such that for any (α, x_1) with $\|\alpha - \rho\|_{H^r} < 2\varepsilon$, $\|x_1 - x_0\|_{H^s} < 2\varepsilon$

$$\|(\Pi_{x_1} - \Pi_{x_0}) D G^2_{x_0}(u,v)\| < \tfrac{1}{2}, \tag{5.32}$$

where $(\alpha, x_1) = G_{x_0}(u,v)$ and $G^2_{x_0}$ is the second component of G_{x_0}. That we accomplish (5.32) follows from 5.20 and 5.22.

We shall assume that G_{x_0} has been restricted to a small enough neighborhood $V \times U$ of $(0,0)$ to insure that $G_{x_0}(V \times U)$ is contained in the 2ε neighborhood of $(\rho, 0)$.

We now have

THEOREM 5.33. *If (α, x_1) is another zero of $\tilde{X}_\alpha$ in a 2ε neighborhood of (ρ, x_0), $x_1 \in \mathcal{I}_\rho$ and if G_{x_1} is another coordinate mapping determined by the choice of a different extension of $\tilde{X}_\alpha$ such that $\overline{G}_{x_1}$ gives a coordinatization of the zeros of $\tilde{X}_\alpha$ on $\mathcal{I}_\rho$ in a neighborhood of (α, x_1) then, if defined,*

$$D(\overline{G}_{x_1}^{-1} \circ \overline{G}_{x_0}) \in \mathrm{GL}_c^+ .$$

PROOF. By construction $D(G_{x_1}^{-1} \circ G_{x_0}) \in \mathrm{GL}_c^+(\mathbf{F} \times \mathbf{E})$, $\mathbf{F} = H^r(S^1, \mathbf{R}^n)$.

Now the matrix of $D(G_{x_1}^{-1} \circ G_{x_0})$ again has the form

$$\begin{pmatrix} D(\overline{G}_{x_1}^{-1} \circ \overline{G}_{x_0}) & \sim \\ 0 & * \end{pmatrix} \in \mathrm{GL}_c^+(\mathbf{F} \times \mathbf{E}). \tag{5.34}$$

We must show that $D(\overline{G}_{x_1}^{-1} \circ \overline{G}_{x_0}) \in \mathrm{GL}_c^+(\mathbf{F})$, and so we necessarily have to know what $*$ looks like. Let

$$f_0(\alpha, e) = f(\alpha, e) + \Pi_{x_0} e, \qquad f_1(\alpha, e) = f(\alpha, e) + \Pi_{x_1} e.$$

By construction

$$f_0 \circ G_{x_0}(\alpha, e) = e, \qquad f_1 \circ G_{x_1}(\alpha, e) = e$$

and

$$f_0 G_{x_0}(\alpha, e) = f G_{x_0}(\alpha, e) + \Pi_{x_0} G_{x_0}(\alpha, e) = e.$$

Now again from the construction of the G's in 3.9 $*$ will be $D_e(f_1 \circ G_{x_0})$; i.e. the derivative in the "$\mathbf{E}$" direction of this composition. But

$$\begin{aligned} D_e(f_1 \circ G_{x_0}) &= D_e(f G_{x_0} + \Pi_{x_1} G^2_{x_0}) \\ &= D_e(e + [\Pi_{x_1} - \Pi_{x_0}] G^2_{x_0}) \\ &= \mathrm{id}_E + (\Pi_{x_1} - \Pi_{x_0}) D_e(G^2_{x_0}), \end{aligned}$$

where id_E is the identity map on E, and $G^2_{x_0}$ again denotes the second component of G_{x_0}. But by construction $\|(\Pi_{x_1} - \Pi_{x_0}) D_e(G^2_{x_0})\| < \frac{1}{2}$ from which it follows that

$(*)$ is in $\mathrm{GL}_c^+(\mathbf{E})$. The fact that 0 falls to the lower left-hand corner of (5.34) is an immediate consequence of the fact that each G_{x_i} sends any point of the form $(u, 0)$ to a zero of $\tilde{X}_\alpha$.

Thus from (5.34) we can conclude that $D(\overline{G}_{x_1}^{-1} \circ \overline{G}_{x_0}) \in \mathrm{GL}_c^+(\mathbf{F})$ as desired. This proves 5.33.

Putting the results of 5.30 and 5.33 together we obtain two of the three main results (5.35 and (5.36)) of this paper.

THEOREM 5.35. *There exists an orientable Fredholm structure $\tilde{\Phi}$ on the manifold $\Sigma_0^0 \subset \mathcal{I}$ such that (cf. 5.7):*

(1) *The projection map $\overline{\pi}_0^0 \colon \Sigma_0^0 \to \mathcal{A}$ is admissible with respect to $\tilde{\Phi}$.*

(2) $(\overline{\pi}_0^0)^* \Phi = \tilde{\Phi}$, *$\Phi$ the natural Fredholm structure on $\mathcal{A}$.*

(3) *If α is a regular value for $\overline{\pi}_0^0$ (or π_Σ) there exists a finite number of nondegenerate minimal surfaces, say $x_1, \ldots, x_m$, spanning α such that* $\operatorname{sgn} D\overline{\pi}_0^0(x_i) = \operatorname{sgn} DX_\alpha(x_i)$.

(4) *Although $\mathcal{A}$ (cf. remarks following* 5.8) *is not connected we may still construct a degree theory for $\pi_\Sigma \colon \Sigma \to \mathcal{A}$, and hence a degree for π_Σ. One then obtains that* $\deg \pi_\Sigma = 1$.

PROOF. (1)–(3) is a consequence of 3.9, 5.30 and 5.33. To see (4) we choose a regular value α for π_Σ. Thus there are a finite number of nondegenerate minimal surfaces $x_1, \ldots, x_m$ all (in three point condition) spanning Γ^α. Define

$$\deg(\pi_\Sigma, \alpha) = \sum_i \operatorname{sgn} D\overline{\pi}_0^0(x_i).$$

The main difficulty is to show the independence of $\deg(\pi_\Sigma, \alpha)$ on α. If $\mathcal{A}$ were connected this would now follow from §2. We know that the total curvature on α (cf. (4.0)) is

$$K(\alpha) < \pi(s-2). \tag{5.36}$$

Let α^* be the standard embedding of the unit circle S^1 into $\mathbf{R}^2$. Then we know that $\pi_\Sigma^{-1}(\alpha^*)$ consists of only one point, namely the identity map of the disc to itself, say id. From [**29**, p. 68] we have that

$$\operatorname{sgn} D\pi_\Sigma(\mathrm{id}) = \operatorname{sgn} D\pi_{\Sigma_0^0}(\mathrm{id}) = 1.$$

Let β be a C^∞ wire close to α which is also a regular value and which also satisfies

$$K(\beta) < \pi(s-2). \tag{5.36$'$}$$

Then the arguments of §2 show that $\deg(\pi_\Sigma, \alpha) = \deg(\pi_\Sigma, \beta)$.

Now connect β by a continuous path β_t of C^∞ wires to α^* with $\beta_0 = \beta$, $\beta_1 = \alpha^*$ (here we use the connectivity of C^∞ embeddings of S^1 into $\mathbf{R}^n$, $n \geq 4$). However, along this path the curvature condition (5.36$'$) may be violated. Let $\tilde{K} = \sup_{t \in [0,1]} K(\beta_t)$ and choose

$$\overline{s} > \tilde{K}/\pi + 2 \quad \text{and} \quad \overline{r} \gg s. \tag{5.37}$$

Now work within the space of wires $H^{\overline{r}}(S^1, \mathbf{R}^n)$ and the space of surfaces $H^{\overline{s}}(S^1, \mathbf{R}^n)$. Clearly since all the β_t's are C^∞ they are of class $H^{\overline{r}}$. Repeat the

construction of $\mathcal{A}$, $\mathcal{N}$, $\mathcal{N}^*$, Σ etc. to obtain $\mathcal{A}'$, $\mathcal{N}'$, $\mathcal{N}'^*$, Σ', essentially the "same" objects except in a new regularity class. But in this new regularity class β can be connected to α^* by a path satisfying the curvature restrictions (5.37) and the methods described in §2 yield that

$$\deg(\pi_\Sigma, \beta) = \deg(\pi_{\Sigma'}, \beta) = \deg(\pi_{\Sigma'}, \alpha^*) = 1.$$

Thus $\deg(\pi_\Sigma, \alpha) = 1$ for *any* regular value α and this is of course the definition of $\deg \pi_\Sigma$.

THEOREM 5.38. *There exists an open dense set of wires*

$$\hat{\mathcal{A}} \subset \mathcal{A}, \qquad \mathcal{A} \subset H^r(S^1, \mathbf{R}^n), \quad n \geq 4,$$

such that if $\alpha \in \mathcal{A}$ there are only a finite number of nondegenerate minimal surfaces $x_1, \ldots, x_m$ spanning α. Let $D^2E_\alpha(x_i)\colon T_{x_i}\mathcal{N}(\alpha) \times T_{x_i}\mathcal{N}(\alpha) \to R$ denote the Hessian of Dirichlet's functional at x_i. Let λ_i be the maximal subspace in $T_{x_i}\mathcal{O}_{x_i}(\mathcal{G})^\perp$ on which $D^2E_\alpha(x_i)$ is negative definite. Then

$$\sum_i (-1)^{\lambda_i} = 1. \tag{5.39}$$

PROOF. This follows immediately from 5.35 and the fact that

$$\operatorname{sgn} DX_\alpha(x_i)|T_{x_i}\mathcal{O}_{x_i}(\mathcal{G}) = (-1)^{\lambda_i}.$$

This equality is a consequence of the relationship between the Hessian of Dirichlet's functional and the derivative of our vector field, namely

$$D^2E_\alpha(x_i)(h,k) = \langle\langle DX_\alpha(x_i)h, k\rangle\rangle.$$

Since $DX_\alpha(x_i)$ is of the form identity I plus a compact operator K, the

$$\operatorname{sgn} DX_\alpha(x_i)|T_{x_i}\mathcal{O}_{x_i}(\mathcal{G})^\perp$$

is the number, counted with multiplicities, of the eigenvalues of $K|T_x\mathcal{O}_{x_i}(\mathcal{G})^\perp$ strictly less than negative 1; but this is precisely λ_i.

COROLLARY 5.40. *Let α be any wire in $\mathbf{R}^n$, $n \geq 4$. Then there exists a minimal surface spanning its image Γ^α.*

PROOF. If no, then $\deg \pi_\Sigma = 0$.

Thus this theory also yields the existence of a minimal surface spanning an arbitrary embedded wire in $\mathbf{R}^n$, $n \geq 4$. But we can recover the existence theorem in $\mathbf{R}^3$.

COROLLARY 5.41. *Given any embedding $\alpha \in H^r(S^1, \mathbf{R}^3)$, there exists a minimal surface spanning its image Γ^α.*

PROOF. Find a sequence of wires $\alpha_n \in H^r(S^1, \mathbf{R}^4)$ such that $\alpha_n \to \alpha$ in H^r. By 5.40 there exists a sequence of minimal surfaces x_n spanning Γ^{α_n}. By Hildebrandt's regularity result the x_n are uniformly bounded in the $H^{s+1}(S^1, \mathbf{R}^n)$ norm ($r \gg s$). Thus we can find a subsequence $x_{n_j} \to x_0$ in H^s. Clearly x_0 spans Γ^α and is a minimal surface.

6. A strong version of the theorem of Morse-Shiffman-Tompkins. Suppose we have a wire $\rho: S^1 \to \mathbf{R}^n$ and two isolated minimal surfaces $x_1: D \to \mathbf{R}^n$ and $x_2: D \to \mathbf{R}^n$ spanning Γ^ρ. If these two surfaces are strict minima in the topology determined by $C^0[D, \mathbf{R}^n] \cap H^1(D, \mathbf{R}^n)$, then in [**17, 25**], Morse and Tompkins and independently Shiffman proved that there existed a third minimal surface which was "unstable"[4] (in the sense of not being a strict relative minimum).

Now suppose that we had two isolated strict minima in some stronger topology, say H^s. Then the methods of Morse-Tompkins-Shiffman break down completely.

However, using the techniques of this paper, it is possible to show that the existence of two H^s (or $C^{k,\alpha}$) minima impies the existence of a third surface which is not a strict minima for Dirichlet's functional. This result is essentially a consequence of formula (5.39) and we briefly first give a sketch of the proof with more details to follow.

First in the generic case for a "good" wire ρ for which all the minimal surfaces which span Γ^ρ are nondegenerate if you have n-minima you must have at least $(n-1)$ nonstable (again in the sense of energy) minimal surfaces. This is an immediate consequence of formula (5.39). The difficult part is the nongeneric case.

Suppose $x_1 \in \mathcal{I}$ and $x_2 \in \mathcal{I}$ are two strict minima satisfying the three point condition of the last section. Although they need not be in Σ_0^0 (they may have boundary branch points and so could be in Σ_ν^0 for some ν) there will be a "local degree" for $\pi_\Sigma: \Sigma \to \mathcal{A}$, $\Sigma = \bigcup_{\lambda,\nu} \Sigma_\nu^\lambda$ about each x_i. Since they are isolated minima one shows that this local degree for each x_i must be 1. If there were no other minimal surfaces then the total degree of π_Σ would be 2 (since it is the sum of the local degrees). This contradicts the fact that $\deg \pi_\Sigma = 1$ (cf. 5.35(4)).

Thus there must exist a third. If there are infinitely more, then by the regularity theorem [**16**] the solutions must have a limit point x_3 which cannot be an isolated strict minimum. If all the other solutions are isolated there must be at least one, say x_*, such that the local degree of π_Σ about x_* is negative, and consequently cannot be a local minima.

These ideas will become clearer as we proceed with the proof of

THEOREM 6.1. *If x_1 and $x_2 \in \mathcal{I} \subset \mathcal{N}(\rho)$ are two H^s minimal surfaces spanning a wire Γ^ρ, $\rho \in H^r(S^1, \mathbf{R}^n)$, $n \geq 4$, which are isolated strict local H^s minima for Dirichlet's functional $E_\rho: \mathcal{N}(p) \to R$, then there must exist at least one other minimal surface x spanning Γ^ρ. If the set of minimal surfaces spanning Γ^ρ is discrete at least one of these other minimal surfaces cannot be a minimum.*

If p is a wire with the property that all minimal surfaces spanning Γ^ρ are nondegenerate then the existence of n-minima implies the existence of $(n-1)$ nonminima.

PROOF. The proof of the last statement as already remarked follows immediately from (5.39).

We begin with the following result from [**29**]:

THEOREM 6.2. *The weak Riemannian structure $\langle\langle\ ,\ \rangle\rangle$ (cf. 5.9) on $\mathcal{N}(\rho)$ induces a smooth C^2 geodesic spray. The corresponding exponential map $\exp_u: T_u\mathcal{N}(\rho) \to \mathcal{N}(\rho)$, $u \in \mathcal{N}(\rho)$, gives a local diffeomorphism of a neighborhood of zero in $T_u\mathcal{N}(\rho)$*

[4]In [**5**] the word unstable is used consistently to mean nonsmooth or noncontinuous dependence of the surface on the boundary wire α.

onto a neighborhood of u in $\mathcal{N}(\rho)$. Moreover with respect to the $\mathcal{G}$-action on $\mathcal{N}(\rho)$ and on $T_u\mathcal{N}(\rho)$

$$g_{\#}(\exp_u h) = \exp_{g_{\#}(u)} gh.$$

Let $\tau_\alpha\colon \mathcal{N}(\rho) \to \mathcal{N}(\alpha)$ be the bundle trivialization $\tau_\alpha(u) = \alpha(\rho^{-1}(u))$. Then we know from 5.17 that $D\tau_\alpha$ has an adjoint with respect to the weak Riemannian structures induced on $T\mathcal{N}(\alpha)$ and $T\mathcal{N}(\rho)$. Using τ_α we may regard the family of functions $E_\alpha\colon \mathcal{N}(\alpha) \to R$ has a family on $\mathcal{N}(\rho)$ via the transformation

$$E^*_\alpha(u) = E_\alpha(\tau_\alpha(u)). \tag{6.3}$$

Since each $D\tau_\alpha(u)\colon T_u\mathcal{N}(\alpha) \to T_u\mathcal{N}(\rho)$ has an adjoint $D\tau^*_\alpha(u)$, the family

$$X^*_\alpha(u) = D\tau^*_\alpha X_\alpha(\tau_\alpha(u)) \tag{6.4}$$

will be a family of vector fields *on* $\mathcal{N}(\rho)$ whose zeros will represent (via τ_α) the minimal surfaces spanning Γ^α. $X^*_\alpha(u)$ will be the gradient of E^*_α on $\mathcal{N}(\rho)$.

The family X^*_α on $\mathcal{N}(\rho)$ will no longer be Palais-Smale but the derivative of any X^*_α at each zero will be Fredholm of index zero (we call this a Fredholm family) but we know more, namely $DX^*_\alpha(u)\colon T_u\mathcal{N}(\rho) \to T_u\mathcal{N}(\rho)$ will be of the form

$$A + K. \tag{6.5}$$

Let A be invertible and positive w.r.t. $\langle\langle\ ,\ \rangle\rangle$; i.e. $\langle\langle Av, v\rangle\rangle > 0$ if $v \neq 0$. If $\alpha = \rho$, then $X_\alpha = X_\rho$ and A will be the identity. If α is close to ρ, A will be close to the identity.

Now let $u \in \mathcal{N}(\rho) \cap H^{s+1}(S^1, \mathbf{R}^n)$. Then the weak complement of $T_u\mathcal{O}_u(\mathcal{G})$, $T_u\mathcal{O}_u(\mathcal{G})^\perp$, is a subspace of $T_u\mathcal{N}(\rho)$. Let $\mathcal{E} = \exp_u(V)$, V a neighborhood of 0 in $T_u\mathcal{O}_u(\mathcal{G})^\perp$. For V small $\mathcal{E}$ is a submanifold of $\mathcal{N}(\rho)$.

Define a new family Y^*_α on $\mathcal{E}$ by

$$Y^*_\alpha(\omega) = P_\omega X^*_\alpha(\omega), \tag{6.6}$$

P_ω the weak Riemannian projection of $T_\omega\mathcal{N}(\rho)$ onto $T_\omega\mathcal{E}$. Then as in [**29**, p. 102] for $\mathcal{E}$ sufficiently small and α sufficiently close to ρ we may conclude that Y^*_α is a Fredholm family of index zero. The zeros of Y^*_α will coincide with the zeros of X^*_α and will be nondegenerate iff the corresponding zeros of X_α on $\mathcal{E}$ are nondegenerate.

Finally Y^*_α will be the gradient of the restriction of E^*_α to $\mathcal{E}$.

It follows further from all these definitions that at a zero u of Y^*_α on $\mathcal{E}$, the derivative $DY^*_\alpha(u)\colon T_u\mathcal{E} \to T_u\mathcal{E}$ will have the form

$$A + K, \tag{6.7}$$

A positive invertible, K compact linear. If α is close to ρ, then A will be close to the identity. We would now like to introduce the notion of a signum for linear maps of the form (6.7) so that we can generalize the theory of local index to a larger class of vector fields than the Palais-Smale class. This was, in fact, carried out in [**30**] and we know the next results from this source.

DEFINITION 6.8. Let $\mathbf{E}$ be a Banach space and let $R_c(\mathbf{E})$ denote those linear operators of the form $R + K$, K compact linear, and R an invertible operator with the property that $tR + (1-t)I$ is invertible for all $0 \le t \le 1$. Let $GR_c(\mathbf{E})$ denote those operators in $\mathrm{GL}(\mathbf{E}) \cap R_c(\mathbf{E})$; i.e. those operators in $R_c(\mathbf{E})$ which are invertible. Then from [**30**] we have the following theorem.

THEOREM 6.9. *$\pi_0(GR_c(\mathbf{E})) = 2$, and thus $GR_c(\mathbf{E})$ has two components, say $GR_c^+(\mathbf{E})$ and $GR_c^-(\mathbf{E})$.*

Thus for each nondegenerate zero u of Y_α^* we may assign a signum to $DY_\alpha^*(u)$ according to the rule

$$\operatorname{sgn} DY_\alpha^*(u) = \begin{cases} +1 & \text{if } DY_\alpha^*(u) \in GR^+(T_u\mathcal{E}), \\ -1 & \text{if } DY_\alpha^*(u) \in GR^-(T_u\mathcal{E}). \end{cases} \tag{6.10}$$

We than have the important

THEOREM 6.11. *Let u be a zero of Y_α^* for α sufficiently close to ρ. Then $\tau_\alpha(u)$ is a zero of X_α and*

$$\operatorname{sgn} DY_\alpha^*(u) = \operatorname{sgn} DX_\alpha^*(\tau_\alpha(u)). \tag{6.12}$$

PROOF. A straightforward calculation.

In [**30**] the author developed an Euler-characteristic for vector fields on a manifold M modelled on $\mathbf{E}$ whose zeros were compact and such that the derivatives at a zero m were in $R_c(T_mM)$.

We shall use the results of this theory without paraphrasing them. From this theory it follows, for ρ fixed and u an isolated zero of Y_ρ^*, that Y_ρ^* has a local rotation number about this zero. This rotation number is defined as follows. If u is isolated, then in some coordinate system we can take an open ball B centered at u and small enough so that $0 \notin Y_\rho^*(\partial B)$. Then a local integer degree

$$\deg(Y_\rho^*, \overline{B}, 0) \tag{6.13}$$

will be defined along the lines of Elworthy-Tromba[5] It follows that for α close to ρ,

$$0 \notin Y_\alpha^*(\partial B) \tag{6.14}$$

and

$$\deg(Y_\alpha^*, \overline{B}, 0) = \deg(Y_\rho^*, \overline{B}, 0). \tag{6.15}$$

By the Index Theorem we may find α arbitrarily close to ρ so that all minimal surfaces spanning Γ^α are nondegenerate.

Now suppose that there are only two isolated minima x_1 and x_2 and *not* a third for $E_\rho\colon \mathcal{N}(\rho) \to R$. Let B_1 and B_2 be two balls in some coordinate system about these points. Let α be so close to ρ that Y_α^* is defined on exponential surfaces $\mathcal{E}_1$ and $\mathcal{E}_2$ about x_1 and x_2, (6.14) holds and such that all minimal surfaces spanning Γ^α are nondegenerate.

Furthermore as in [**29**, p. 77] or from the regularity results that for α sufficiently close to ρ all minimal surfaces spanning Γ^α must be represented as zeros of Y_α^* in $B_1 \cup B_2$, say $x_1^1, x_2^1, \ldots, x_{m_1}^1, \ldots, x_1^2, x_2^2, \ldots, x_{m_2}^2$, the definition of local degree implies that

$$\deg(Y_\alpha^*, \overline{B}_i, 0) = \sum_{j=1}^{m_i} \operatorname{sgn} DY_\alpha^*(x_j^i). \tag{6.16}$$

[5]For B small let y near zero be a regular value for Y_ρ^*. Then $(Y_\rho^*)^{-1}(y)$ consists of a finite number of points $u_1, \ldots, u_m \in B$. $\deg(Y_\rho^*, \overline{B}, 0) = \sum_i \operatorname{sgn} DY_\rho^*(u_i)$ and this is independent of y.

By (6.12) this implies that

$$\deg \pi_\Sigma = \sum_{i=1}^{2} \deg(Y_\alpha^*, \overline{B}_i, 0) = \deg \overline{\pi}_0^0. \tag{6.17}$$

Thus by (6.15)

$$\deg \pi_\Sigma = \sum_{i=1}^{2} \deg(Y_\alpha^*, B_i, 0). \tag{6.18}$$

If we can show that for each B_i

$$\deg(Y_\rho^*, \overline{B}_i, 0) = 1 \tag{6.19}$$

we would obtain the result that $\deg \pi_\Sigma = 2$, which contradicts 5.35.

That (6.19) holds is a consequence of the fact that each x_i is a strict H^s minimum for E_ρ and hence a strict minimum for E_ρ restricted to an exponential surface $\mathcal{E}_i$ about each x_i. Recall that Y_ρ^* is the gradient of the restriction of E_ρ to $\mathcal{E}_1 \cup \mathcal{E}_2$. Using this fact we have from [**2**] the normal form result which uses the techniques of [**31**] and applies it to the splitting lemma of Gromoll-Meyer [**12**].

THEOREM 6.20. *There is a neighborhood U_i of each x_i in $\mathcal{E}_i$ and a coordinate chart $\varphi_i: U_i \to T_{x_i}\mathcal{N}(\rho)$ such that $\varphi_i(U_i) \subset \mathbf{F}_i$ as an open subset of a codimension three subspace $\mathbf{F}_i$ of $T_{x_i}\mathcal{N}(\rho)$ with $\varphi_i(x_i) = 0$ and $D\varphi_i(x)$ having an adjoint $D\varphi_i(x)^*: T_{x_i}\mathcal{N}(\rho) \to T_x\mathcal{N}(\rho)$ w.r.t. the weak Riemannian structures on $T_{x_i}\mathcal{N}(\rho)$ and $T_x\mathcal{N}(\rho)$ respectively such that $x \to D\varphi_i(x)^*$ is smooth.*

Let $\tilde{E}_\rho$ denote the induced map on $\varphi_i(U_i)$ and let J_i be the kernel of $DY_\rho^*(x_i)$. Let $\mathbf{H}_i = J_i^\perp$. Then there is a local origin preserving diffeomorphism Φ_i of a neighborhood of $T_{x_i}\mathcal{N}(\rho)$ such that

(1) $D\Phi_i(0) = I$, the identity,

(2) $D\Phi_i(x)$ has a smooth adjoint $x \to D\Phi_i(x)^*$.

Moreover there is an origin preserving differentiable map h_i defined on a neighborhood W_i of 0 in J_i such that for $(x, y) \in \mathbf{H}_i \times J_i$ close to 0

$$\tilde{E}_\rho \circ \Phi_i(x, y) = \tilde{E}_\rho(0) + \tfrac{1}{2} D^2 \tilde{E}_\rho(0)(x, x) + \tilde{E}_\rho(h_i(y), y), \tag{6.21}$$

where $\frac{1}{2} D^2 \tilde{E}_\rho(0)$ denotes the Hessian of $\tilde{E}_\rho$ at $0 \in T_{x_i}\mathcal{N}(\rho)$.

Using this result we are now ready to compute $\deg(Y_\rho^*, \overline{B}_i, 0)$!

Let $\xi_i: W_i \to \mathbf{R}$ be defined by $\xi_i(y) = \tilde{E}_\rho(h_i(y), y)$. In this coordinate system it is easy to see

$$\nabla(\tilde{E}_\rho \circ \Phi_i)(p, q) = (p, \nabla \xi_i(q)). \tag{6.22}$$

But

$$\nabla(\tilde{E}_\rho \circ \Phi_i)(x) = D\Phi_i^* Y_\rho^*(\Phi_i(x)).$$

Let $B_i^\#$ be a sufficiently small ball about 0 so that $\deg(\nabla(\tilde{E}_\rho \circ \Phi_i), \tilde{B}_i, 0)$ is defined, with $B_i^\# \subset \Phi_i^{-1}(B_i)$. Then elementary degree arguments show that

$$\deg(\nabla(\tilde{E}_\rho \circ \Phi_i), \overline{B}_i^\#, 0) = \deg(Y_\rho^*, \overline{B}_i, 0). \tag{6.23}$$

From this and (6.22) it follows that

$$\deg(Y_\rho^*, \overline{B}_i, 0) = \deg(\nabla \xi_i, \overline{B}_i^{\#} \cap J_i, 0). \tag{6.24}$$

Notice that the right-hand side of this equality is just the Brouwer degree of the gradient of a smooth function about an absolute minimum. It is well known that this degree must be one.

This concludes the proof of 6.1 and we end the section with a remark.

REMARK 6.25. Since elliptic regularity works for $C^{k,\alpha}$ as well as the Sobolev spaces H^s, our version of Morse-Shiffman-Tompkins would also go through if one assumed two isolated $C^{k,\alpha}$ strict minima ($k \geq 7$).

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