ON THE SPECTRA OF COMPACT NILMANIFOLDS

BY

JEFFREY S. FOX

ABSTRACT. We show the equivalence of the Howe-Richardson multiplicity formula for compact nilmanifolds and the formula obtained by Corwin and Greenleaf using the Selberg trace formula.

Introduction. Let $G$ be a connected simply connected nilpotent Lie group and suppose $G$ contains a discrete cocompact subgroup $\Gamma$. Let $\rho = \text{ind}_G^G(1)$. Then $\rho$ is a direct sum of irreducible representations each occurring with finite multiplicity; we will write $\rho = \bigoplus m(\pi)\pi$. A basic problem in representation theory is to determine $m(\pi)$ and give a criterion for $m(\pi)$ not to be zero. Moore first studied this problem in [M] and later Howe [H] and Richardson [R] independently gave a closed formula for $m(\pi)$ that generalized the classical Frobenius reciprocity formula for finite groups. Using the Poisson summation and Selberg trace formulas, Corwin and Greenleaf [C-G] gave a formula for $m(\pi)$ that depended only on the coadjoint orbit in $g^*$ corresponding to $\pi$ via Kirillov theory and the structure of $\Gamma$, but the connection between the two formulas was not clear. In §1 we consider the case when $\Gamma$ is a lattice subgroup of $G$, i.e. $\log(\Gamma)$ is an additive subgroup of the Lie algebra of $G$, and show there is a simple relationship between the two. In §2 we show how Frobenius reciprocity can be used to reduce the general case to the lattice subgroup case.

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1. Let $G$ be a connected simply connected nilpotent Lie group. We denote the Lie algebra of $G$ by $\mathfrak{g}$ and the dual of $g$ by $g^*$. Let $\exp: \mathfrak{g} \to G$ be the exponential map and $\log: G \to \mathfrak{g}$ its inverse. We let $\text{Ad}$ be the adjoint action of $G$ on $\mathfrak{g}$ and $\text{Ad}^*$ the coadjoint on $g^*$. If $\pi$ is an irreducible unitary representation we write $\mathfrak{o}(\pi) \subseteq \mathfrak{g}^*$ for the coadjoint orbit associated to $\pi$ via Kirillov theory. Let $\Gamma \subseteq G$ be a discrete cocompact subgroup of $G$. If $Q$ denotes the rational numbers, then $\Gamma$ determines a $Q$ structure on $\mathfrak{g}$ by $\mathfrak{g}_Q = \text{span}_Q(\log(\Gamma))$. We say $g \in G$ is rational iff $g = \exp(X)$ with $X \in \mathfrak{g}_Q$ and let $G_Q$ denote the set of rational points. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we say $\mathfrak{h}$ is rational if $\mathfrak{h} \cap \mathfrak{g}_Q$ contains a basis of $\mathfrak{h}$ over $\mathbb{R}$. This is equivalent to $\exp(\mathfrak{h}) = H$ having $H \cap \Gamma$ for a discrete cocompact subgroup. If $f \in \mathfrak{g}^*$, say, $f$ is
rational if \( f(\mathfrak{a}_Q) \subseteq Q \). For a very complete and detailed discussion of this see [C-G]. One should also note that if one views \( G \) as an affine algebraic group, then the existence of a cocompact \( \Gamma \) is equivalent to \( G \) being defined over \( Q \) and the notion of a rational point in the sense of algebraic groups is equivalent to the definition given above.

For the rest of this section we suppose that \( \Lambda = \text{log}(\Gamma) \) is an additive subgroup of \( \mathfrak{a} \). Let \( \Lambda^\perp = \{ f \in \mathfrak{a}^* | f(\Lambda) \subseteq \mathbb{Z} \} \). Let \( \pi \) be an irreducible representation of \( G \) and \( \mathcal{O} \subseteq \mathfrak{a}^* \) the coadjoint orbit corresponding to \( \pi \). According to [M], we have \( m(\pi) > 0 \) if and only if \( \mathcal{O} \cap \Lambda^\perp \neq \varnothing \), so we will suppose this intersection is nonempty. Since \( \mathcal{O} \cap \Lambda^\perp \) is \( \text{Ad}^*(\Gamma) \) invariant we can write it as a union of \( \text{Ad}(\Gamma) \) orbits. To each such orbit \( \Omega \subseteq \mathcal{O} \cap \Lambda^\perp \) one can associate a number \( C(\Omega) \) as follows:

Let \( f \in \mathcal{O} \) and \( \mathfrak{a}(f) = \{ X \in \mathfrak{a} | \text{ad}^*(X)f = 0 \} \). Then \( \mathfrak{a}(f) \) is a rational subalgebra, so the \( \mathbb{Z} \)-rank of \( \mathfrak{a}(f) \cap \Lambda \) is equal to \( \text{dim}(\mathfrak{a}(f)) \). Choose a \( \mathbb{Z} \)-basis \( X_1, \ldots, X_n \) of \( \Lambda \) (consequently it is an \( \mathbb{R} \) basis for \( \mathfrak{a} \)) such that \( X_1, \ldots, X_n \) span \( \mathfrak{a}(f) \) over \( \mathbb{R} \). Let \( A_f \) be the matrix with entries \( f([X_i, X_j]) \), \( s < i, j \leq n \). Then \( |\det(A_f)| \) is independent of the basis satisfying the above conditions and depends only on the \( \Gamma \) orbit of \( f \) in \( \mathcal{O} \cap \Lambda^\perp \). Set \( C(\Omega) = |\det(A_f)|^{-1/2} \). Then \( C(\Omega) \) is a positive rational number. The multiplicity formula of Corwin and Greenleaf can be written as

\[
m(\pi) = \sum_{\Omega \in \mathcal{O} \cap \Lambda^\perp / \text{Ad}^*(\Gamma)} C(\Omega).
\]

For details see [C-G].

We now describe the Howe-Richardson formula for \( m(\pi) \). If \( m(\pi) > 0 \) there exists a rational element \( f \in \mathcal{O}(\pi) \), fix such an element. Let \( \mathfrak{h} \subseteq \mathfrak{a} \) be a rational polarization for \( f \) and \( H = \exp(\mathfrak{h}) \) the connected subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). For \( L \in H \) let \( X_f(\mathfrak{h}) = \exp(2\pi i f(\text{log}(h))) \). Then \( X_f(\mathfrak{h}) \) is a character of \( H \) and \( \pi \) is equivalent to \( \text{ind}_{H}^{G}(X_f(\mathfrak{h})) \). Let \( X = \{ (\text{Ad}(g)\mathfrak{h}, \text{Ad}^*(g)f) | g \in G \} \) and define an equivalence relation on \( X \) by \( (\text{Ad}(g_1)\mathfrak{h}, \text{Ad}^*(g_1)f) \sim (\text{Ad}(g_2)\mathfrak{h}, \text{Ad}^*(g_2)f) \) iff \( \text{Ad}(g_1)\mathfrak{h} = \text{Ad}(g_2)\mathfrak{h} \) and \( \text{Ad}^*(g_1)f = \text{Ad}^*(g_2)f |_{\text{Ad}^*(g_2)\mathfrak{h}} \). Let \( \mathcal{C}(\mathfrak{h}, f) \) denote the set of equivalence classes. Define \( L(\mathfrak{h}, f) \subseteq \mathcal{C}(\mathfrak{h}, f) \) by \( (\text{Ad}(g)\mathfrak{h}, \text{Ad}^*(g)f) \in L(\mathfrak{h}, f) \) iff \( \text{Ad}(g)\mathfrak{h} \) is a rational subalgebra of \( \mathfrak{a} \) and \( X_{\text{Ad}^*(g)f} |_{gHg^{-1} \cap \Gamma} = 1 \). There is a natural action of \( \Gamma \) on \( L(\mathfrak{h}, f) \) by

\[
\gamma \cdot (\text{Ad}(g)\mathfrak{h}, \text{Ad}^*(g)f) = (\text{Ad}(\gamma g)\mathfrak{h}, \text{Ad}^*(\gamma g)f)
\]

and the number of \( \Gamma \) orbits in \( L(\mathfrak{h}, f) \) is the multiplicity \( m(\pi) \). In what follows we will assume that \( \mathfrak{h} \) and \( f \) have been chosen such that \( \mathfrak{h} \) is rational and \( X_f |_{H \cap \Gamma} = 1 \).

Suppose \( X_{\text{Ad}^*(g)f} |_{gHg^{-1} \cap \Gamma} = 1 \). Then \( \text{Ad}^*(g)f(\text{Ad}(g)\mathfrak{h} \cap \Lambda) \subseteq \mathbb{Z} \); thus we can find \( \phi \in (\text{Ad}(g)\mathfrak{h})^\perp \) such that \( \text{Ad}^*(g)f + \phi \in \Lambda^\perp \). From \( \text{Ad}^*(gHg^{-1})|_{\text{Ad}^*(g)f} = \text{Ad}^*(g)f + (\text{Ad}(g)\mathfrak{h})^\perp \) we have \( \text{Ad}^*(g)f + \phi = \text{Ad}^*(\gamma)(\text{Ad}^*(g)f) = \text{Ad}^*(\gamma g)f \) for some \( \gamma \in gHg^{-1} \). Since \( \text{Ad}(g)\mathfrak{h}, \text{Ad}^*(g)f \) is equivalent to \( (\text{Ad}(\gamma g)\mathfrak{h}, \text{Ad}^*(\gamma g)f) \) we have that every equivalence class in \( L(\mathfrak{h}, f) \) has a representative \( (\text{Ad}(g)\mathfrak{h}, \text{Ad}^*(g)f) \) such that \( \text{Ad}^*(g)f \in \Lambda^\perp \). We can now define a surjective map \( \alpha: \mathcal{O} \cap \Lambda^\perp \rightarrow L(\mathfrak{h}, f) \). If \( \phi \in \Lambda^\perp \), then both \( \phi \) and \( f \) are rational.
points in \( \mathcal{O} \); it follows [B or M] that there exists \( g \in G_0 \), the rational points of \( G \), such that \( \text{Ad}^*(g)f = \phi \). Define \( \alpha(\phi) = \alpha(\text{Ad}^*(g)f) = (\text{Ad}(g)h, \text{Ad}^*(g)f) \). Since \( g \) is rational, \( \text{Ad}(g)h \) is a rational subalgebra of \( \mathfrak{g} \) and \( \phi \in \Lambda^1 \) automatically says \( X_{\text{Ad}^*(g)f}|_{gH \cap \Gamma} = 1 \). Let \( G(f) = \{ g \in G| \text{Ad}^*(g)f = f \} \); since \( G(f) \subseteq H \) for any polarization \( H \) we see that \( \alpha \) is well defined. Since \( \alpha \) is a \( \Gamma \) equivariant map we can write the Corwin-Greenleaf formula as follows:

\[
m(\pi) = \sum_{\omega \in I.(h, f)/\text{Ad}^*(\Gamma)} \sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega).
\]

The equivalence of the Howe-Richardson formula and the Corwin-Greenleaf formula for \( \Gamma \) a lattice subgroup follows from above once we show

**Theorem 1.** With notation as above \( \sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega) = 1. \)

Before we begin with the proof of Theorem 1 we need the following lemmas from [H and C-G].

**Lemma 1.** Let \( \Gamma \subseteq G \) be a discrete cocompact subgroup of \( G \), suppose \( \mathfrak{z} \) is center of \( \mathfrak{g} \) is one-dimensional and let \( z \in \log(\Gamma) \cap \mathfrak{z} \) be a generator. Then there exists \( y \in \mathfrak{g} \) such that if \( W = \text{span} \ of \ y \) and \( z \) over \( \mathbb{R} \), then \( y \) and \( z \) generate \( \log(\Gamma) \cap W \). Let \( \mathfrak{g}_1 \) is the centralizer of \( y \) in \( \mathfrak{g} \). Then \( \mathfrak{g}_1 \) is rational and of codimension 1 in \( \mathfrak{g} \). \( \mathfrak{g}_1 \) is the Kirillov codimension 1 subalgebra [K]. There is \( x \in \log(\Gamma) \) such that if \( \Gamma_1 = \Gamma \cap \exp(\mathfrak{g}_1) \), then \( \Gamma_1 \) and \( \exp(x) \) generate \( \Gamma \). If \( L = \text{span} \ over \mathbb{R} \ of \ x, y, z \), then \( L \) is a three-dimensional Heisenberg algebra and \( \exp(x), \exp(y), \exp(z) \) generate \( \exp(L) \cap \Gamma \). Finally, there exists \( a \in \mathbb{Z}, a \neq 0 \), so that \( [x, y] = az \) and \( a \) is independent of the choice of \( x \) satisfying the above conditions.

**Definition 1.** Let \( \Gamma \) be a discrete torsion-free nilpotent group. A weak Malcev basis for \( \Gamma \) is a set \( \{ d_1, \ldots, d_p \} \subseteq \Gamma \) such that:

(i) For any \( d \in \Gamma \) there is a decomposition \( d = d_1^{n_1} \cdots d_p^{n_p} \), where \( n_i \in \mathbb{Z} \).

(ii) The set \( \Gamma_i = d_1^{n_1} \cdots d_i^{n_i} \cdots d_p^{n_p} \) is a subgroup with \( \Gamma_{i-1} \) normal in \( \Gamma_i \) for \( i = 1, 2, \ldots, p \).

(iii) \( \Gamma_i/\Gamma_{i-1} \cong \mathbb{Z} \) for \( i = 2, \ldots, p \).

**Definition 2.** Let \( G \) be a simply connected nilpotent Lie group. A weak Malcev basis for \( G \) is a set \( \{ X_1, \ldots, X_p \} \subseteq \mathfrak{g} \) such that:

(i) For \( X \in G \exists t_i \in \mathbb{R}, i = 1, \ldots, p \), such that \( X = \gamma_1(t_1) \cdots \gamma_p(t_p) \), where \( \gamma_i(t) = \exp(tX_i) \).

(ii) The set \( G_i = \gamma_1(\mathbb{R}) \cdots \gamma_i(\mathbb{R}) \) is a closed subgroup of \( G \) with \( G_{i-1} \) normal in \( G_i \) for each \( i \).

(iii) \( G_i/G_{i-1} \cong \mathbb{R} \).

Weak Malcev basis is an adaptation of Malcev's coordinates of the 2nd kind as necessitated by inducing from nonnormal subgroups in the Kirillov model [Ma].

If \( \Gamma \subseteq G \) is a discrete cocompact subgroup of \( G \) we say a weak Malcev basis \( \{ X_1, \ldots, X_p \} \) of \( \mathfrak{g} \) is subordinate to \( \Gamma \) if \( \{ \exp(X_1), \ldots, \exp(X_p) \} \) is a weak Malcev basis for \( \Gamma \).
Lemma 2. Let $\Gamma$ be a discrete cocompact subgroup of $G$; if \( \{ d_1, \ldots, d_n \} \) is a weak Malcev basis for $\Gamma$, then \( \{ X_i = \log(d_i) | i = 1, \ldots, n \} \) is a weak Malcev basis of $G$ subordinate to $\Gamma$.

Lemma 3. Let $G$ and $\Gamma$ be as above, $M \subseteq G$ a closed connected subgroup of $G$ such that $M/M \cap \Gamma$ is compact. If \( \{ X_1, \ldots, X_s \} \) is a weak Malcev basis of $M$ subordinate to $M \cap \Gamma$, then it can be extended to a weak Malcev basis of $G$ subordinate to $\Gamma_0$.

Lemma 4. If $\Gamma$ is a lattice subgroup of $G$ and \( \{ X_1, \ldots, X_n \} \) is a weak Malcev basis of $G$ subordinate to $\Gamma$, then \( \{ X_1, \ldots, X_n \} \) forms a $\mathbb{Z}$ basis of $A = \log(\Gamma)$.

Proof of Theorem 1. We proceed by induction on $\dim(G)$. If $\dim(G) = 1$ the statement is trivial and therefore suppose $\dim(G) > 1$. Let $\hat{\gamma}$ be the center of $\hat{g}$; if $\dim(\ker(f) \cap \hat{\gamma}) > 1$ we can divide out the corresponding central subgroup and proceed to a lower dimensional case. Consequently we can assume $\dim(\hat{\gamma}) = 1$, and if $x, y, z$ and $\hat{g}_1$ are as in Lemma 1, then $f(z) = \lambda \neq 0$. Note that for $f_1, f_2 \in \mathcal{O}(\pi)$, $f_1(z) = \text{Ad}^*(g)f_2(z) = f_2(z) = \lambda$.

We can assume $\omega = \Gamma \cdot (\mathfrak{h}, f)$ simply by relabeling. To find $\Gamma$ orbits in $\mathcal{O} \cap \Lambda^\perp$ such that $\alpha(\Omega) = \omega$ we proceed as follows: The point $(H, f) \in \mathcal{O}$ has $\Gamma$ and $H$ for its stability group; consequently $\Gamma \cap H$ preserves the fiber $\alpha^{-1}((H, f)) = (f + \mathbb{h}^\perp) \cap \Lambda^\perp$. A set of representatives for $\Gamma$ orbits in $\mathcal{O} \cap \Lambda^\perp$ that map into $\omega$ via $\alpha$ is given by a set of representatives of $\Gamma \cap H$ orbits in $(f + \mathbb{h}^\perp) \cap \Lambda^\perp$.

As usual, there are two cases we must consider.

Case I. Suppose $\mathfrak{h} \subseteq \mathfrak{g}_1$. Let $\tilde{f}$ be the restriction of $f$ to $\mathfrak{g}_1$. Then $\mathfrak{h}$ is a rational polarization for $\tilde{f}$ and $\pi = \text{ind}_{\mathfrak{g}_1}^{\mathfrak{g}}(X_j)$ occurs in $L^2(G_\mathfrak{g} / \Gamma_\mathfrak{g})$ by the criterion in [C-G or M], i.e., $\tilde{f} \in \Lambda_1^\perp \subseteq \mathfrak{g}^*$. Let $\Lambda_1^\perp = \{ \phi \in \mathfrak{g}^* | \phi(x) = 0 \forall X \in \mathfrak{h} \}$ and let $r: \mathfrak{g}^* \to \mathfrak{g}_1^*$ be the restriction map—so $r$ is $\text{Ad}^*$ equivariant. If $L = (f + \mathbb{h}^\perp) \cap \Lambda^\perp$ and $L_1 = (\tilde{f} + \mathbb{h}^\perp) \cap \Lambda_1^\perp$, then $\Gamma \cap H = \Gamma_1 \cap H$ is an equivariant surjective map. If $\mathcal{S}_1 \subseteq L_1$ is a set of $\Gamma \cap H$ orbit representatives, then $(r|_{L_1})^{-1}(\mathcal{S}_1)$ contains a set of $\Gamma \cap H$ orbit representatives in $L$, say $\mathcal{S}_1$. Let $\bar{\phi} \in \mathcal{S}_1, \phi \in (r|_{L_1})^{-1}(\bar{\phi})$, $G_1(\phi)$ be the stability group of $\bar{\phi}$ in $G$, and $G(\phi)$ be the stability group of $\phi$ in $G$. Then $G_1(\phi) = G(\phi) \cdot \{ \exp(tY) | t \in \mathbb{R} \}$ since the $\Gamma \cap \{ \exp(tY) | t \in \mathbb{R} \}$ orbit of $(r|_{L_1})^{-1}(\bar{\phi})$ is the same as a $\Gamma \cap \{ \exp(tY) | t \in \mathbb{R} \}$ orbit in $(r|_{L_1})^{-1}(\phi)$. An element $\phi \in (r|_{L_1})^{-1}(\bar{\phi})$ is determined by its value on $x \in \mathfrak{g}$. Suppose $\phi(x) = b$. Then $\text{Ad}^*(\exp(nY))\phi(x) = b + na\lambda$ and we see there are $|a\lambda| \{ \exp(nY) | n \in \mathbb{Z} \}$ orbits in $(r|_{L_1})^{-1}(\bar{\phi})$.

Let $\phi \in (r|_{L_1})^{-1}(\bar{\phi})$ for some $\bar{\phi} \in \mathcal{S}_1$. We want to compute $C(\Gamma \cdot \phi)$. To do this we need a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that the $\mathbb{Z}$ span is $\Lambda$ and $X_1, \ldots, X_s$ span $\mathfrak{g}_1(\phi)$ over $\mathbb{R}$. Using Lemmas 3 and 4 we can find a basis $X_1 = z$, $X_2 = y$, $\ldots$, $X_{n-1}$ of $\mathfrak{g}_1$ such that $X_1, \ldots, X_{n-1}$ span $\Lambda$ over $\mathbb{Z}$, $X_1, X_2, \ldots, X_s$ span $\mathfrak{g}_1(\phi)$, and $X_1, X_3, X_4, \ldots, X_s$ span $\mathfrak{g}_1(\phi) \subseteq \mathfrak{g}_1(\phi)$. We have $X_1, X_3, \ldots, X_{n-1}, y, x$ span $\Lambda$ over $\mathbb{Z}$ and $X_1, X_3, \ldots, X_s$ span $\mathfrak{g}_1(\phi)$. Using this basis we can compute $C(\Gamma \cdot \phi) = C(\mathcal{S}_1)$. Recall that $C(\phi) = |\det(A(\phi))|^{1/2}$, where $A(\phi) = (\phi([x, y]))$, $s \leq i, j \leq n$. Since $[Y, \mathfrak{g}_1] = 0$, if we expand $A(\phi)$ on that row and column, using $\phi([x, y]) = \lambda a$, we get $\det(A(\phi)) = |\lambda a|^2 \det(A(\phi))$. Consequently, we have $C(\phi) = |\lambda a|^{-1} \cdot C(\phi)$ for all $\phi \in \mathcal{S}_1$.  

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By our induction hypothesis \( \sum_{\phi \in S} C(\phi) = 1 \), so we get

\[
\sum_{\phi \in S} C(\phi) = \sum_{\phi \in S} \left( \sum_{\phi \in \langle \alpha \rangle^{-1}(\phi) \cap S} C(\phi) \right) = \sum_{\phi \in S} C(\phi) \left( \sum_{\phi \in \langle \alpha \rangle^{-1}(\phi) \cap S} |\lambda a|^{-1} \right) = \sum_{\phi \in S} C(\phi) = 1. \]

Thus Case I is verified.

**Case II.** Now suppose \( f \notin \mathfrak{g} \) and set \( \mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_1 \), \( \mathfrak{h} = \text{span}(\mathfrak{h}_0, Y) \). Then \( \tilde{\mathfrak{h}} \subseteq \mathfrak{g}_1 \) and is a polarization for \( f \). As before, let \( r: \mathfrak{g}^* \rightarrow \tilde{\mathfrak{g}}_1^* \) be the restriction map. Then \( r|_{f+\mathfrak{h}^+}: f + \mathfrak{h}^+ \rightarrow \tilde{\mathfrak{g}}_1^* \) is injective and \( r(f + \mathfrak{h}^+) = \bigcup_{n \in \mathbb{Z}} (f + ny^* + \mathfrak{h}^+) \cap \Lambda^+ \). Consequently we have \( r((f + \mathfrak{h}^+) \cap \Lambda^+) = \bigcup_{n \in \mathbb{Z}} (f + ny^* + \mathfrak{h}^+) \cap \Lambda^+ \). A set \( S \) of \( \Gamma \cap H \) orbit representatives in \( (f + \mathfrak{h}^+) \cap \Lambda^+ \) can be written as \( U_{0 \leq b < |\lambda a|} S_b \), where \( S_b \) is a set of \( \Gamma \cap H \) orbit representatives in \( (f + by^* + \mathfrak{h}^+) \cap \Lambda^+ \) and \( b \in \mathbb{Z} \). As before, if \( \phi \in r^{-1}(S_b) \cap \Lambda^+ \), then \( C(\phi) = |\lambda a|^{-1} C(\phi) \), where \( \phi = r(\phi) \). Thus we have

\[
\sum_{\phi \in S} C(\phi) = \sum_{0 \leq b < |\lambda a|} \left( \sum_{\phi \in r^{-1}(S_b) \cap \Lambda^+} C(\phi) \right) = \sum_{0 \leq b < |\lambda a|} |\lambda a|^{-1} \left( \sum_{\phi \in S_b} C(\phi) \right) = \sum_{0 \leq b < |\lambda a|} |\lambda a|^{-1} = 1. \qquad \text{Q.E.D.}
\]

**Corollary 1 (Moore [M]).** \( m(\pi) \leq \# \{ \mathcal{O} \cap \Lambda^+/\text{Ad}^*(\Gamma) \} \).

**Proof.** By the above proof, \( C(\phi)^{-1} = |\lambda a| C(\phi) \); thus we can reason by induction to conclude that \( C(\phi)^{-1} \) is an integer. \qquad \text{Q.E.D.}

In [R] Richardson constructed a polarization for \( f \) such that in the inductive reasoning one never has Case II occurring. We note that if \( \mathfrak{h} \) is a Richardson polarization, then the above proof shows that the \( C(\mathcal{O}) \)s are the same for every \( \mathcal{O} \) such that \( \alpha(\mathcal{O}) = \omega \). Since \( \sum_{\alpha(\mathcal{O}) = \omega} C(\mathcal{O}) = 1 \), \( C(\mathcal{O}) \) equals the number of \( \Gamma \cap H \) orbits in \( f + \mathfrak{h}^+ \cap \Lambda^+ \). This observation was pointed out to me by Larry Corwin.

2. If \( \pi \) is a unitary representation of \( G \) on the Hilbert space \( H(\pi) \), we let \( H^\infty(\pi) = \{ u \in H(\pi) | \pi(q) u \text{ is a } C^\infty \text{-mapping} \} \). There is a representation of \( \mathfrak{g} \), the Lie algebra of \( G \), on \( H^\infty(\pi) \). If \( X \in \mathfrak{g} \), \( u \in H^\infty(\pi) \) define

\[
\pi(X) u = \frac{d}{dt} (\pi(\exp(tX)u))|_{t=0}. \]

Then \( X \rightarrow \pi(X) \) is a Lie algebra representation of \( \mathfrak{g} \), so it extends to a representation of \( \mathfrak{u}(\mathfrak{g}) \). Given \( a \in \mathfrak{u}(\mathfrak{g}) \) define a seminorm on \( H^\infty(\pi) \) by \( \rho_a(u) = ||\pi(a)u|| \). Then \( H^\infty(\pi) \) has the structure of a Fréchet space with respect to these seminorms and we let \( H^{-\infty}(\pi) \) be the topological dual of \( H^\infty(\pi) \). For details see [P-1]. We write \( \pi^\infty \) for the restriction of \( \pi \) to \( H^\infty(\pi) \) and \( \pi^{-\infty} \) for the dual representation of \( G \) on \( H^{-\infty}(\pi) \). The following is in [P-1].
Theorem 2.1 (Penney). Let $G$ be a Lie group, $\Gamma$ a discrete cocompact subgroup and $\pi$ an irreducible representation of $G$. Then

$$\text{Hom}_G(\pi, \text{ind}_\Gamma^L(1)) = \text{Hom}_\Gamma(1, \pi^{-\infty}).$$

If we set $\Gamma(\pi) = \{ D \in H^{-\infty}(\pi) | \pi^{-\infty}(\lambda)D = D \text{ \forall } \lambda \in \Gamma \}$, then the above theorem says $\dim(\text{Hom}_G(\pi, \text{ind}_\Gamma^L(1))) = \dim(\Gamma(\pi))$.

When $G$ is a connected simply connected nilpotent Lie group, the lift maps of Richardson can be viewed as providing a basis of $\Gamma(\pi)$ [R]. If $\pi = \text{ind}_H^G(\chi_f)$, then $H^{-\infty}(\pi)$ corresponds to all Schwartz functions on $G/H$ [K] and for each $\Gamma$ orbit in $L(h, f)$ one can construct an element of $\Gamma(\pi)$. If $(\text{Ad}(g)h, \text{Ad}^*(g)f)$ is a point of $L(h, f)$, then we can construct $D_g \in H^{-\infty}(\pi)$ as follows: For $\phi \in H^{-\infty}(\pi)$ let $D_g(\phi) = \sum_{\gamma \in \Gamma/\Gamma \cap gHg^{-1}} \phi(\gamma g)$. The $D_g$'s are linearly independent for $g$'s in different $\Gamma : H$ double cosets and they span $\Gamma(\pi)$ [P-2, F].

Now suppose $\Gamma_0 \subseteq \Gamma$ is a normal subgroup of finite index and we know the truth of the Howe-Richardson formula for $\Gamma_0$.

Let $L_0(h, f)$ be defined using $\Gamma_0$ and $L(h, f)$ be defined using $\Gamma$. Of course $L(h, f) \subseteq L_0(h, f)$, so we will suppose $L_0(h, f)$ is not empty. Let $D_g \in \Gamma_0(\pi) \subseteq H^{-\infty}(\pi)$. For $\gamma \in \Gamma$, $\phi \in H^{-\infty}(\pi)$ we have

$$\pi^{-\infty}(\gamma)D_g(\phi) = \sum_{\delta \in \Gamma_0 \cap \gamma Hg^{-1}} \phi(\gamma \delta g) = \sum_{\delta \in \Gamma_0 \cap \gamma Hg^{-1}} \phi(\delta g).$$

Thus $\pi^{-\infty}(\gamma)$ stabilizes $C \cdot D_g$ iff $\gamma \in \Gamma_0 \cdot (\Gamma \cap gHg^{-1})$, then

$$\pi^{-\infty}(\gamma)D_g = \chi_f(g^{-1}\gamma^{-1}g^{-1})D_g = \widetilde{\chi}_g.f(\gamma) \cdot D_g$$

(where $\widetilde{\chi}_g.f$ extends to a character of $\Gamma_0 \cdot (\Gamma \cap gHg^{-1})$ by being trivial on $\Gamma_0$). If we set $W_g = \text{span}\{ \pi^{-\infty}(\gamma)D_g | \gamma \in \Gamma \}$, then we see that $\pi^{-\infty}|_{\Gamma}$ acting on $W_g$ is exactly $\text{ind}_{\Gamma_0}(\Gamma \cap gHg^{-1})^L(\chi_q.f)$. Let $S$ be a set of representatives for $\Gamma$ orbits in $L_0(h, f)$, so given $g \in S$ we get $(\text{Ad}(g)h, \text{Ad}^*(g)f)$ or equivalently a $D_g \in H^{-\infty}(\pi)$. From above we see that the representation $\pi^{-\infty}|_{\Gamma}$ acting on $\Gamma_0(\pi)$ is

$$\bigoplus_{g \in S} \text{ind}_{\Gamma_0}(\Gamma \cap gHg^{-1})^L(\widetilde{\chi}_g.f).$$

Since $\Gamma(\pi) \subseteq \Gamma_0(\pi)$, we have that

$$\text{nnHom}_\Gamma(1, \pi^{-\infty}) \simeq \bigoplus_{g \in S} \text{Hom}_\Gamma(1, \text{ind}_{\Gamma_0}(\Gamma \cap gHg^{-1})^L(\widetilde{\chi}_g.f)) \simeq \bigoplus_{g \in S} \text{Hom}_{\Gamma_0}(1, \widetilde{\chi}_g.f).$$

Thus

$$\dim\left(\text{Hom}_\Gamma(1, \text{ind}_{\Gamma_0}(\Gamma \cap gHg^{-1})^L(\widetilde{\chi}_g.f))\right) = \begin{cases} 1 & \text{if } \chi_{g.f}|_{gHg^{-1} \cap \Gamma} \equiv 1, \\ 0 & \text{otherwise}. \end{cases}$$

Finally those $g \in S$ such that $\chi_{g.f}|_{gHg^{-1} \cap \Gamma} \equiv 1$ are parametrized by $L(h, f)$. Thus we have

**Proposition.** If $\Gamma_0 \subseteq \Gamma$ is a normal subgroup and the Howe-Richardson formula for $\Gamma_0$ is known, then the Howe-Richardson formula for $\Gamma$ is true.
Remark. It is shown in [M or C-G] that for a given $\Gamma$ it is always possible to find a normal $\Gamma_0$ such that $\Gamma_0$ is a lattice subgroup.

References


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907