TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 290, Number 2, August 1985

A. C. GROUPS: EXTENSIONS, MAXIMAL SUBGROUPS, AND AUTOMORPHISMS

BY

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ABSTRACT. In §1 we extend the results of [3] on centralizers to r.e. subgroups and show, e.g., that every a.c. group has an ∞-ω-equivalent subgroup of the same power which is embedded maximally in itself; and we pursue a natural typology of maximal subgroups. §2 shows that if A is a countable group of automorphisms of a countable a.c. group G such that $A \supset \operatorname{Inn} G$, then there exists $\tau \in \operatorname{Aut} G$ such that the HNN extension $(A, \tau\colon \tau^{-1}g\tau = \tau(g)$ for all $g \in \operatorname{Inn} G)$ is a subgroup of $\operatorname{Aut} G$. We show in §3 that every a.c. group with a countable skeleton has a proper extension to an a.c. group having any skeleton that contains the original one and any f.g. group which contains the countable a.c. group equivalent to the original one as an r.e. subset. This uses Ziegler's construction [7]. Finally, in §4, also using Ziegler's construction we show that there exists an a.c. group A of any power and having any countable skeleton which has a free subgroup M such that for all $x \in A - M$ and $y \in A$ there exist free generators $a, b, c \in M$ such that $y = (ax)^b(ax)^c$.

1. Centralizers and maximal subgroups. First we will review some facts all of which will be generalized presently. In general we will assume familiarity with such basic properties of a.c. groups as can be found in [**4**, Chapter IV, §8].

Let G be a nontrivial a.c. group. In [**3**, Lemma 1] it was proved that

(1.0) If $A_1, \ldots, A_n \subset G$ are finitely generated (f.g.), then $\mathscr{C}_G(A_1 \cap \cdots \cap A_n) = \langle \mathscr{C}_G(A_1), \ldots, \mathscr{C}_G(A_N) \rangle$;

and from this was deduced [**3**, Theorem 3]

(1.1) If $A \subset G$ is finite and $\mathscr{C}_G(A) \subset T \subset G$, then $\mathscr{C}_G(A_0) \subset T \subset \mathscr{N}_G(A_0)$ for some $A_0 \subset A$.

($\mathscr{C}_G$ and $\mathscr{N}_G$ denote the centralizer and normalizer in G.) From (1.1) was deduced

(1.2) If $S \subset G$ is finite and characteristically simple, then $\mathscr{N}_G(S)$ is maximal in G.

PROOF OF (1.2). $N = \mathscr{N}_G(S) \subset T \subset G$ implies by (1.1) that $\mathscr{C}_G(S_0) \subset T \subset \mathscr{N}_G(S_0)$ for some $S_0 \subset S$. Hence $N \subset \mathscr{N}_G(S_0)$, which implies that S_0 is characteristic in S since every automorphism of S is induced by some element of G, and the two possibilities $S_0 = 1$ and $T = G$ or $S_0 = S$ and $T = N$ follow. □

Received by the editors November 9, 1983.

1980 *Mathematics Subject Classification*. Primary 20F99, 20E06; Secondary 20E28, 20E15, 20F28.

Key words and phrases. Algebraically closed groups, automorphism groups, maximal subgroups, HNN extensions.

This matter was pursued no further in [**3**]. However, (1.1) easily implies the existence of other maximal subgroups as well.

(1.3) If G is an a.c. group, $A \subset B \subset G$ are finite, A is maximal in B and the identity is the only automorphism of B which induces the identity on A, then $\mathscr{C}_G(B)$ is maximal in $\mathscr{C}_G(A)$.

PROOF OF (1.3). Suppose $\mathscr{C}_G(B) \subset T \subset \mathscr{C}_G(A)$. From (1.1) we have $\mathscr{C}_G(A_0) \subset T \subset \mathscr{N}_G(A_0)$ for some $A_0 \subset B$. Thus $\mathscr{C}_G(A_0) \subset \mathscr{C}_G(A)$ which implies $A \subset A_0$ since G is a.c. Thus either $A_0 = A$ or $A_0 = B$. If $A_0 = A$, then $T = \mathscr{C}_G(A)$ is immediate; if $A_0 = B$, then $T \subset \mathscr{C}_G(A) \cap \mathscr{N}_G(B)$ which implies (using the last hypothesis) that $T \subset \mathscr{C}_G(B)$ and equality follows. □

Let $\mathrm{Sk}(G)$ = skeleton of G be the class of f.g. subgroups of G. If a.c. groups have the same skeleton we will say they are *equivalent*. In (1.3), if A and B are centerless, then $\mathscr{C}_G(A)$ and $\mathscr{C}_G(B)$ are equivalent to G, and if G is countable, then $\mathscr{C}_G(B) \subset \mathscr{C}_G(A)$ is a maximal embedding of G *into itself* since any two equivalent countable a.c. groups are isomorphic. In particular it suffices to take $A = \mathrm{Sym}(n-1)$ and $B = \mathrm{Sym}(n)$ to obtain such an embedding. Most maximal embeddings (1.3) are structurally very different from those obtained in (1.2) because:

(1.4) Suppose $M \subset N$, where $M = \mathscr{C}_G(B)$ and $N = \mathscr{C}_G(A)$ is a maximal embedding of (1.3) and $B \neq A\mathscr{C}_G(A)$. Then, for all f.g. $D \subset N$ we have $\mathscr{C}_N(D) \not\subset M$.

PROOF OF (1.4). Suppose $D \subset \mathscr{C}_G(A)$ is f.g. We have $DA \neq \langle D, B\rangle \subset G$ since $B \neq A\mathscr{C}_B(A)$. Hence, since G is a.c., there exists $t \in \mathscr{C}_G(DA) - \mathscr{C}_G(B)$, i.e., $t \in \mathscr{C}_N(D) - M$. □

(1.5) DEFINITION. If M is a maximal subgroup of an a.c. group G, then

(i) M is of Type 1 if $\mathscr{C}_G(A) \subset M$ for some f.g. $A \subset M$;

(ii) M is of Type 2 if $\mathscr{C}_G(A) \not\subset M$ for all f.g. $A \subset M$.

Note that (1.4) implies that the maximal embeddings of (1.3), provided $B \neq A \oplus \mathbf{Z}_2$, are of the second type; while (1.2) gives natural examples of the first type. Among other things we hope to show that (1.5) is an interesting classification.

(1.6) PROPOSITION. *Suppose G is an a.c. group, $M \subset G$ is maximal and $A \subset M$ is f.g. Then, $\mathscr{C}_G(A) \not\subset M$ if and only if*

(1.7) *There exist $x_1, \ldots, x_n \in M$ such that $A^{x_1} \cap \cdots \cap A^{x_n} = 1$.*

PROOF. ($\Leftarrow$) If $\mathscr{C}_G(A) \subset M$ and (1.7) holds, then $\mathscr{C}_G(A^{x_j}) \subset M$, $1 \leqslant j \leqslant n$, and so (1.0) would imply $M \supset \mathscr{C}_G(A^{x_1} \cap \cdots \cap A^{x_n}) = G$.

($\Rightarrow$) Since $A \times A$ is embeddable in G [**3**, Proposition 3] and one direct factor can be conjugated to another in G, there exists $x \in G$ such that $A \cap A^x = 1$. Choose $y \in \mathscr{C}_G(A) - M$. Since M is maximal in G, $x \in \langle M, y\rangle$; say $x = m_1 y^{k_1} \cdots m_n y^{k_n}$ with $m_i \in M$. Put $x_i^{-1} = m_1 \cdots m_i$, $1 \leqslant i \leqslant n$. We claim that $I = A \cap A^{x_1} \cap \cdots \cap A^{x_n} = 1$. To check this, suppose $1 \neq a \in A$. If all the conjugates $a^{m_1}, \ldots, a^{m_1 \cdots m_n}$ lie in A, then $a^x \in A$ also since y centralizes A. Thus some $a^{x_i^{-1}} \notin A$ and hence $a \notin I$, proving $I = 1$. □

The interesting feature of (1.6) is that it shows that the *isomorphism-type* of M determines whether M is of Type 1 or of Type 2. In particular a Type 1 maximal M has a f.g. subgroup A such that

(1.8) Every finite set of conjugates of A in M intersects nontrivially.

These are precisely the f.g. $A \subset M$ for which $\mathscr{C}_G(A) \subset M$; and if $A_1,\ldots,A_n$ are finitely many such A, then $A_1 \cap \cdots \cap A_n$ also satisfies (1.8) (in place of A) by (1.0).

We will postpone a detailed discussion of this situation until after the main theorems of this section.

The following result was also proved in [**3**, Theorem 4]:

(1.9) Suppose $A_1,\ldots,A_n$ are f.g., $A_1 \cap \cdots \cap A_n$ is finite and $\mathfrak{A}_i \subset \mathrm{Aut}(A_i)$ $(1 \leqslant i \leqslant n)$. Let K be the largest subgroup of A invariant under $\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_n$. Then $\mathscr{N}_G^{\langle \mathfrak{A}_1,\ldots,\mathfrak{A}_n\rangle}(K) = \left\langle \mathscr{N}_G^{\mathfrak{A}_1}(A_1),\ldots,\mathscr{N}_G^{\mathfrak{A}_N}(A_n)\right\rangle = J$, where $\mathscr{N}_G^{\mathfrak{A}}(A) = \{g \in G \mid g$ induces an $\mathfrak{A}$-automorphism on A by conjugation$\}$.

PROOF OF (1.9). Put $B = A_1 \cap \cdots \cap A_n$. Since $1 \in \mathfrak{A}_i$, we have $\mathscr{C}_G(A_i) \subset \mathscr{N}_G^{\mathfrak{A}_i}(A_i)$ and hence by (1.0) $\mathscr{C}_G(B) \subset J$. Since B is finite, (1.1) implies that for some $B_0 \subset B$ we have

$$\mathscr{C}_G(B_0) \subset J \subset \mathscr{N}_G(B_0). \tag{1.10}$$

The second inclusion in (1.10) implies $B_0 \subset K$ since B_0 is evidently $(\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_n)$-invariant. The first inclusion in (1.10) implies $K \subset B_0$ because otherwise, since G is a.c., there would be elements in $\mathscr{C}_G(B_0)$ which did not normalize K and hence could not lie in J. Hence $J = \mathscr{N}_G^{\mathfrak{A}}(K)$ for some $\mathfrak{A} \subset \mathrm{Aut}(K)$ and we clearly have $\mathfrak{A} = \langle \mathfrak{A}_1,\ldots,\mathfrak{A}_n\rangle$. □

Regretfully, in [**3**] this proof of Theorem 4 from Theorem 3 was not given. Instead, an overly complicated direct proof of Theorem 4 was given, when in fact Theorem 3 can be proved very pleasantly (see (1.13)). This carelessness was fully the fault of the present author. At any rate, all of these results will now be generalized to r.e. subgroups.

(1.11) DEFINITION. A subset $S \subseteq G$ is *enumeration reducible* (*e.r.*) *in* G if there exists $A = \langle x_1,\ldots,x_n\rangle \subset G$ such that $S \subseteq A$ and S equals a set of words $X(S)$ on $\{x_1,\ldots,x_n\}$ which is enumeration reducible to the set $W(A)$ of all relations on $\{x_1,\ldots,x_n\}$ which are trivial in A. Put $r_0(G) = \{S \subseteq G \mid S$ is e.r. in $G\}$ and $r(G) = \{H \in r_0(G) \mid H$ is a *subgroup* of $G\}$. The relation of enumeration reducibility between (coded) sets of natural numbers is $\leqslant_e$.

(1.12) PROPOSITION. *If* $S_1, S_2 \in r_0(G)$, *then* $S_1 \cap S_2 \in r_0(G)$. *Hence* $r(G)$ *is a sublattice of the subgroup lattice of* G.

PROOF. Let $S_i \subseteq A_i$ and $X(S_i)$ $(i = 1,2)$ be as given in (1.11). Let $A = \langle x_1,\ldots,x_n\rangle \subset G$, where $\{x_1,\ldots,x_n\}$ is the union of the generating sets of A_1 and A_2. Thus $X(s_i) \leqslant_e W(A_i)$ and $S_i = X(S_i) \subseteq A$, as this is the content of (1.11). Put $T = S_1 \cap S_2$ and $T^* =$ the set of all words on $\{x_1,\ldots,x_n\}$ which lie in $T \subseteq A$. Thus $T \in r_0(G)$ will follow if we prove that $T^* \leqslant_e W(A)$. Suppose we have an enumeration of

$W(A)$. Since $X(S_i) \leqslant_e W(A)$ we also have enumerations of $X(S_i) = S_i \subseteq A$. As we enumerate these sets we can list, as we find them, all words $w = w_1 w_2^{-1} \in W(A)$, where $w_i \in X(S_i)$. As we find a w as above we add the words $w_1 = w_2$ to our list of T^*. We also include in our list of T^*, as we find them, all words in $x_1, \ldots, x_n$ which equal any member of T^* that we have previously listed. Hence $T \in r_0(G)$. An easier similar proof shows that $r(G)$ contains the join of any two of its members. □

For reference we now state the

GENERAL HIGMAN EMBEDDING THEOREM (C. F. MILLER III (UNPUBLISHED) AND M. ZIEGLER [**8**]). *Suppose the group K has a presentation $(X, t_n\,(n \in \omega)\colon W(A) \cup R)$, where X is a finite generating set of the group A, $R \leqslant_e W(A)$, and the relations R are consistent over A (and may involve t's and x's). Then K is a subgroup of a f.g. group H which has a similar presentation $(X, s_1, \ldots, s_m\colon W(A) \cup F)$, where F is finite; furthermore, $\{t_n | n \in \omega\}$ is a recursive subset of H.*

The proof of this is an application of the standard Higman Embedding Theorem using the definition of the enumeration-reducibility relation in terms of r.e. sets [**6**, p. 146] and Neumann-type relations.

We extend (1.0) to

(1.13) THEOREM. *If $A_1, \ldots, A_n \in r(G)$ and G is a.c., then $\mathscr{C}_G(A_1 \cap \cdots \cap A_n) = \langle \mathscr{C}_G(A_1), \ldots, \mathscr{C}_G(A_n)\rangle$.*

PROOF. Put $A = A_1 \cap \cdots \cap A_n$ and $B = \langle A_1, \ldots, A_n\rangle$. Let H = the HNN extension $(B, t_1, \ldots, t_n;\ t_i$ centralizes $A_i)$. An easy argument using Britton's lemma shows that if $v = (t_1 \cdots t_n)^2$, then $H \supset \langle B, v\rangle \cong$ the HNN extension $(B, v;\ v^{-1}av = a$ $(a \in A))$. Let $x \in \mathscr{C}_G(A)$ with $|x| = \infty$ and $\langle x\rangle \cap B = 1$. Let P have the presentation $(\langle B, x\rangle,\ y_1, y_2;\ y_1$ and y_2 centralize A and $x = y_1 y_2)$. Thus $P = \langle B, x\rangle *_{\langle x = y_1 y_2\rangle \oplus A} (A \oplus \langle y_1\rangle * \langle y_2\rangle)$. By considering reduced products in P, it is easily checked that $P \supset \langle B, y_i\rangle \cong$ the HNN extension $(b, y_i;\ y_i^{-1} a y_i = a\ (a \in A))$ for $i = 1$ and 2. So we can amalgamate P with two copies of H, H' and H'', via the relatins $a_i = a_i' = a_i''$ $(a \in A_i,\ 1 \leqslant i \leqslant n)$, $v' = y_1$, and $v'' = y_2$. This shows the relations $\rho = \{t_i'$ and t_i'' centralize A_i and $x = v'v''\}$ to be consistent over $\langle B, x\rangle \subset G$. Since $A_i \in r(G)$ we can obtain, via the General Higman Embedding Theorem, finitely many relations over a f.g. subgroup of G which imply ρ. Hence $x \in \langle \mathscr{C}_G(A_1), \ldots, \mathscr{C}_G(A_n)\rangle$. To complete the proof we must show that $\mathscr{C}_G(A)$ is generated by elements x as above. Let $c \in \mathscr{C}_G(A)$. Since B is contained in a f.g. subgroup of G, $\langle B, c\rangle \oplus \langle z\rangle \subset G$ for some z of infinite order and z and $c + z$ satisfy the requirements for x above. □

For later use we will observe that we have also proved

(1.14) If $w \in \mathscr{C}_G(A)$, there exist elements t_{ij} where $1 \leqslant j \leqslant 4$, $1 \leqslant i \leqslant n$ such that $t_{ij} \in \mathscr{C}_G(A_i)$ and $w = \prod_{j=1}^{4} (t_{1j} \cdots t_{nj})^2$.

This is so because in our proof we have $w = x_1 x_2$ where each $x = v'v''$ and each v has the form $(t_1 \cdots t_n)^2$.

Now we can generalize Theorem 3 of [3] (and (1.1)) to

(1.15) THEOREM. *Suppose G is an a.c. group, $A \subset B$ with $A, B \in r(G)$ and the subgroup lattice $\mathscr{L}(A, B) = \{H | H = B^{x_1} \cap \cdots \cap B^{x_n} \cap B \ (x_1, \ldots, x_n \in \mathscr{N}_G(A))\}$ satisfies MIN. Then, for all T such that $\mathscr{C}_G(B) \subset T \subset \mathscr{N}_G(A)$, there exists $B_0 \in \mathscr{L}(A, B)$ such that $\mathscr{C}_G(B_0) \subset T \subset \mathscr{N}_G(B_0)$.*

Note that $B \in r(G)$ implies $\mathscr{L}(A, B) \subseteq r(G)$ by (1.12); and if $X \in \mathscr{L}(A, B)$, then $\mathscr{L}(A, X) \subseteq \mathscr{L}(A, B)$.

PROOF. Induct on $X \in \mathscr{L}(A, B)$ to prove that $\mathscr{C}_G(X) \subset T \subset \mathscr{N}_G(A)$ implies $\mathscr{C}_G(B_0) \subset T \subset \mathscr{N}_G(B_0)$ for some $B_0 \in \mathscr{L}(A, X)$. Note that the case $X = A$ is trivial. Let $Y \in \mathscr{L}(A, B)$ and assume the result for all $X \in \mathscr{L}(A, Y)$ with $X \neq Y$. Suppose $\mathscr{C}_G(Y) \subset T \subset \mathscr{N}_G(A)$. If $T \subset \mathscr{N}_G(Y)$ we take $B_0 = Y$ and are done. So assume there exists $x \in T - \mathscr{N}_G(Y)$. Put $Y_0 = Y \cap Y^x$. Thus $Y_0 \in \mathscr{L}(A, Y)$ since $x \in T \subset \mathscr{N}_G(A)$, and $Y_0 \neq Y$ since $\mathscr{L}(A, Y)$ has MIN. We have $T \supset \langle \mathscr{C}_G(Y), \mathscr{C}_G(Y^x) \rangle = \mathscr{C}_G(Y_0)$ by (1.13). Now, by our inductive hypothesis, there exists $B_0 \in \mathscr{L}(A, Y_0)$ such that $\mathscr{C}_G(B_0) \subset T \subset \mathscr{N}_G(B_0)$ as required. □

The desired generalization of (1.3) is

(1.16) THEOREM. *Suppose G is an a.c. group, $A \subset B \subset G$ with $A, B \in r(G)$, A is maximal in B and the centralizer of A in* $\mathrm{Aut}(B)$ *is trivial. Then, $\mathscr{C}_G(B)$ is maximal in $\mathscr{C}_G(A)$.*

PROOF. This is the same as the proof of (1.3) using (1.13) in place of (1.1).To prove that $\mathscr{C}_G(B) \neq \mathscr{C}_G(A)$ we use the General Higman Embedding Theorem. □

We can also generalize Theorem 4 of [3] (and (1.9) above) as follows. If $H \subset G$, define $\mathrm{Aut}_G(H) = \mathscr{N}_G(H)/\mathscr{C}_G(H) \subset \mathrm{Aut}(H)$. Notice that if G is a.c. and H is f.g., then $\mathrm{Aut}_G(H) = \mathrm{Aut}(H)$.

(1.17) THEOREM. *Assume the hypotheses of* (1.15) *and suppose $B = B_1 \cap \cdots \cap B_n$, where $B_i \in r(G)$ and $\mathfrak{A}_i \subset \mathrm{Aut}_G(B_i)$ $(1 \leqslant i \leqslant n)$ such that $\mathfrak{A}_i$ normalizes A. Let $K \in \mathscr{L}(A, B)$ be the largest subgroup invariant under $\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_n$. Then*

$$\mathscr{N}_G^{\langle \mathfrak{A}_1, \ldots, \mathfrak{A}_n \rangle}(K) = \left\langle \mathscr{N}_G^{\mathfrak{A}_1}(B_1), \ldots, \mathscr{N}_G^{\mathfrak{A}_n}(B_n) \right\rangle = J.$$

PROOF. Note that K is the intersection of conjugates of B in G by elements which normalize A. Since $\mathscr{L}(A, B)$ has MIN, we indeed have $K \in \mathscr{L}(A, B)$. Now (1.15) implies that $H_0 \in \mathscr{L}(A, B)$ exists such that $\mathscr{C}_G(H_0) \subset J \subset \mathscr{N}_G(H_0)$. Our conclusion now follows exactly as in the proof of (1.9). □

We are now in a position to prove

(1.18) THEOREM. *Suppose G is a nontrivial a.c. group of power κ. Then G has an equivalent a.c. subgroup M of power κ which is embedded maximally in itself. In fact there is a proper embedding $\varphi(M) < M$ such that for all $y \in M - \varphi(M)$ and $z \in M$ there exist $a_i, b_i \in \varphi(M)$ such that $z = \prod_{i=1}^4 (a_i^y b_i)^2$.*

PROOF. Let Π = the restricted symmetric group on a countable set I. Let $\Pi \subset A$ be an embedding of Π into a f.g., recursively presented group A such that Π is a recursive subgroup of A (i.e., there is an algorithm which, given a restricted

permutation on I, computes an element of A equal to it). Since G is a.c., using Higman's Embedding Theorem, we can pass to an image $\tilde{A}$ of A with $\Pi \subset \tilde{A} \subset G$. In particular $\Pi \in r(G)$. If $x \in I$ let Π_x be the stabilizer of $\{x\}$ in Π. It is well known that Π_x is maximal in Π and since $\mathrm{Aut}(\Pi) = \mathrm{Sym}(I)$, the hypotheses of (1.15) are satisfied and we conclude that $\mathscr{C}_G(\Pi)$ is maximal in $\mathscr{C}_G(\Pi_x)$. It is well known that these subgroups are a.c. and equivalent to G (cf. the identical construction in [7, §3]). Since Π and Π_x are isomorphic by a recursive isomorphism, there is an element of G which induces this isomorphism by conjugation, and so $\mathscr{C}_G(\Pi)$ and $\mathscr{C}_G(\Pi_x)$ are also conjugate in G, and hence isomorphic. Thus $M = \mathscr{C}_G(\Pi_x)$ is embedded maximally in itself. To see that M has cardinal κ we use Macintyre's argument: since $\tilde{A} \times \tilde{A}$ is embeddable in G, (1.0) implies that $\mathscr{C}_G(\tilde{A})$ together with an isomorphic copy of itself generate all of G; hence it has cardinal κ. To show that the embedding satisfies the final property of our theorem, suppose $y \in \mathscr{C}_G(\Pi_x) - \mathscr{C}_G(\Pi)$. Thus $\Pi \cap \Pi^y = \Pi_x$ by maximality and the fact that $y \notin \mathscr{N}_G(\Pi)$. Thus $\mathscr{C}_G(\Pi_x) = \langle \mathscr{C}_G(\Pi), \mathscr{C}_G(\Pi)^y \rangle$ by (1.13) and if $z \in M$ our conclusion now follows from (1.14). □

Before continuing the discussion of (1.5) we will give one further useful result on centralizers.

(1.19) THEOREM. *Suppose G is an a.c. group and $A, B \in r(G)$ with $A < B$. Then $\mathscr{C}_G(A) \subset J = \langle B^t | t \in \mathscr{C}_G(A) \rangle$.*

PROOF. First we note that it will suffice to prove

(1.20) There exists $w \in J$ such that $|w| = \infty$ and $\langle w \rangle \oplus A$ exists in G.

This is because every $c \in \mathscr{C}_G(A)$ is the product of two such w as above (see the proof of (1.13)), and one such w can be conjugated to another by some $t \in \mathscr{C}_G(A)$ since $A \in r(G)$.

We will deduce (1.20) from

(1.21) There exists $t \in \mathscr{C}_G(A)$ and $z \in \langle B, B^t \rangle$ such that $z \notin \langle A, A^z \rangle$.

Let us assume (1.21). Put $D = \langle B, B^t \rangle \subset G$ and let $\overline{D}$ be an isomorphic copy of D over A, i.e., $\bar{a} = a$ for all $a \in A$. Let P be the amalgamated free product

$$D *_{\langle A, A^z \rangle = \langle A, A^{\bar{z}} \rangle} \overline{D}.$$

The subgroups $\langle A, A^z \rangle$ and $\langle A, A^{\bar{z}} \rangle$ are identified by the restriction of $d \to \bar{d}$ ($d \in D$). Since $z \notin \langle A, A^z \rangle \subset D$, we clearly have $|z^{-1}\bar{z}| = \infty$ and $\langle z^{-1}\bar{z} \rangle \oplus A$ exists in P. Since G is a.c. and $A, B \in r(G)$, using the General Higman Embedding Theorem, we obtain $s \in \mathscr{C}_G(A)$ such that $|z^{-1}z^s| = \infty$ and $\langle z^{-1}z^s \rangle \oplus A$ exists in G and we put $w = z^{-1}z^s$ to satisfy (1.20), noting that $\langle D, D^s \rangle \subset J$.

All we need now is to establish (1.21).

PROOF OF (1.21). Let $\overline{B}$ be an isomorphic copy of B over A and let $P = B *_A \overline{B}$. Let $y \in B - A$. Noting that $A^y \cap A = A^{\bar{y}} \cap A$ we claim that in P we have

$$y^{-1}\bar{y} \notin \langle A^y, A^{\bar{y}} \rangle. \tag{1.22}$$

The reason for this is that since $y \notin A^y$ we can choose left transversals T of A in B and $\bar{T}$ of A in $\bar{B}$ such that T contains a transversal of $A^y \cap A$ in A^y and $y^{-1} \in T$; while $\bar{T}$ contains a transversal of $A^{\bar{y}} \cap A$ in $A^{\bar{y}}$ and $\bar{y}^{-1} \in \bar{T}$. Then (1.22) follows from the Normal Form Theorem for elements of P. Now conjugating (1.22) by y^{-1} gives

$$\bar{y}y^{-1} \notin \left\langle A, A^{\bar{y}y^{-1}} \right\rangle. \tag{1.23}$$

Since G is a.c. and $A, B \in r(G)$, again using the Higman Embedding Theorem, we obtain a homomorphism φ: $P \to G$ such that $\varphi(b) = b$ for all $b \in B$, (1.23) holds with $\bar{y}$ replaced by $\varphi(\bar{y})$ and there exists $t \in \mathscr{C}_G(A)$ such that $\varphi(\bar{b}) = t^{-1}bt$ for all $b \in B$. Putting $z = \varphi(\bar{y})y^{-1}$ we have proved (1.21). □

(1.24) COROLLARY. *Suppose G is an a.c. group and $A, B \in r(G)$ with $A \subset B$. Let $K = \bigcap\{A^t | t \in B\}$ and suppose that $\mathscr{L}(K, B)$ satisfies MIN. Then $J = \langle B^t | t \in \mathscr{C}_G(A)\rangle = \mathscr{N}_G^{\mathfrak{A}}(K)$ where $\mathfrak{A} = \mathrm{Aut}_B(K)$.*

PROOF. By (1.19) we have $\mathscr{C}_G(A) \subset J$. Note that $K \in r(G)$ since $\mathscr{L}(K, B)$ has MIN. Hence $\mathscr{C}_G(B) \subset J \subset \mathscr{N}_G(K)$ implies (by (1.15)) that $\mathscr{C}_G(B_0) \subset J \subset \mathscr{N}_G(B_0)$ for some $B_0 \in \mathscr{L}(K, B)$. The proof that $B_0 = K$, etc., is identical to (1.9) after noting that $\mathscr{C}_G(K) \subset J$ (using (1.13)). □

The preprint **[2]**, which was referred to in **[3]**, will not be published in its original form. This paper contains many results of **[2]** as well as more recent work. In **[2]** the following characterization was obtained for Type 2 maximal subgroups of countable a.c. groups (see (1.5)).

(1.25) Suppose G is a countable a.c. group and $M \subset G$. Then G has a maximal subgroup $\tilde{M} \cong M$ such that, for all f.g. $A \subset \tilde{M}$, $\mathscr{C}_G(A) \not\subset \tilde{M}$ *if and only if* for all f.g. $A \subset M$ and $s \in G - A$, there exists $x \in M *_A \langle A, s\rangle$ such that $A \cap A^x = 1$. Furthermore, whenever $\tilde{M}$ exists as above, there are $2^{\aleph_0}$ distinct such $\tilde{M}$ satisfying $\mathscr{C}_G(A) \not\subset \tilde{M}$ for all f.g. $A \subset G$.

PROOF OF "ONLY IF". For this direction we need not assume G to be countable. Suppose $\tilde{M}$ is a Type 2 maximal subgroup of G, $A \subset \tilde{M}$ and $r \in G - A$. Put $B = \langle A, r\rangle \subset G$. Then, for some $t \in \mathscr{C}_G(A)$ we have $B^t \not\subset \tilde{M}$, for otherwise $\mathscr{C}_G(A) \subset \tilde{M}$ by (1.19), contrary to our hypothesis. Hence $r^t \notin \tilde{M}$ and WLOG we can assume $r \notin \tilde{M}$. Thus $G = \langle \tilde{M}, r\rangle$ since $\tilde{M}$ is maximal. Since $A \times A$ is embeddable in G and one factor can be conjugated to the other there exists $x \in \langle \tilde{M}, r\rangle$ such that $A \cap A^x = 1$. Since $\langle \tilde{M}, r\rangle$ is a natural quotient of $\tilde{M} *_A \langle A, r\rangle = P$, we also have $A \cap A^x = 1$ in P since the natural homomorphism is the identity on both factors.

Let φ: $\tilde{M} \to M$ be an isomorphism and let $\langle A, r\rangle \cong \langle \varphi(A), s\rangle$ by an isomorphism mapping $r \to s$ and $a \to \varphi(a)$ $(a \in A)$. Thus $P \cong M *_{\varphi(A)} \langle \varphi(A), s\rangle$. If we start with $A \subset M$, then we use $\varphi^{-1}(A) \subset \tilde{M}$ in the above argument in place of A and we get $x \in M *_A \langle A, s\rangle$ as desired. □

We will not prove the reverse implication in (1.25) in this paper. In **[2]** this was proved using a rather complicated series of amalgamations, but one could also use small cancellation theory to construct a maximal embedding of M into G. This is

easy to appreciate since the condition $A \cap A^x = 1$ used iteratively is sufficient to employ the small cancellation theory over amalgamated free products. It turned out, after an additional technical lemma, that the amalgamation technique used in [**2**] to prove (1.25) is also an alternative to small cancellation theory in the construction of Jonsson groups. Thus, the proof of (1.25) rightfully belongs in a future paper which will present a new construction of Jonsson groups together with other applications of this method.

It is easy to see that every M which (1.25) asserts to be maximally embedded in G satisfies (1.7) for all f.g. $A \subset M$ (as we know it must because it is embedded as a Type 2 maximal subgroup) because $A \oplus \langle y \rangle \subset G$ for some $y \neq 1$, and hence $M *_A (A \oplus \langle y \rangle)$ has an element x such that $A \cap A^x = 1$, and the same argument used to prove Proposition (1.6) establishes (1.7) here also. In general it seems quite impossible to prove that (1.7) for all f.g. $A \subset M$ is sufficient to embed M maximally in G, although for special cases, such as where M is locally finite, this is the case: a countable locally finite group $M \neq 1$ is maximal in every countable a.c. group if and only if M has no nontrivial finite normal subgroup.

The last statement in (1.25) follows from the construction of $\tilde{M}$ alluded to above (which is an inductive embedding procedure in which branchings can be made at each step). The significance of the last condition in (1.25) will be clearer upon comparison with (1.26). However, just the number of such embeddings provides a curious contrast to many Type 1 maximal subgroups: if $M = \mathcal{N}_G(A)$, where $A \subset G$ is finite and characteristically simple and $\tilde{M} \cong M$ is also maximal in G, then evidently $\tilde{M} = \mathcal{N}_G(\tilde{A})$ with $\tilde{A} \cong A$, whence there are only $\aleph_0$ such $\tilde{M}$; another example (see [**2**, Proposition 1]) is $M = \{x \in G | \mathbf{Z} \cap \mathbf{Z}^x \neq 1\}$ where $\mathbf{Z} \subset G$ is infinite cyclic—if $\tilde{M} \cong M$ is maximal, then $\tilde{M} = \{x \in G | \tilde{\mathbf{Z}} \cap \tilde{\mathbf{Z}}^x \neq 1\}$ where $\mathbf{Z} \cong \tilde{\mathbf{Z}} \subset \tilde{M}$ is infinite cyclic since (1.6) implies $\mathscr{C}_G(\tilde{\mathbf{Z}}) \subset \tilde{M}$.

These observations invite a closer examination of Type 1 maximal subgroups. The natural object to study in this regard is the subgroup lattice

$$\mathscr{M}(A, G) = \{M | \mathscr{C}_G(A) \subset M \subset G\},$$

where G is an a.c. group and $A \in r(G)$. First we will observe that $\mathscr{M}(A, G)$, for many A, contains examples of Type 2 maximals as well as Type 1. Then we will characterize members of $\mathscr{M}(A, G)$ in terms of the isomorphisms they induce in A.

(1.26) THEOREM. *Suppose H is a group with a solvable word problem and which has a series $H = H_1 \supset \cdots \supset H_n \supset \cdots$ such that every H_n is a nontrivial r.e. subgroup of H and, for all $x \in H$, there exists $n \geqslant 1$ such that $H_n \cap H_n^x = 1$. Let G be a nontrivial a.c. group and let $H \subset G$ be an embedding such that $H_n \in r(G)$ for all $n \geqslant 1$. Then there exists $\mathscr{C}_G(H) \subset M \subset G$ such that M is a Type 2 maximal subgroup of G, i.e., $\mathscr{C}_G(A) \not\subset M$ for all f.g. $A \subset M$. (Note that $M \supset \mathscr{C}_G(A)$, where $H \subset A \subset G$ and A is f.g.) In particular we can take H to be a free group of countably infinite rank.*

PROOF. First we note that H is embedded in G (actually, an isomorphic copy of G) in the required way. Our hypothesis means that H has a generating set $X = \{x_i | i \geqslant 1\}$ with respect to which H has a solvable word problem and the subgroups H_n equal

r.e. sets of words, and so we need only embed $H \subset \langle a_1, a_2 \rangle \subset G$ in such a way that X is a recursive subset of words on $\{a_1, a_2\}$ (e.g., [**4**, p. 188]) to ensure that $H_n \in r(G)$ for all $n \geqslant 1$. Put $\tilde{M} = \bigcup\{\mathscr{C}_G(H_n) | n \geqslant 1\}$. We have $\tilde{M} \neq G$ since, e.g., there exists $g \in G$ such that $H \cap H^g = 1$. We claim that, for all $1 \neq x \in H$, we have $G = \langle x, \tilde{M} \rangle$ because by hypothesis $H_n \cap H_n^x = 1$ for some $n \geqslant 1$ and since $H_n \in r(G)$ we have $G = \langle \mathscr{C}_G(H_n), \mathscr{C}_G(H_n^x) \rangle \subseteq \langle x, \tilde{M} \rangle$ by (1.13). Let $\tilde{M} \subset M \subset G$, where M is maximal with respect to $M \cap H = 1$. Thus M is maximal in G and $\mathscr{C}_G(H) \subset M$. Suppose $A \subset M$ is f.g. and $\mathscr{C}_G(A) \subset M$. We must have $A \cap H = 1$ for otherwise $\langle A, M \rangle = G$. Thus $M \supset \langle \mathscr{C}_G(A), \mathscr{C}_G(H) \rangle = G$ by (1.13), a contradiction proving $\mathscr{C}_G(A) \not\subset M$. Thus M is of Type 2. □

(1.27) THEOREM. *Let G be an a.c. group and let $A \in r(G)$. If $x \in G$ define $\varphi_x = \{(a, a^x) | a \in A$ and $a^x \in A\}$. φ_x is called "the isomorphism which x induces in A". If $M \in \mathscr{M}(A, G)$ define $\Phi_M = \{\varphi_x | x \in M\}$. Then:*

(1) *for all $M, N \in \mathscr{M}(A, G)$, $\Phi_M = \Phi_N$ implies $M = N$; hence*

(2) *$|\mathscr{M}(A, G)| \leqslant 2^{2^{\aleph_0}}$ and if $\mathrm{Sk}(G)$ is countable, then $|\mathscr{M}(A, G)| \leqslant 2^{\aleph_0}$; and*

(3) *if $\mathrm{Sk}(G)$ is countable and $A \subset H \subset G$, where H is a countable a.c. group equivalent to G, then the map $\zeta: \mathscr{M}(A, G) \to \mathscr{M}(A, H)$ defined by $\zeta(M) = M \cap H$ is a bijection of these lattices.*

PROOF OF (1). Assume $\Phi_M = \Phi_N$ and let $x \in M$. Choose $y \in N$ such that $\varphi_x = \varphi_y$. Thus $x^{-1}y \in \mathscr{C}_G(\text{range } \varphi_x) = C$. Since range $\varphi_x = A \cap A^x$ and $M \supset \langle \mathscr{C}_G(A), \mathscr{C}_G(A^x) \rangle = \mathscr{C}_G(A \cap A^x)$ by (1.13), we have $C \subset M$ (and by symmetry $C \subset N$). Thus $x^{-1}y \in C$ implies $x \in N$. So $M \subset N$ and equality follows by symmetry.

Since A is countable, there are at most $2^{2^{\aleph_0}}$ possibilities for Φ_M where $M \in \mathscr{M}(A, G)$. If $\mathrm{Sk}(G)$ is countable, then there are only countably many possible φ_x and $2^{\aleph_0}$ possible Φ_M, proving (2).

PROOF OF (3). Suppose $M \in \mathscr{M}(A, G)$. Note that $M = \{x \in G | \varphi_x \in \Phi_M\}$ because if $x \in G$ and $\varphi_x \in \Phi_M$, say $\varphi_x = \varphi_y$ with $y \in M$, then $x^{-1}y \in C \subset M$ (as above), and so $x \in M$. Thus, recalling that $A \subset H \subset G$, we have

$$M \cap H = \{x \in H | \varphi_x \in \Phi_M\}.$$

Next we show that $\Phi_{M \cap H} = \Phi_M$. Let $x \in M$ and $\langle x, H \rangle \subset H_1 \subset G$, where H_1 is countable and equivalent to G. Since $A \in r(G)$ there is an isomorphism Γ of H_1 with H such that $\Gamma(a) = a$ for all $a \in A$. It follows that $\varphi_x = \varphi_{\Gamma(x)} \in \Phi_{M \cap H}$. So, $\Phi_M \subset \Phi_{M \cap H}$ and equality follows.

Thus, we have shown that the map

$$M \xrightarrow{\alpha} \Phi_M = \Phi_{M \cap H} \xrightarrow{\beta} M \cap H$$

with domain $\mathscr{M}(A, G)$ is 1-1 into $\mathscr{M}(A, H)$. To prove (3), we must show that $\eta = \beta\alpha$ is onto $\mathscr{M}(A, H)$. Suppose $N \in \mathscr{M}(A, H)$ and put $M = \{x \in G | \varphi_x \in \Phi_N\}$. We must check that M is a subgroup of G, for then. clearly, $M \in \mathscr{M}(A, G)$ and $\eta(M) = N$. Suppose $x, y \in M$ and let $\langle x, y, A \rangle \subset H_1 \subset G$ and Γ be as before; so $\Gamma(x), \Gamma(y) \in N = \{z \in H | \varphi_z \in \Phi_N\}$ and hence $\Gamma(xy) \in N$ which implies $xy \in M$, showing M to be a subgroup. □

REMARK. Suppose $\mathscr{C}_G(A) \subset M \subset G$, i.e., $M \in \mathscr{M}(A, G))$ where $A \in r(G)$. Put $\Delta = \{\operatorname{dom} \varphi_x | x \in M\}$. Since $\operatorname{dom} \varphi_x = \operatorname{range} \varphi_x^{-1}$ we also have range $\varphi_x \in \Delta$ for all $x \in M$. It is easy to see that $\Delta \subseteq r(G)$: Let $A \subset B \subset G$, where B is f.g. and $A \leqslant_e W(B)$, and let $x \in M$. As we enumerate $W(\langle B, x\rangle)$ we can enumerate A and A^x and form a list of all pairs (a, a^x) such that $a, a^x \in A$. Thus φ_x, $\operatorname{dom} \varphi_x \leqslant_e W(\langle B, x\rangle)$ and $\operatorname{dom} \varphi_x \in r(G)$. Putting Δ_0 = finite intersections of members of $\Delta \subseteq r(G)$, we have $M \supset \bigcup\{\mathscr{C}_G(D) | D \in \Delta_0\}$ by (1.13) since $\mathscr{C}_G(\operatorname{dom} \varphi_x) \subset M$ for all $x \in M$. We cannot hope to prove that $\Delta_0 = \Delta$; nor can we prove that Φ_M is closed under function composition. It is not at all apparent which families of isomorphisms between $r(G)$-subgroups of A can occur as Φ_M for some $M \in \mathscr{M}(A, G)$. This must depend on G as well as on A except in very special cases.

Suppose M is a Type 2 maximal subgroup of G. We will call M Type 2a if $\mathscr{C}_G(A) \subset M$ for some $A \in r(G)$; otherwise we call M Type 2b. Theorem (1.26) shows that Type 2a maximals occur in $\mathscr{M}(A, G)$ for many pleasant A; while (1.25) shows that every Type 2a maximal can be embedded into G also as a Type 2b (in $2^{\aleph_0}$ distinct ways) if G is countable. It would appear to be an impossible task to characterize the isomorphism-types of those Type 2 maximals which can be embedded as Type 2a, although most Type 2 maximals can be ruled out since Type 2a have a very complicated structure. (Recall that (1.25) does neatly characterize all Type 2 maximals in countable G.) For example, (1.4) shows that all countable G occur quite naturally as Type 2b maximals of themselves (and this also follows from (1.25)). But I do not know if any G occurs as a Type 2a maximal of itself–the construction in (1.26) of Type 2a maximals uses Zorn's Lemma and it is not clear what isomorphism-types one can get, even though one begins with $\tilde{M} \cong G$. It also seems intractible to determine whether there exist Type 1 maximals $M \subset G$ such that there are $2^{\aleph_0}$ distinct maximal embeddings of M into G. We can combine this question and the identical question restricted to Type 2a maximals by asking: Is there a countable a.c. group G and some $A \in r(G)$ such that $\mathscr{M}(A, G)$ contains $2^{\aleph_0}$ isomorphic maximal subgroups? In fact this question is interesting when A is free Abelian of infinite rank.

If A is finite, Theorem (1.15) shows that $\mathscr{M}(A, G)$ is finite and that all maximals in $\mathscr{M}(A, G)$ are Type 1 (they are all normalizers of finite characteristically simple subgroups of A). We can also obtain this result whenever $A \in r(G)$ is solvable.

(1.28) THEOREM. *Suppose G is an a.c. group, $M \subset G$ is maximal and $\mathscr{C}_G(H) \subset M$, where $H \in r(G)$ is solvable. Then M is of Type* 1 (*see* (1.5)) *and $\mathscr{C}_G(F) \subset M$ for some Abelian $F \in r(G)$ with $F \subset H$.*

PROOF. As a lemma we will first prove the theorem when H is Abelian. For this we need the construction of [**1**, p. 433] of a group $B = \langle a, b\rangle$ such that ζB (the center of B) is free Abelian of rank $\aleph_0$ with free generators $\{c_n = [a, a^{b^n}] | n \geqslant 1\}$ and $B/\zeta B \cong \langle a\rangle \operatorname{wr} \langle b\rangle \cong \mathbf{Z} \operatorname{wr} \mathbf{Z}$. B is recursively presented and has a solvable word problem. Since $H \in r(G)$ we have $H \leqslant_e W(A)$ for some f.g. A with $H \subset A \subset G$. As we enumerate $W(A)$ we can also enumerate the set of all relations holding in H among its elements $h_1, h_2, \ldots$ (in the order in which we enumerate them $\leqslant_e W(A)$),

i.e., the set of all words $w(h_1,\ldots,h_n)$ which are trivial in H. As we list these we can enumerate a presentation for the group $D = B/N$, where

$$N = \langle w(c_1,\ldots,c_n) | w(h_1,\ldots,h_n) = 1 \text{ in } H\rangle \subset \zeta B.$$

We clearly have $\zeta D = \zeta B/N \cong H$, noting that $B/\zeta B$ is centerless and recalling that we are assuming H to be Abelian. Using the General Higman Embedding Theorem we can construct a group P which is finitely presented over A and such that $A, D \subset P$. Let $\langle P, t\rangle =$ the HNN extension $(P, t\colon c_n^t = h_n\ (n \geqslant 1))$. Again using the G.H.E.T. we embed $\langle P, t\rangle \subset Q$ with Q finitely presented over A. Since G is a.c. there is a homomorphism $\tilde{}\,: Q \to G$ such that $\tilde{a} = a$ for all $a \in A$. Thus $(\zeta\tilde{D})^{\tilde{t}} = H$ in $\tilde{Q}$ and hence $H = \zeta(E)$ where $E = \tilde{D}^{\tilde{t}} = \langle \tilde{a}, \tilde{b}\rangle^{\tilde{t}}$. Hence $E \subset \mathscr{C}_G(H) \subset M$, E is f.g. and $\mathscr{C}_G(E) \subset \mathscr{C}_G(H) \subset M$. This shows that M is a Type 1 maximal subgroup of G.

Now assume that $H \in r(G)$ is solvable. If $N \neq 1$ is any solvable group let $\hat{N} =$ the last nontrivial term in the derived series of N. We will choose H WLOG so that

(1.29) $\quad H \in r(G)$ has minimum solvable class such that $\mathscr{C}_G(H) \subset M$.

The proof will consist in showing that (under the assumption (1.29)) we have $\mathscr{C}_G(\hat{H}) \subset M$ and then appealing to the proof given above since $\hat{H}$ is Abelian and $H \in r(G)$ implies $\hat{H} \in r(G)$ also since $\hat{H}$ is a verbal subgroup of H. Let $y_1,\ldots,y_{n+1} \in \mathscr{C}_G(\hat{H})$ and $x_1,\ldots,x_n \in M$ and put $w = y_1x_1 \cdots y_nx_ny_{n+1}$. We will show that $\hat{H} \cap (\hat{H})^w \neq 1$. From this will follow $G \neq \langle \mathscr{C}_G(\hat{H}), M\rangle$ and $\mathscr{C}_G(\hat{H}) \subset M$ (by maximality of M) as desired.

Put $D_i = \operatorname{dom}\varphi_{x_i} = \{a \in H | a^{x_i} \in H\}$ and define inductively $S_n = D_n$ and $S_i = D_i \cap S_{i+1}^{x_i^{-1}}$ for $1 \leqslant i \leqslant n-1$. Using the Remark after the proof of Theorem (1.27) we deduce easily that $S_i \in r(G)$ and $\mathscr{C}_G(S_i) \subset M$ for $1 \leqslant i \leqslant n$. Hence, by our hypotheses on H and M, every S_i has the same solvable class as H $(1 \leqslant i \leqslant n)$. Now, we have $\hat{S}_1 \subset \hat{H}$ since S_1 and H have the same solvable class; and since $S_1^{x_1} \subset S_2$ we have $\hat{S}_1^{x_1} \subset \hat{S}_2$ also. Proceeding in this manner and using the facts $S_i^{x_i} \subset S_{i+1}$ from which follows $\hat{S}_i^{x_i} \subset \hat{S}_{i+1}$, since all groups involved have the same solvable class, and the fact that $y \in \mathscr{C}_G(\hat{H})$ we find that $\hat{S}_1^w \subset \hat{H}$ from which follows $\hat{H} \cap (\hat{H})^w \neq 1$ as desired. □

Let us note that Theorem (1.26) implies that Theorem (1.28) cannot be extended to locally finite p-groups which have invariant solvable series of length ω. A counterexample is

$$H = (\mathbf{Z}_p \operatorname{wr} \mathbf{Z}_p) \operatorname{wr} \mathbf{Z}_p \cdots$$

(the wreath power of $\mathbf{Z}_p$ indexed by the natural numbers) because H satisfies the hypotheses of (1.26) with $H_n = ((1 \operatorname{wr} 1) \operatorname{wr} \cdots 1) \operatorname{wr} \mathbf{Z}_p \operatorname{wr} \mathbf{Z}_p \cdots$ $(n-1$ trivial factors). (The nth term of the solvable ω-series referred to above is the normal closure in H of the first $n-1$ factors of H.)

Very likely there are Type 1 maximals $M \in \mathscr{M}(A, G)$ where $A \in r(G)$ for which "reduction to an Abelian subgroup" does not occur, i.e., $M \in \mathscr{M}(H, G)$ with $H \in r(G)$ implies H is non-Abelian; but we must leave this exotic possibility open.

Finally, we can at least observe how to construct many maximals.

(1.30) PROPOSITION. *Suppose* $A \in r(G)$ *and put* $r_A(G) = \{H \subset A | H \in r(G)\}$. *Suppose* Π *is an ultrafilter in the lattice* $r_A(G)$ *(see* (1.12)) *and let* M *consist of all* $x \in G$ *which preserve* Π, *i.e.*, $x \in M \Leftrightarrow D_x = \{a \in A | a^x \in A\} \in \Pi$ *and* $P^x \in \Pi$ *for all* $P \in \Pi$ *such that* $P \subset D_x$. *Then* M *is a maximal subgroup of* G.

PROOF. To verify that M is a subgroup, it is clear that $x \in M$ implies $x^{-1} \in M$ because $D_{x^{-1}} = D_x^x$ and conjugation takes members of $r(G)$ to members of $r(G)$. Suppose $x, y \in M$. Then

$$D_{xy} \supset D_x \cap (D_y)^{x^{-1}} \supset D_x \cap (D_y \cap D_{x^{-1}})^{x^{-1}} = Q \in \Pi$$

and hence $D_{xy} \in \Pi$ since $D_{xy} \in r_A(G)$. If $P \subset D_{xy}$ with $P \in \Pi$, then $(P \cap Q)^{xy} \in \Pi$ and hence $P^{xy} \in \Pi$. Hence $xy \in M$.

Clearly $M \neq G$ since $A \cap A^x \neq 1$ for all $x \in M$. Note that obviously $\mathscr{C}_G(P) \subset M$ for all $P \in \Pi$. To show that M is maximal suppose $y \in G - M$.

Case 1. $D_y \notin \Pi$. So for some $P \in \Pi$ we have $D_y \cap P = 1$ since $D_y \in r_A(G)$. Hence $P^y \cap A = 1$, and $\langle M, y\rangle \supset \langle \mathscr{C}_G(A), \mathscr{C}_G(P^y)\rangle = G$ by (1.13) since $P^y \in r(G)$.

Case 2. $D_y \in \Pi$, but for some $P \subset D_y$ with $P \in \Pi$ we have $P^y \notin \Pi$. Hence $P^y \cap Q = 1$ for some $Q \in \Pi$ and so $\langle M, y\rangle = G$ as in Case 1. □

Unfortunately we apparently cannot show that every maximal $M \in \mathscr{M}(A, G)$ is of the type described in (1.30). In general, if $M \in \mathscr{M}(A, G)$ and we define $\Pi = \{P \in r_A(G) | \mathscr{C}_G(P) \subset M\}$, then Π is a filter in $r_A(G)$ and Φ_M (see (1.27)) consists of certain isomorphisms which preserve Π. If M is maximal, then Φ_M consists of all the isomorphisms which preserve Π, but we cannot show that Π is an ultrafilter. If $M \in \mathscr{M}(A, G)$ is maximal it is easy to see that, if M is of Type 1, then for some $P \in \Pi$ there are isomorphisms $\varphi_1, \ldots, \varphi_n \in \Phi_M$ such that the HNN extension $K = (P, t_1, \ldots, t_n\colon a^{t_i} = \varphi_i(a)$ for all $1 \leqslant i \leqslant n$ and $a \in P \cap \operatorname{dom} \varphi_i)$ has a homomorphic image $\bar{K} = \langle \bar{t}_1, \ldots, \bar{t}_n\rangle$ such that $\bar{P} = P \subset \bar{K}$. For this to happen, it is necessary that $\Pi_P = \{Q \cap P | Q \in \Pi\}$ be preserved by all inner automorphisms of P. A sufficient condition beyond P being Abelian is very inscrutable.

It is also worth mentioning here that in **[2]** under the assumption $\diamondsuit_{\aleph_1}$ (a combinatorial consequence of Godel's Axiom of Constructability) an a.c. group G of power $\aleph_1$ was constructed with any given countable skeleton such that every maximal $M \subset G$ satisfies $\mathscr{C}_G(A) \subset M$ for some f.g. $A \subset G$. The proof is similar to that of (1.25) and Jonsson groups.

2. Automorphism groups. In this section we will prove

(2.0) THEOREM. *Suppose* G *is a countable a.c. group and* $\operatorname{Inn} G \subset A \subset \operatorname{Aut} G$ *where* A *is countable. Then there exists* $\tau \in \operatorname{Aut} G$ *such that the HNN extension* $(A, \tau\colon \tau^{-1} g \tau = \tau(g)\ (g \in \operatorname{Inn} G))$ *is a subgroup of* $\operatorname{Aut} G$.

This is a very easy consequence of the following more mundane result.

(2.1) LEMMA. *Given the hypotheses of* (2.0) *there exists an involution* $\sigma \in \operatorname{Aut} G - \operatorname{Inn} G$ *such that the free amalgamated product* $\langle \sigma \rangle \operatorname{Inn} G *_{\operatorname{Inn} G} A$ *is a subgroup of* $\operatorname{Aut} G$.

PROOF OF (2.0). Using (2.1) choose involutions $\sigma_1, \sigma_2 \in \operatorname{Aut} G - \operatorname{Inn} G$ such that

$$A_1 = \langle \sigma_1 \rangle \operatorname{Inn} G *_{\operatorname{Inn} G} A \subset \operatorname{Aut} G \quad \text{and} \quad A_2 = \langle \sigma_2 \rangle \operatorname{Inn} G *_{\operatorname{Inn} G} A_1 \subset \operatorname{Aut} G.$$

It is easily checked that $\tau = \sigma_1\sigma_2$ satisfies (2.0) by considering a Britton-reduced product w in the generators $A \cup \{\tau\}$. The only case requiring examination is when w has a segment $\tau^i g \tau^j$ where $ij > 0$ and $g \in \operatorname{Inn} G$. Assuming $i = j = 1$ we have $w = \cdots \sigma_1\sigma_2(g\sigma_1)\sigma_2 \cdots$. The bracketing given displays the A_2-reduced form of w and it follows that $w \neq 1$ in $\langle A, \tau \rangle \subset A_2$. □

To construct the σ of (2.1) we need to develop some properties of outer automorphisms of a.c. groups. We begin with

(2.2) LEMMA. *If G is an a.c. group, $A \subset G$ is f.g. and $\tau \in \operatorname{Aut} G$ satisfies $\tau(x) \in \langle A, x \rangle$ for all $x \in G$, then $\tau \in \operatorname{Inn} G$.*

PROOF. Assume that A and τ have the indicated property. Since the hypothesis remains true if we enlarge A, we can assume that A is centerless [**3**, Corollary to Theorem 6] (we can enlarge A to $A \operatorname{wr} \mathbf{Z} \subset G$). Since $\mathscr{C}_G(A)$ is a.c., it is generated by involutions. Hence our hypothesis guarantees that $\tau(x) = x$ for all $x \in \mathscr{C}_G(A)$.

Let $U = A \operatorname{wr} \langle u \rangle$ and $W = A \operatorname{wr}(\langle u \rangle \oplus \langle v \rangle)$, where $|u| = |v| = 2$. Since $U, W \in \operatorname{Sk}(G)$ we can assume $A \subset U \subset W \subset G$. The embeddings here are taken so that A is the 1-coordinate of the base group of U and of W. Our hypothesis implies that $\tau(A) = A$ and that the base group $A \oplus A^u$ of U is also τ-invariant. Put $V = A \operatorname{wr} \langle v \rangle \subset W$. Since $\tau(u) \in U$ and $\tau(v) \in V$, we have $\tau(u) = u(a + b^u)$ and $\tau(v) = v(c + d^v)$, where $a, b, c, d \in A$. Thus

$$\tau(uv) = \tau(u)\tau(v) = uv\left(c + b^{uv} + (ad)^v\right)$$

and we conclude $a = d^{-1}$ since we must have $\tau(uv) \in A \operatorname{wr} \langle uv \rangle = \langle uv \rangle (A \oplus A^{uv})$. Thus $\tau(uv) = uv(c + b^{uv})$, and by symmetry we can also conclude $c = b^{-1}$, $a = b^{-1}$, etc. Thus $\tau(u) = u(a + (a^{-1})^u) = u^a$ and $\tau(v) = v^a$.

Now, putting $H = \{g \in G | \tau(g) = g^a\}$ we have shown that $\langle \mathscr{C}_G(A), u \rangle \subset H$. Hence $\mathscr{C}_G(A^u) \subset H$ also and we conclude that $H = G$ by (1.13) since $A \cap A^u = 1$. □

Now we can give the property of outer automorphisms that is crucial for proving (2.1).

(2.3) LEMMA. *Suppose $A < B$ and C are f.g. subgroups of the a.c. group G and $\tau \in \operatorname{Aut} G - \operatorname{Inn} G$. Then there exists in G an isomorphic copy $\tilde{B}$ of B over A (that is, $\tilde{a} = a$ for all $a \in A$) such that $\tau(\tilde{B}) \not\subset \langle \tilde{B}, C \rangle$.*

PROOF. Using Lemma (2.2) we choose a f.g. $D \subset G$ such that $\langle C, A, \tau(A) \rangle \subset D$ and $\tau(D) - D \neq \varnothing$. We will show that $\tilde{B}$ exists such that $\tau(\tilde{B}) \not\subset \langle \tilde{B}, D \rangle$. To this end we will first prove

(2.4) If $\tau(\tilde{B}) \subset \langle \tilde{B}, D \rangle$ for all $\tilde{B} \subset G$ isomorphic to B over A, then $\tau(x) \in \langle D, x \rangle$ for all $x \in \mathscr{C}_G(A)$.

PROOF OF (2.4). Suppose $x \in \mathscr{C}_G(A)$ and $\tau(x) \notin \langle D, x\rangle$. By (1.19) we have

$$\langle A, x\rangle \subset H = \langle B_1, \ldots, B_n\rangle \subset G,$$

where each B_i is isomorphic to B over A. Since G is a.c. and $\tau(x) \notin \langle D, x\rangle$ there exists $g \in \mathscr{C}_G(\langle D, x\rangle)$ such that $\tau(x) \notin \langle H, D\rangle^g = \langle H^g, D\rangle$. But since each B_i^g is isomorphic to B over A the hypothesis of (2.4) implies

$$\tau(x) \in \langle \tau(B_1^g), \ldots, \tau(B_n^g)\rangle \subset \langle H^g, D\rangle,$$

a contradiction proving (2.4).

Now, assuming the conclusion of (2.4), we will reach a contradiction proving the lemma. (Since we will bring the condition $\tau(D) - D \neq \varnothing$ to bear which depends on (2.2) we are not directly generalizing (2.2) here.) Put $H = \langle \tau^{-1}(D), D, \tau(D)\rangle \subset G$ and $K = \langle x\rangle \oplus D$. Since $\tau(D) - D \neq \varnothing$, $H *_D K$ satisfies

$$[y, \tau(D)] \neq 1 \quad \text{for all } y \in (\langle x\rangle \oplus D) - D. \tag{2.5}$$

It follows that there is a subgroup $P = \langle H, x\rangle \subset G$ which also satisfies (2.5) and $x \notin H$. The reason for this is that the set of $y \in (\langle x\rangle \oplus D) - D$ is obviously r.e. in G and so the General Higman Embedding Theorem allows us to force all the inequations (2.5) to hold in G. Thus $\langle x\rangle \oplus D \subset P \subset G$ and we must have $\tau(x) \notin D$ because $\tau^{-1}(D) \subset H$, whereas $x \notin H$. Now (2.4) implies $\tau(x) \in (\langle x\rangle \oplus D) - D$ and hence $\langle x\rangle \oplus D \cong \tau(x) \oplus \tau(D)$ contradicts the inequality (2.5) with $y = \tau(x)$. □

PROOF OF LEMMA (2.1). We will construct σ inductively. Suppose σ_n has already been defined on some f.g. $B_n \subset G$ so that $\sigma_n(B_n) = B_n$ and $|\sigma_n| = 2$. To ensure that $G = \bigcup_{n \geqslant 1} B_n$ we can enlarge B_n to include the nth element g_n in some list of G. Since the symmetrical free amalgamated product $P = \langle B_n, g_n\rangle *_{B_n} \langle B_n, \tilde{g}_n\rangle$ has an automorphism of order 2 extending σ_n we can obtain a homomorphism $\varphi\colon P \to G$ with $\varphi \equiv 1$ on $\langle B_n, g_n\rangle$ so that $\varphi(P)$ also has this automorphism and take $\varphi(P)$ as the new B_n.

Since we will have $|\sigma| = 2$ any element $w \notin \operatorname{Inn} G$ of $\langle \sigma\rangle \operatorname{Inn} G *_{\operatorname{Inn} G} A$ is conjugate to one of the form $w = \sigma\alpha_m\sigma \cdots \alpha_1\sigma\alpha_0$ where $\alpha_i \in A - \operatorname{Inn} G$ $(0 \leqslant i \leqslant m)$.

Let w be given as above. We will show that σ_n can be extended to σ_{n+1} defined on $B_{n+1} \supset B_n$ in such a way that $\sigma_{n+1}(B_{n+1}) = B_{n+1}$, $|\sigma_{n+1}| = 2$, and

(2.6) There exists $z \in B_{n+1} - B_n$ such that the elements z, $\alpha_0(z)$, $\sigma\alpha_0(z), \ldots, w(z) \in B_{n+1} \subset G$ are all distinct.

If in the construction of σ we include steps (2.6) for every such w we will have $w \neq 1$ in Aut G and Lemma (2.1) will follow. We can choose $z \in G - B_n$ so that $\alpha_0(z) \notin \langle B_n, z\rangle$ by Lemma (2.2) since $\alpha_0 \notin \operatorname{Inn} G$. Put $B_{n,0} = \langle B_n, z\rangle$ and $\mu_i = \alpha_i\sigma \cdots \alpha_1\sigma\alpha_0$. Assume inductively that $0 \leqslant i \leqslant m$ and

$(2.7)_i$ $\alpha_0(z)$, $\sigma\alpha_0(z), \ldots, \mu_i(z)$ have been defined and are distinct and $B_n \subset B_{n,i} \subset G$ has been defined so that $B_{n,i}$ is f.g. and $\alpha_0(z), \ldots, \alpha_i^{-1}\mu_i(z) \in B_{n,i}$, but $\mu_i(z) \notin B_{n,i}$; and

$(2.8)_i$ $\sigma_{n,i} \subset B_{n,i} \times B_{n,i}$ has been defined so that $\sigma_n \subset \sigma_{n,i}$ and $\sigma_{n,i}$ is extendable to an (inner) automorphism of G having order 2.

We will show how to accomplish the next step by defining $\sigma(\mu_i(z))$ and $\alpha_{i+1}(\sigma(\mu_i(z)))$ (or just the first if $i = m$).

Put $J = \langle B_{n,i}, \mu_i(z)\rangle \subset G$. Let $\langle J, y\rangle \in \mathrm{Sk}(G)$ with $y \notin J$ be such that $\sigma_{n,i+1} = \sigma_{n,i} \cup \{(\mu_i(z), y)\}$ can be extended to an automorphism of order 2 of some supergroup of $\langle J, y\rangle$. (The hypothesis $(2.8)_i$ guarantees that some such group $\langle J, y\rangle$ exists and since G is a.c. we can obtain an image of this group containing J in $\mathrm{Sk}(G)$ which also admits this automorphism.) Now we use Lemma (2.3) to embed $\langle J, y\rangle$ into G over J in such a way that $\alpha_{i+1}(\langle J, y\rangle) \not\subset \langle J, y\rangle$ (in the notation of (2.3) A is J, B is $\langle J, y\rangle$ and C is 1). Hence $\alpha_{i+1}(y) \notin \langle J, y\rangle$. Thus

$$\mu_i(z) \overset{\sigma_{n,i+1}}{\to} y \overset{\alpha_{i+1}}{\to} \alpha_{i+1}(y) = \mu_{i+1}(z) \notin \langle J, y\rangle$$

and if we define $B_{n,i+1} = \langle J, y\rangle$ then the inductive properties $(2.7)_{i+1}$ and $(2.8)_{i+1}$ are satisfied.

After obtaining $B_{m,m+1}$ satisfying $(2.7)_{m+1}$ and $(2.8)_{m+1}$ (with $\mu_{m+1} = w$), we embed $B_{n,m+1} \subset B_{n+1} \subset G$ so that $\sigma_{n,m+1} \subset \sigma_{n+1} =$ an automorphism of B_{n+1} having order 2.

3. Extensions. Before proving the result mentioned in the abstract, which requires just a very slight generalizing of the construction as presented in [7], we will use the results of §1 to prove that the two natural cases of the construction [7, §§3, 5], namely using finitary permuations and piecewise linear order preserving maps, always yield nonisomorphic a.c. groups. We think this to be the most pleasant construction of nonisomorphic equivalent a.c. groups in arbitrary powers ([3] gives a very pleasant construction in power $\aleph_1$).

First we will describe the construction and the properties that we need. Let M denote a nontrivial countable a.c. group, let I be a countably infinite relational structure, and let Π be a group of automorphisms of I such that every isomorphism of finite subsets of I is induced by some member of Π; let $\Pi \subset M$ be an e.r. embedding, i.e., $\Pi \in r(M)$. (Usually Π has a solvable word problem and $\Pi \subset M$ is a recursive embedding.) Let $P(I)$ denote the *finite* subsets of I. If $s \in P(I)$, put $\Pi_s = \{\sigma \in \Pi | \sigma(x) = x \text{ for all } x \in s\}$. We will assume also that every Π_s is centerless and that $\Pi_s \in r(M)$. These assumptions ensure that $G = \bigcup\{\mathscr{C}_M(\Pi_s) | s \in P(I)\}$ is an a.c. group equivalent to M. Finally suppose $I \subseteq J$ is an extension of I with $|J| = \kappa$ such that every finite subset of J is isomorphic to a subset of I. Then, an a.c. group $G = G(M, J)$ is obtained which is equivalent to M and which is the direct limit of subgroups $\{M_s | s \in P(J)\}$ satisfying for all $s, t \in P(J)$

(3.0) $\qquad M_s \cong M; \qquad s \subseteq t$ implies $M_s \subset M_t$; and

(3.1) $\qquad$ If $s \subseteq t$, then the embedding $M_s \subset M_t$ is isomorphic to $\mathscr{C}_M(\Pi_{s'}) \subset \mathscr{C}_M(\Pi_{t'})$, where $s' \subseteq t' \in P(I)$ are arbitrary such that $'$ is an isomorphic mapping of t onto t'.

This description corresponds to [7, Lemma 2.5] with the condition $M_s \subset \mathscr{C}_M(\Pi_s)$ strengthened to equality. If $K \subseteq J$, put $G_K = \bigcup\{M_s | s \in P(K)\} \subset G(M, J)$. Thus $G_s = M_s$ for all $s \in P(J)$. The construction also satisfies

(3.2) $\qquad$ If $K, L \subseteq J$ and $K \cong L$, then $G_K \cong G_L$ (an isomorphism of G_K with G_L is induced via any isomorphism of K with L).

We need a final hypothesis to ensure that $|G(M, J)| = |J| = \kappa$. Various conditions will suffice for this. We choose a rather strong one which is satisfied in both cases we will consider, namely, we assume that Π_s has a solvable decision problem in Π for all $s \in P(I)$ (and that Π has a solvable word problem), and that $\Pi_s \subset \Pi_t$ implies $t \subseteq s$ for all $s, t \in P(I)$. These hypotheses ensure that

(3.3) For all $a \in I$ there exists $g \in \mathscr{C}_M(\Pi_a)$ such that $[g, h] \neq 1$ for all $h \in \Pi - \Pi_a$; and

(3.4) For all $a \in J$ there exists $g \in M_{\{a\}} - M_{J-\{a\}}$, and hence $M_K \subset M_L$ implies $K \subset L$ for all $K, L \subseteq J$.

From (3.4) it follows that $|G(M, J)| = |J| = \kappa$. For the general functorial properties of this construction, from which all of the above is easily deduced, the reader is asked to consult [**7**, §2].

Two natural cases of this construction considered in [**7**] are as follows.

(3.5) $I = \omega$ (natural numbers) and Π = all finitary permutations of ω.

In (3.5) the isomorphism-type of $G(M, J)$ will depend only on the cardinality of J and we will denote such a group by $G(\kappa)$, where $\kappa \geqslant \omega$ is a cardinal.

(3.6) $I = Q$ (rationals) and $\Pi = \Pi_Q$ = the piecewise linear order-preserving permutations of Q.

In (3.6) J can be any fully ordered set ξ and we will denote such a group by $G_Q(\xi)$. Notice that by (3.2) the order type of ξ uniquely determines the isomorphism type of $G_Q(\xi)$. Note that both $G(\kappa)$ and $G_Q(\xi)$ are equivalent to our fixed countable a.c. group M. As promised, we will prove

(3.7) THEOREM. *If $\kappa > \omega$ and ξ is any ordered set, then $G(\kappa) \not\cong G_Q(\xi)$.*

The principal property required in the proof will be as follows.

(3.8) LEMMA. *If ξ is an ordered set, then the only subgroups of $G_Q(\xi)$ containing $G_\varnothing = M_\varnothing \cong M$ are the obvious ones, namely $\{G_K | K \subseteq \xi\}$.*

The proof of (3.8) will make use of the following two properties of $\Pi_Q = \Pi$.

(3.9) If $s \in P(Q)$, then the only subgroups of Π containing Π_s are the Π_t where $t \subseteq s$;

(3.10) If $t \subseteq s \in P(Q)$ and $\tau \in \mathrm{Aut}(\Pi_t)$ satisfies $\tau(\alpha) = \alpha$ for all $\alpha \in \Pi_s$, then $\tau = 1_{\Pi_t}$.

In [**2**] proofs of these were given, but we will omit them here since we have been informed by Andrew Glass that both of these properties follow in rather general circumstances from the general theory of ordered permutation groups [**9**]. Indeed (3.9) is true for any highly transitive group of order-preserving permutation of an ordered set (cf. Theorem 4.1.5 of [**9**]) and (3.10) can be proved by making a direct adaption of the proof of Theorem 2.D of [**9**]. Here our concern is with the consequences of properties such as these for the construction in [**7**].

PROOF OF LEMMA (3.8). In this proof we will write $G(\xi)$ for $G_Q(\xi)$. Every ordered set ξ is contained in a densely ordered set δ and δ is the union of subsets $\{\delta_\alpha\}$ order-isomorphic to Q. So we have $G(\xi) \subset G(\delta) = \bigcup\{G_{\delta_\alpha}\}$, where $G_{\delta_\alpha} \cong G(Q) = \bigcup\{M_s | s \in P(Q)\}$ (this is actually an isomorphism of direct limits). Suppose that we have established

(3.11) If $G_\varnothing \subset F \subset G(Q)$, then $F = G_K$ for some $K = Q$.

Then, if $G_\varnothing \subset H \subset G(\xi) \subset G(\delta)$, we have for all α, $G_\varnothing \subset H \cap G_{\delta_\alpha} = G_{K_\alpha}$ for some $K_\alpha \subseteq \delta_\alpha$ and hence $H = \bigcup\{G_{K_\alpha}\} = G_K$, where $K = \bigcup\{K_\alpha\}$, and we conclude that $K \subseteq \xi$ by (3.4). Thus (3.8) will follow if we prove the special case (3.11).

To prove (3.11) we first prove

(3.12) Suppose $G_\varnothing \subset F \subset G(Q) = \bigcup\{G_s | s \in P(Q)\}$. Then there is a subset $K \subseteq Q$ such that, for all $s \in P(Q)$, we have $G_s \subset F$ iff $s \subseteq K$; that is, K is the *maximum* subset of Q such that $G_K \subset F$.

PROOF OF (3.12). Put $K = \bigcup\{s \in P(Q) | G_s \subset F\}$. To prove that $G_K \subset F$ we must check that, for all $s, t \in P(Q)$, if $G_s, G_t \subset F$, then $G_{s \cup t} \subset F$. The extension $G_s \subset G_{s \cup t}$ is isomorphic to $\mathscr{C}_M(\Pi_s) \subset \mathscr{C}_M(\Pi_{s \cup t})$ and similarly for $G_t \subset G_{s \cup t}$. Since Π_s, $\Pi_t \in r(M)$ (1.13) implies

$$\langle \mathscr{C}_M(\Pi_s), \mathscr{C}_M(\Pi_t) \rangle = \mathscr{C}_M(\Pi_s \cap \Pi_t) = \mathscr{C}_M(\Pi_{s \cup t})$$

whence $\langle G_s, G_t \rangle = G_{s \cup t} \subset F$, as required.

Now to prove (3.11) suppose $G_\varnothing \subset F \subset G(Q)$. We wish to show that $F = G_K$ where $K \subseteq Q$ is as in (3.12). Let $s \in P(Q)$. By (3.9) only finitely many subgroups lie between Π_s and $\Pi_{K \cap s}$. Since $G_{K \cap s} = \mathscr{C}_M(\Pi_{K \cap s})$, (1.15) implies

(3.13) $\mathscr{C}_M(V) \subset F \cap \mathscr{C}_M(\Pi_s) \subset \mathscr{N}_M(V)$ for some V with $\Pi_s \subset V \subset \Pi_{K \cap s}$,

and (3.9) implies that $V = \Pi_t$ for some t with $K \cap s \subseteq t \subseteq s$. Now we use property (3.10) to replace $\mathscr{N}_M(V)$ in (3.13) by $\mathscr{C}_M(V)$: Suppose $g \in F \cap \mathscr{C}_M(\Pi_s)$. By (3.13) g normalizes $V = \Pi_t$; thus g induces an automorphism on Π_t which centralizes Π_s, whence g must centralize Π_t by (3.10). So we have proved $F \cap \mathscr{C}_M(\Pi_s) = \mathscr{C}_M(\Pi_t)$ and we conclude that $t \subseteq K$, i.e., $t = K \cap s$, and consequently $F = G_K$. □

PROOF OF THEOREM (3.7). Suppose $\kappa > \omega$ and φ: $G(\kappa) \to G_Q(\xi)$ is an (onto) isomorphism. To avoid ambiguity we write $G(\kappa) = \bigcup\{N_s | s \in P(\kappa)\}$ and $G_Q(\xi) = \bigcup\{G_s | s \in P(\xi)\}$ for the canonical direct limit systems. There exists a countable subset $A \subseteq \kappa$ such that $G_\varnothing \subset \varphi(N_A) \subset G_Q(\xi)$. By Lemma (3.8) the only subgroups between $G_\varnothing$ and $G_Q(\xi)$ are of the form G_T, where $T \subseteq \xi$ and these are all a.c. groups equivalent to M. Hence every subgroup of $G(\kappa)$ containing N_A must be of this type. To reach a contradiction we will display a subgroup V with $N_A \subset V \subset G(\kappa)$ such that V is not an a.c. group. By the categoricity of this direct limit construction, the embeddings $N_A \subset G(\kappa)$, $N_\omega \subset G(\kappa)$ and $N_{\omega - \{0,1\}} \subset G(\kappa)$ are isomorphic (assuming WLOG that A is infinite), and we will define V to be an intermediate group in the last one. Let $\tau = (01) \in \Pi \subset M$ and put $V = \mathscr{C}_{G(\kappa)}(\tau)$. For all $s \in P(\omega - \{0,1\})$ we have $N_s = \mathscr{C}_M(\Pi_s) \subset V$ since $\tau \in \Pi_s$. Thus $N_{\omega - \{0,1\}} \subset V$ and, since τ is in the center of V, V is not an a.c. group. □

Finally, we will give a more precise statement of the result on extensions mentioned in the abstract and show how to slightly modify the construction of [7] in order to prove it.

(3.14) THEOREM. *Suppose $M \subset N$ are a.c. groups such that M is countable and $M \in r(N)$. Suppose J is any a.c. group equivalent to M. Then there exists an a.c. group G equivalent to N such that $J < G$.*

Actually, the conclusion of this theorem could have been stated $J \subset G$ because this containment *must* be proper since M is not equivalent to N by Theorem 1 of [3] since M is contained in a f.g. subgroup of N. For exactly the same reason (3.14) will give no information at all about embedding J in an a.c. group having the same skeleton as J—this remains an open question. Notice that, given M, in order to obtain a suitable N one must merely increase the skeleton of M by some f.g. group which contains M as an e.r. subgroup. Of course this introduces many more subgroups into the skeleton as well (see Proposition 3 of [3]). It would be interesting to know if there ever exists a minimal skeleton for N (such that $M \in r(N)$). Perhaps the characterization determined in [8] can answer this question for certain M (such as the M whose skeleton consists of all the f.g. recursively presented groups).

To prove (3.14) we want the "relational structure" I to be M and we want Π to also be M. To do this we replace the finite subsets $P(I)$ by

$$\Delta(M) = \{ D \mid D \text{ is a f.g. centerless subgroup of } M \}. \tag{3.15}$$

This is a direct limit system of M by the Corollary to Theorem 6 in [3]. Now $\Pi = M$ (acting on itself by conjugation) has the homogeneity property required for the construction, namely

(3.16) If $f\colon D \to E$ is an isomorphism of members of $\Delta(M)$, then there exists $g \in \Pi = M$ which induces f.

Now we define the homogeneous direct limit system on which M "operates". Considering the embedding $M \subset N$ we define

$$M_D = \mathscr{C}_N(\mathscr{C}_M(D)) \quad \text{for all } D \in \Delta(M). \tag{3.17}$$

Suppose $D \in \Delta(M)$. Since D is f.g. and $M \in r(N)$, it is easy to see that $\mathscr{C}_M(D) \in r(N)$ also: to enumerate $\mathscr{C}_M(D)$ we enumerate both M and the relations $W(M)$ holding among elements of M. We discover that $g \in M$ lies in $\mathscr{C}_M(D)$ as soon as we find that all of the commutators of g with the generators of D lie in $W(M)$. From $\mathscr{C}_M(D) \in r(N)$ it follows that $\mathscr{C}_N(\mathscr{C}_M(D))$ is an a.c. group equivalent to N (using, as usual, the General Higman Embedding Theorem). Thus $G(M) = \bigcup\{ M_D = \mathscr{C}_N(\mathscr{C}_M(D)) \mid D \in \Delta(M)\}$ is an a.c. group equivalent to N and we have a natural containment $M \subset G(M)$ since $D \subset \mathscr{C}_N(\mathscr{C}_M(D))$ for all $D \in \Delta(M)$ and, as previously noted, $M = \bigcup\Delta(M)$. We obtain an operation of $\Pi = M$ on the system $\{ M_D \mid D \in \Delta(M)\}$ *exactly* as in [7, §3], namely if $f\colon D \to E$ is an isomorphism of members of $\Delta(M)$, then $\mathfrak{A}(f)\colon M_D \to M_E$ is well defined by $\mathfrak{A}(f)(x) = g^{-1}xg$, where $g \in M$ is an arbitrary element which induces f on D by conjugation; and this collection of $\mathfrak{A}(f)$'s compose and restrict as required of an operation.

So it follows that if $M \subset J$, where J is an a.c. group equivalent to M, this operation extends (uniquely) to an operation of J on a direct limit system $\{M_D | D \in \Delta(J)\}$ (using the same proof as that of Lemma 2.3 of [7]). If $D \subset E \in \Delta(J)$, the embedding $M_D \subset M_E$ can be obtained by choosing *any* isomorphisms α and β of D and E with members $\alpha(D), \beta(E) \in \Delta(M)$ and then using the inclusion map j: $D \to E$ to obtain $f = \beta^{-1} j \alpha^{-1}$: $\alpha(D) \to \beta(E)$ and $\mathfrak{A}(f)$: $M_{\alpha(D)} = \mathscr{C}_N(\mathscr{C}_M(\alpha(D))) \to M_{\beta(E)}$ which defines the direct limit embedding $M_D \to M_E$ once we fix all of our choices for α and β. Now we must observe that the images of the subgroups $\Delta(M)$ (where $M \subset G(M) \subset G(J)$ is the natural embedding mentioned previously) under all of the induced maps into the direct limit group $G(J) = \bigcup\{M_D | D \in \Delta(J)\}$ form a subgroup of $G(J)$ isomorphic to J. The proof of this requires the same commutative diagram chasing which shows that the direct limit system is well defined. Indeed this property is rather transparent since the inclusion embeddings j: $D \subset E$ for $D, E \in \Delta(J)$ are reproduced in a subgroup of $G(J)$ via the embeddings $f = \beta^{-1} j \alpha^{-1}$ (above) which are restrictions of the operator maps $\mathfrak{A}(f)$ because $\mathfrak{A}(f)$ is defined by conjugation by an element of M which induces f.

The above proof does suggest a possible method for properly enlarging J to an a.c. group equivalent to M. To do this we would have to somehow obtain a "proper operation of M on itself". For example we would accomplish this if we could construct an embedding φ: $M \to M$ satisfying: $G = \bigcup\{\mathscr{C}_M(\mathscr{C}_{\varphi(M)}(\varphi(D))) | D \in \Delta(M)\}$ is an a.c. group and $\varphi(M)$ is a proper subgroup of G where $\varphi(M) \subset G$ is the natural embedding given by $\varphi(D) \subset \mathscr{C}_M(\mathscr{C}_{\varphi(M)}(\varphi(D)))$.

Since this requirement does not appear to be terribly stringent, it gives considerable hope for a positive answer to this question. Yet no means for constructing φ seems to be at hand.

4. Free maximal subgroups. In this section we prove

(4.0) THEOREM. *Suppose I is a countable relational structure and Π is a group of automorphisms of I such that every isomorphism of finite subsets of I is induced by some element of Π. Further suppose that $\mathscr{C}_\Pi(\Pi_s) = 1$ and Π_s has a solvable decision problem in Π for all $s \in P(I)$ (in particular, Π has a solvable word problem). Let $\Pi \subset M$ be a recursive embedding of Π into a countable a.c. group M (i.e., an e.r. embedding in which all of the degrees involved are recursive). Let J be an extension of I having the same finite substructures as I. Then, the homogeneous limit group $N(J)$ as in the construction of* [7] *has a maximal subgroup which is free. In fact there exists $X \subseteq N(J)$ which freely generates $\langle X \rangle$ in $N(J)$ and such that, for all $y \in N(J) - \langle X \rangle$ and $z \in N(J)$, there exists $a, b, c \in X$ such that $z = (ay)^b (ay)^c$.*

Here we have used the notation of §3 except that the direct limit group $G(J)$ is denoted by $N(J)$ (as in [7]).

COROLLARY. *$G_Q(\zeta)$ for all ordered sets $\zeta \supseteq Q$ (as in* §3*) has a free maximal subgroup as described in* (4.0).

PROOF. With $\Pi_Q = \Pi$ as in §3 one easily checks that $\mathscr{C}_\Pi(\Pi_s) = 1$ and that Π_s has a solvable decision problem in Π for all $s \in P(Q)$. □

First we will prove

(4.1) THEOREM. *Every countable a.c. group M has a free maximal subgroup. In fact there exists $X \subseteq M$ which freely generates $\langle X \rangle$ in M and satisfies: for all $y \in M - \langle X \rangle$ and $z \in M$, there exists $a, b, c \in X$ such that $z = (ay)^b(ay)^c$.*

Note that $\langle X \rangle$, a countable free group, is an example of a Type 2b maximal subgroup of M. This is the simplest illustration of (1.25).

PROOF. We inductively construct X. Let $M \times M = \{(y_i, z_i) | i \in \mathbf{N}\}$. At the ith step of the construction we will add three more free generators—say a, b, c—to X_i (the previously chosen free generators) which will satisfy

(4.2) If $y_i \notin \langle X_i \rangle$, then $z_i = (ay_i)^b(ay_i)^c$ and $X_{i+1} = X_i \cup \{a, b, c\}$ freely generates $\langle X_{i+1} \rangle$ in M.

Thus $X = \bigcup_{i \geqslant 1} X_i$ will satisfy (4.1). To accomplish (4.2) put $M_0 = \langle X_i, y_i, z_i \rangle \subset M$ and consider the group H with presentation $(M_0, a, b, c, u, v\colon uv = z_i, (ay_i)^b = u, (ay_i)^c = v)$, where a, b, c, u, v are new symbols. H is an HNN extension with stable letters b and c of the group $G = (M_0, a, u, v\colon uv = z_i)$. One checks easily that, in G, $L = \langle X_i \rangle * \langle a \rangle$ exists and $\langle u \rangle \cap L = \langle v \rangle \cap L = 1$, and if $y_i \notin \langle X_i \rangle$, then $\langle ay_i \rangle \cap L = 1$ also. It follows that the conjugated subgroups in the HNN extension H all intersect L trivially, so Britton's Lemma implies that $L * \langle b \rangle * \langle c \rangle$ exists in H, i.e., $X_i \cup \{a, b, c\}$ are free generators in H. Now we can find elements of M (which we also call a, b and c) which satisfy the relations of H, and since free groups have solvable word problems (and are embeddable in M), we can arrange $X_i \cup \{a, b, c\}$ to be free generators in M also (we can include relations which conjugate these to a free set contained in M); thus (4.2) follows. □

The proof of (4.0) reduces easiliy to a certain construction in the countable case based on the above proof. Let $N = \bigcup\{\mathscr{C}_M(\Pi_s) | s \in P(I)\}$ be a countable a.c. group expressed as a direct limit as in Ziegler's construction, and assume that (Π, I) satisfies the hypotheses of (4.0). To prove (4.0) it will suffice to produce a subset $X \subseteq N$ satisfying

(4.3) X freely generates $\langle X \rangle$ in N,

(4.4) $X = X^\sigma$ for all $\sigma \in \Pi$, and putting $C_s = \mathscr{C}_M(\Pi_s)$,

(4.5) for all $s \in P(I)$, $y \in C_s - \langle X \rangle$, and $z \in C_s$, there exist $a, b, c \in X \cap C_s$ such that $z = (ay)^b(ay)^c$.

Notice that (4.5) implies that, for all $s \in P(I)$, $\langle C_s \cap X \rangle$ is maximal in C_s, and hence equals $C_s \cap \langle X \rangle$. It will follow by simple diagram chasing that if $N(J) = \bigcup\{N_s | s \in P(J)\}$ then, if the above $X \subseteq \bigcup\{N_s | s \in P(I)\}$ is extended to $\overline{X} \subseteq N(J)$ via the canonical isomorphisms induced by the chosen bijections $f_s\colon s' \to s$ [7, Lemma 2.3] where $s \in P(J)$ and $s' \in P(I)$, then

(4.6) $\overline{X}$ freely generates $\langle X \rangle$ in N, and

(4.7) for all $s \in P(J)$, $y \in N_s - \langle \overline{X} \rangle$ and $z \in N_s$, there exist $a, b, c \in N_s \cap \overline{X}$ such that $z = (ay)^b(ay)^c$.

Theorem (4.0) follows from (4.6) and (4.7).

To adapt the proof of (4.1) to the context of Ziegler's construction, we need a technical lemma.

(4.8) LEMMA. *Suppose $P \subset \Pi \subset G$ are groups, $\mathscr{C}_\Pi(P) = 1$, $X \subseteq G$ freely generate $\langle X \rangle$ in G, Π normalizes $\langle X \rangle$ and $\Pi \cap \langle X \rangle = 1$. Further suppose*

(4.9) *for all $\sigma \in \Pi$ and $x \in \langle X \rangle$, $\sigma^x \in \Pi$ iff σ and x commute.*

Then

(4.10) *Suppose $\overline{G} = (G, a;\ a$ centralizes $P) = G *_P (P \oplus \langle a \rangle)$ and $y \in \mathscr{C}_G(P) - \langle X \rangle$. Then, in $\overline{G}$, $\overline{X} = \{a^\sigma | \sigma \in \Pi\} \cup X$ freely generates $\langle \overline{X} \rangle$, $\langle \overline{X} \rangle \cap \Pi = 1$, (4.9) holds with $\overline{X}$ in place of X, $\langle ay \rangle \cap \Pi \langle \overline{X} \rangle = 1$ and, for all $\Gamma \in \langle \overline{X} \rangle$ and $\sigma \in \Pi$, $\sigma^\Gamma \notin \langle ay \rangle P - P$;*

(4.11) *Suppose $z \in \mathscr{C}_G(P)$ and $\overline{G} = (G, u, v;\ u$ and v centralize P and $z = uv)$. Then, in $\overline{G}$, $\langle u \rangle \cap G = \langle v \rangle \cap G = 1$;*

(4.12) *Suppose $t, u, v \in G$ have infinite order and centralize P, and, for $w = t, u, v$, $\langle w \rangle \cap \Pi \langle X \rangle = 1$, and, for all $x \in \langle X \rangle$ and $\sigma \in \Pi$, $\sigma^x \notin \langle w \rangle P - P$. Let $\overline{G} = (G, b, c;\ b$ and c centralize P and $t^b = u$, $t^c = v)$. Then, in $\overline{G}$, $\overline{X} = \{b^\sigma, c^\sigma | \sigma \in \Pi\} \cup X$ freely generates $\langle \overline{X} \rangle$, $\langle \overline{X} \rangle \cap \Pi = 1$, and (4.9) holds with $\overline{X}$ in place of X.*

PROOF OF (4.10). Let $1 \in T =$ a right transversal of P in Π. Note that $\overline{X} = \{a^\tau | \tau \in T\} \cup X$. Suppose Γ is a nonempty reduced word in the free group on $\overline{X}$. To check that $\Gamma \neq 1$ in the amalgamated product $\overline{G}$, we write $\Gamma = d_1 a_1 \cdots d_n a_n d_{n+1}$, where $1 \neq a_i \in \langle a \rangle$, possibly $d_1, d_{n+1} = 1$, but $1 \neq d_i = x$, $\sigma x \tau^{-1}$, or $\sigma \tau^{-1}$ with $1 \neq x \in \langle X \rangle$ and $\sigma, \tau \in T$ for $1 < i \leqslant n$. Since $\langle X \rangle \cap \Pi = 1$ we have $d_i \notin \Pi$ for $1 < i \leqslant n$. So the above product is reduced in $\overline{G}$, implying $\Gamma \neq 1$ in $\overline{G}$. It also follows that if some factor a^σ occurs in Γ, then $\Gamma \notin G$, proving $\langle \overline{X} \rangle \cap \Pi = 1$.

To prove (4.9) with $\overline{X}$ in place of X, suppose $\sigma \in \Pi$ and $\Gamma \in \langle \overline{X} \rangle$, and put

(4.13) $\Gamma = x_1(t_1^{-1} a^{i_1} t_1) x_2 \cdots x_n (t_n^{-1} a^{i_n} t_n) x_{n+1}$, where $t_j \in T$, $i_j \neq 0$, possibly $x_1, x_{n+1} = 1$, but $1 \neq x_j \in \langle X \rangle$ for $1 < j \leqslant n$.

The proof that $\sigma^\Gamma \in \Pi \Leftrightarrow \sigma$ and Γ commute runs as follows. If $\sigma^{x_1} \notin \Pi$, then $d = t_1 \sigma^{x_1} t_1^{-1} \notin \Pi$ also, and either $\Gamma = x_1$ or σ^Γ has $\overline{G}$-reduced form

$$\sigma^\Gamma = \cdots t_1^{-1} a^{-i_1} \underline{d} a^{i_1} t_1 \cdots \notin G. \tag{4.14}$$

So assume $\sigma^{x_1} \in \Pi$, and hence $\sigma^{x_1} = \sigma$ by (4.9). If $d \notin P$, then σ^Γ has $\overline{G}$-reduced form (4.14); so assume $d \in P$. It follows that $x_1 t_1 a^{i_1} t_1^{-1}$ commutes with σ because a centralizes P. The proof can now be completed by induction on the length of Γ.

For the final conclusions, we will need

$$y \notin \Pi \langle X \rangle. \tag{4.15}$$

PROOF. Suppose $y = \sigma x$ with $\sigma \in \Pi$ and $x \in \langle X \rangle$. Thus $\sigma x \in \mathscr{C}_G(P)$, implying $xPx^{-1} = P^\sigma \subset \Pi$, and so x centralizes P by (4.9). Thus, σ centralizes P also, implying $\sigma = 1$ since $\mathscr{C}_\Pi(P) = 1$. Thus $y = x$, contrary to hypothesis. □

To prove $\langle ay \rangle \cap \Pi\langle \bar{X} \rangle = 1$, suppose $j \neq 0$ and $\Gamma \in \langle \bar{X} \rangle$. We will show $Q = (ay)^j\Gamma \notin \Pi$. Let Γ be written as in (4.13).

Case 1. $j > 0$. $Q = \cdots ayx_1t_1^{-1}a^{i_1}t_1 \cdots$ and $\bar{G}$-reduction will occur $\Leftrightarrow yx_1t_1^{-1} \in P$ which would imply $y \in \Pi\langle X \rangle$, contrary to (4.15).

Case 2. $j < 0$. $Q = \cdots y^{-1}a^{-1}x_1t_1^{-1}a^{i_1}t_1 \cdots$ and $\bar{G}$-reduction will occur $\Leftrightarrow x_1t_1^{-1} \in P \Leftrightarrow x_1 = 1$ and $t_1 = 1$ since $\Pi \cap \langle X \rangle = 1$. This gives

$$Q = \cdots \left(y^{-1}a^{-1}\right) y^{-1}a^{i_1-1}\left(x_2t_2^{-1}a^{i_2}t_2\right) \cdots,$$

where the factors in parentheses may not occur. This product is reduced unless $i_1 = 1$. This gives $Q = \cdots y^{-1}(x_2t_2^{-1}a^{i_2}t_2) \cdots$. Since $y^{-1}(x_2t_2^{-1}) \notin P$ (otherwise $y^{-1} \in \Pi\langle X \rangle = \langle X \rangle\Pi$, contrary to (4.15)), we conclude $Q \notin \Pi$. □

To establish our last conclusion, we will show

(4.16) for all $\Gamma \in \langle \bar{X} \rangle$, $\sigma \in \Pi$ and $j \neq 0$, $Q = (ay)^j\Gamma^{-1}\sigma\Gamma \notin \Pi$.

The proof is very similar to that just given, but we sketch it anyway. Let Γ be as in (4.13). Assume WLOG $d = (x_1t_1^{-1})^{-1}\sigma x_1t_1^{-1} \notin P$ (otherwise we could omit the first factor in parentheses in (4.13)). Also assume $\Gamma \notin \langle X \rangle$ (otherwise (4.16) is trivial). We consider only the case $j < 0$.

$$Q = \cdots y^{-1}a^{-1}x_{n+1}^{-1}\left(t_n^{-1}a^{-i_n}t_n\right) \cdots x_2^{-1}t_1^{-1}a^{-i_1}\underline{d}a^{i_1} \cdots,$$

and reduction occurs $\Leftrightarrow x_{n+1} = t_n = 1$ and $i_n = -1$. This gives

$$Q = \cdots \left(y^{-1}a^{-1}\right) y^{-1} \cdots x_2^{-1}t_1^{-1}a^{-i_1}\underline{d} \cdots,$$

which is reduced since $y^{-1}x_n^{-1}t_{n-1}^{-1} \notin P$ (if $n = 1$ and $j = -1$, then $Q = y^{-1} \notin \Pi$).

PROOF OF (4.11). This is very easy.

PROOF OF (4.12). $\bar{G}$ is an HNN extension of G with stable letters b and c and conjugated subgroups $\langle w \rangle P$, $w = t, u, v$. We again choose T as in the proof of (4.10) and have $\bar{X} = \{b^\tau, c^\tau | \tau \in T\} \cup X$. If Γ is a reduced word in the free group on $\bar{X}$, then any segment of Γ occurring between successive occurrences of the stable letters (after combining similar conjugate sequences such as $\tau^{-1}b\tau\tau^{-1}c\tau = \tau^{-1}bc\tau$) would be of the form $\tau x\sigma^{-1}$ ($\tau, \sigma \in T$, $x \in \langle X \rangle$). If $\tau x\sigma^{-1} \in \langle w \rangle P$, say $\tau x\sigma^{-1} = w^jp$, then $j \neq 0$ (otherwise $x \in \Pi \Rightarrow x = 1 \Rightarrow \tau\sigma^{-1} \in P \Rightarrow \tau\sigma^{-1} = 1$ would have been cancelled already). Now $w^j \in \Pi\langle X \rangle\Pi = \Pi\langle X \rangle$, contrary to hypothesis. Thus $\tau x\sigma^{-1}$ lies outside of the conjugated subgroups, so no stable letter reduction can occur in Γ, and Britton's Lemma implies $\Gamma \neq 1$ in $\bar{G}$. Furthermore, if some generator of the form b^τ or c^τ occurs in Γ, then $\Gamma \notin G$ again by Britton's Lemma, proving $\Pi \cap \langle \bar{X} \rangle = 1$ also.

To prove (4.9) with $\bar{X}$ in place of X, let $\sigma \in \Pi$ and $\Gamma \in \langle \bar{X} \rangle$. Put $\Gamma = x_1\Sigma_1^{\tau_1}x_2 \cdots x_n\Sigma_n^{\tau_n}x_{n+1}$, where $\Sigma_j = b^{\pm 1}, c^{\pm 1}$, $\tau_j \in T$, $1 \neq x_j \in \langle x \rangle$ for $1 < j \leqslant n$ (possibly $x_1, x_{n+1} = 1$). To show $\sigma^\Gamma \in \Pi \Leftrightarrow \sigma$ and Γ commute, we use the same argument as in the proof of (4.10), but we have stable letter reductions to contend with. WLOG $\Gamma \notin \langle X \rangle$ and we have

$$\sigma^\Gamma = \cdots \left(\tau_1^{-1}\Sigma_1^{-1}\tau_1\right)x_1^{-1}\sigma x_1\left(\tau_1^{-1}\Sigma_1\tau_1\right) \cdots.$$

If $\sigma^{x_1} \notin \Pi$, then $d = \tau_1 x_1^{-1} \sigma x_1 \tau_1^{-1} \notin \Pi$ also and we must check that d lies outside of the conjugated subgroups $\langle w \rangle P$. Put $x = \tau_1 x_1 \tau_1^{-1} \in \langle X \rangle$ since Π normalizes $\langle X \rangle$. We have $d = x^{-1} \tau_1 \sigma \tau_1^{-1} x = \pi^x$, $\pi \in \Pi$. So, $d \in \langle w \rangle P$ would imply $\pi^x \in \langle w \rangle P - P$, contrary to hypothesis. So $\sigma^\Gamma = \cdots \Sigma_1^{-1} d \Sigma_1 \cdots$ is reduced and $\sigma^\Gamma \notin G$ by Britton's Lemma. So we can assume $\sigma^{x_1} \in \Pi$. Hence $\sigma^{x_1} = \sigma$ by (4.9). If $d = \tau_1 \sigma \tau_1^{-1} \notin P$, then $d \notin \langle w \rangle P$ (otherwise $\langle w \rangle \cap \Pi \neq 1$) and hence $\sigma^\Gamma \notin G$ as before; if $d \in P$, then $x_1 \tau_1^{-1} \Sigma_1 \tau_1$ commutes with σ since Σ_1 centralizes P. The proof is now completed by induction on the length of Γ. $\square$

It now remains to use Lemma (4.8) to construct $X \leqslant N = \bigcup\{\mathscr{C}_G(\Pi_s) | s \in P(I)\}$ satisfying (4.3), (4.4) and (4.5). We construct X inductively. Let $\{(y_i, z_i, s(i)) | i \geqslant 0\}$ be a list of $\bigcup\{C_s \times C_s \times \{s\} | s \in P(I)\}$, where $C_s = \mathscr{C}_M(\Pi_s)$. Let $\Pi \subset F \subset M$, F f.g., be an effective embedding. Suppose that after i steps of the construction we have found $x_0, \ldots, x_{3i-1} \in M$ such that

(a) $X_i = \{x_0^\sigma, \ldots, x_{3i-1}^\sigma | \sigma \in \Pi\}$ freely generates $\langle X_i \rangle$ in M,

(b) for all $0 \leqslant j < i$, x_{3j}, x_{3j+1} and x_{3j+2} centralize $\Pi_{s(j)}$,

(c) for all $0 \leqslant j < i$, if $y_j \notin \langle X_j \rangle$, then

$$z_j = (x_{3j} y_j)^{x_{3j+1}} (x_{3j} y_j)^{x_{3j+2}},$$

(d) $\langle X_i \rangle \cap \Pi = 1$ (note that $X_0 = \varnothing$, $\langle X_0 \rangle = 1$),

(e) for all $\sigma \in \Pi$ and $x \in \langle X_i \rangle$, $\sigma^x \in \Pi$ iff σ and x commute, and

(f) $\Pi \langle X_i \rangle$ has a solvable word problem with respect to the generators $\Pi \cup X_i$.

Notice that if this construction can be continued to obtain $X = \{x_j^\sigma | j \geqslant 0, \sigma \in \Pi\}$ satisfying (a)–(c) for all $i \geqslant 0$, then (4.3)–(4.5) follow.

To obtain the next triple $\vec{x} = (x_{3i}, x_{3i+1}, x_{3i+2})$, we use Lemma (4.8). If $y_i \in \langle X_i \rangle$, put $\vec{x} = (x_0, x_0, x_0)$. Otherwise, let $\overline{G}_i = (G_i, a, b, c;$ a, b, and c centralize $\Pi_{s(i)}$ and $z_i = (ay_i)^b (ay_i)^c)$, where $G_i = \langle F, x_0, \ldots, x_{3i-1}, y_0, \ldots, y_i, z_i \rangle \subset M$. To apply Lemma (4.8), we view $\overline{G}_i$ as constructed from G_i in 3 steps: (1) $G_i' = (G_i, a;$ a centralizes $P = \Pi_{s(i)})$; (2) $G_i'' = (G_i', u, v;$ u and v centralize P and $z_i = uv)$; (3) $\overline{G}_i = (G_i'', b, c;$ b and c centralize P, $(ay_i)^b = u$, and $(ay_i)^c = v)$. To apply Lemma (4.8), note: In (4.10), G_i is G, X_i is X, and G_i' is $\overline{G}$; in (4.11), G_i' is G, $X_i \cup \{a^\sigma | \sigma \in \Pi\}$ is X, and G_i'' is $\overline{G}$; in (4.12), G_i'' is G, $t = ay$, $X_i \cup \{a^\sigma | \sigma \in \Pi\}$ is X, $\overline{G}_i$ is $\overline{G}$, and $X_i \cup \{a^\sigma, b^\sigma, c^\sigma | \sigma \in \Pi\}$ is $\overline{X}$. Putting $\overline{X}_i = X_i \cup \{a^\sigma, b^\sigma, c^\sigma | \sigma \in \Pi\} \subseteq \overline{G}_i$, the conclusions of (4.12) are

(i) $\overline{X}_i$ freely generates $\langle \overline{X}_i \rangle$ in $\overline{G}_i$,

(ii) $\langle \overline{X}_i \rangle \cap \Pi = 1$ in $\overline{G}_i$, and

(iii) for all $\sigma \in \Pi$ and $x \in \langle \overline{X}_i \rangle$, $\sigma^x \in \Pi$ iff σ and x commute.

Note that $\overline{G}_i$ is recursively presented over the f.g. $G_i \subset M$. We need to observe that there are elements $\vec{x}$ of M which, when corresponded with (a, b, c) respectively, satisfy the relations of $\overline{G}_i$ as well as (i)–(iii) (which hold in $\overline{G}_i$). We can choose $\vec{x}$ so that any given recursively enumerable set of equations and inequations which hold in $\overline{G}_i$ are satisfied also in $\langle G_i, \vec{x} \rangle \subset M$.

So, we will restate (i)–(iii) as a recursive set of inequations. We have

(i) $\Leftrightarrow \{x \neq 1 | x$ is a nonempty freely reduced word on $\overline{X}_i\}$,

(ii) $\Leftrightarrow \{x\sigma \neq 1 | 1 \neq x \in \langle \overline{X}_i \rangle$ and $\sigma \in \Pi\}$, and

(iii) $\Leftrightarrow \{\sigma^x \pi \neq 1 | \sigma, \pi \in \Pi, x \in \langle \overline{X}_i \rangle$, and σ and x do not commute$\}$.

These sets will be recursive provided $\Pi\langle \overline{X}_i\rangle \subset \overline{G}_i$ has a solvable word problem with respect to its natural generators $\Pi \cup \overline{X}_i$. This fact is also needed to permit us to embed $\Pi\langle \overline{X}_i\rangle$ isomorphically in $\langle G_i, \bar{x}\rangle$ in order to meet the inductive hypothesis (f) of the next step. Now,

$$\Pi\langle \overline{X}_i\rangle = \Pi(\langle X_i\rangle * \langle a^\sigma, b^\sigma, c^\sigma | \sigma \in \Pi\rangle)$$

and Π normalizes both of these free factors. Hence, $\Pi\langle \overline{X}_i\rangle$ will have a solvable word problem provided $\Pi\langle a^\sigma, b^\sigma, c^\sigma | \sigma \in \Pi\rangle$ does, and to prove this we need only to be able to decide, given w^σ and w^τ with $\sigma, \tau \in \Pi$ and $w = a, b$, or c, whether or not $w^\sigma = w^\tau$ (i.e., we must be able to distinguish free generators). By our constructions (1), (2), and (3) above (as in Lemma (4.8)), we have $w^\sigma = w^\tau$ in $\overline{G}_i \Leftrightarrow \sigma\tau^{-1} \in P = \Pi_{s(i)}$ which has a solvable decision problem in Π by the hypothesis of Theorem (4.0) and the proof of this theorem is now complete.

ADDED IN PROOF. Graham Higman has pointed out an interesting example of a maximal subgroup. Let $F \subset G$ where G is a.c. and F is free abelian of rank n. Put $M = \{x \in G | F \cap F^x$ has rank $n\}$. Thus M is a maximal subgroup of G, $M \in \mathscr{M}(F, G)$, and M is obtained from the subgroup filter $\Pi = \{P | P \subset F$ and P has rank $n\}$ of F in the manner of our Theorem (1.30), but Π cannot be replaced by any subgroup ultrafilter of F if $n > 1$. (Only the case $n = 1$ is discussed in [**3**, Proposition 1].) Higman also observes that with $C = \bigcup\{\mathscr{C}_G(P) | P \in \Pi\}$ we have $M/C \cong \mathrm{GL}_n(Q)$.

Very recently on a visit to M.S.U. Higman encouraged the author to examine the permutational aspects of maximal subgroups of a.c. groups. A little reflection uncovered an interesting situation. The permutational analysis of naturally occurring maximals (that is, Type 1, Type 2a, and the natural Type 2b of (1.4)) is quite difficult, as Higman has noted, just to verify primitivity. However, in a countable a.c. group G we can construct a free maximal subgroup $F \subset G$ such that the representation of G on the right cosets of F has a rather strong homogeneity property defined as follows. If Σ is a class of f.g. subgroups of G and ρ: $G \to \mathrm{Sym}(X)$ is a permutation representation we say that ρ is *relatively Σ-homogeneous* if for all $A \subset G$ with $A \in \Sigma$ and every $\rho(A)$-isomorphism f among finitely many orbits of $\rho(A)$ there exists $g \in \mathscr{C}_G(A)$ such that $\rho(g)$ extends f. Taking $A = 1$ shows that ρ is highly transitive if it is relatively Σ-homogeneous. Otto Kegel [*Examples of highly transitive permutation groups*, Rend. Sem. Mat. Univ. Padova **63** (1980), 295–300] has shown that the countable universal l.f. group has a relatively homogeneous representation (with respect to all finite subgroups). For countable a.c. groups we can do almost as well.

THEOREM. *Every countable a.c. group $G \neq 1$ has a free subgroup F such that the regular representation $\rho(G)$ on the right cosets of F is relatively homogeneous with respect to the class Σ of all f.g. periodic subgroups of G.*

The construction of F with this property is similar to and not much more difficult than the proof of Theorem (4.1). It should be noted that Kegel's proof, which uses coset-induced permutation extensions, does not seem applicable to arbitrary countable a.c. groups because one cannot effectively construct the transversals in $\mathrm{Sk}(G)$

that one needs to extend the representations of its members. I have not yet determined whether the above theorem can be extended via Ziegler's construction as in §4 to produce $F \subset G$ of arbitrary powers with the same homogeneity property, but this seems very likely. The above theorem as well as a discussion of homogeneous permutation representations in a wider context will be presented in a paper that is now in preparation.

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