P POINTS WITH COUNTABLY MANY CONSTITUTIONS

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ABSTRACT. If the continuum hypothesis (CH) is true, then for any P point ultrafilter D (on the set of natural numbers) there exist initial segments of the Rudin-Keisler ordering, restricted to (isomorphism classes of) P points which lie above D, of order type $\aleph_1$. In particular, if D is an RK-minimal ultrafilter, then we have (CH) that there exist P-points with countably many constellations.

0. Introduction. Our main result is that in the presence of the continuum hypothesis (henceforth denoted CH), there exist P point ultrafilters on $\omega$ with exactly $\aleph_0$ many constellations. Actually, we prove a somewhat stronger theorem about initial segments of the Rudin-Keisler (RK) ordering on the class of P points; in order to state this result, we begin with a few definitions. All ultrafilters here are nonprincipal ultrafilters on $\omega = \{0, 1, 2, \ldots\}$. An ultrafilter D is a P point iff any function $f: \omega \to \omega$ is either constant or finite-to-one on a set in D. P points have been studied extensively, and we shall assume basic results about them and their RK ordering; good references are [B1 and Pu]. If D is a P point, let $<_p, D$ denote the RK ordering on (equivalence classes of) P points which lie above D in RK. An initial segment of $<_p, D$ means a downward closed subset, and the initial segment determined by $E$ is $\{F: D < F < E\}$ (we use $<$ to denote the RK ordering).

In his thesis [Ec], Eck showed (CH) that if D is any P point, then there exist P points $E$ immediately above $D$ in RK in the strong sense that any strict RK predecessor of $E$ is a predecessor of $D$; we call such an $E$ a strong immediate successor (s.i.s) of $D$. Iterating Eck’s theorem $\omega$ times yields the existence (CH), for any P point $D$, of initial segments of $<_p, D$ of order type $\omega$. Our main theorem is the existence (CH) of initial segments of $<_p, D$ of order type $\aleph_1$; the bulk of the article is devoted to its proof.

In [B3], Blass proved the result just stated without the restriction to P points; that is, he showed (CH) that for any ultrafilter D, there exist initial segments of “RK above $D$” of order type $\aleph_1$. The proof involved reformulating the problem in model theoretic terms, and we shall take the same approach. Let $\mathbb{N}$ be the complete first order structure on $\omega$ (i.e. the language for $\mathbb{N}$ contains names for every finitary function and relation on $\omega$). We use the term model to mean “nonstandard model of $Th(\mathbb{N})$”, and we use $^*f$ to indicate the interpretation of the function $f: \omega \to \omega$ in whichever model is under consideration. If $D$ is an ultrafilter, then $D$-prod denotes

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the ultrapower of \( N \) by \( D \), and if \( f \) is a function from \( \omega \) to \( \omega \), then the corresponding element of the universe of \( D \)-prod (called the germ of \( f \)) is denoted \([f]_D\). In general, a model is isomorphic to an ultrapower iff it is finitely generated, which, due to the existence of pairing functions, is the same as saying that the model is generated by a single element in its universe.

There is an intimate relationship between the structure of an ultrapower \( D \)-prod and the RK ordering below \( D \); roughly, finitely generated submodels of \( D \)-prod correspond to RK predecessors of \( D \); the submodel generated by \([f]_D\) corresponds to the RK predecessor \( f(D) \) of \( D \) (the submodel is isomorphic to \( f(D) \)-prod by the map \( f^*: f(D) \)-prod \( \rightarrow \) \( D \)-prod defined by \( f^*(|[g]_D) = [g \circ f]_D \)). The details for this construction are well known and can be found in [B2]. An ultrafilter \( D \) is a \( P \) point iff every (nonstandard) submodel of \( D \)-prod is cofinal in \( D \)-prod, and in general, we refer to models in which all nonstandard submodels are cofinal as “single-skied” (see [Pu] for a discussion of skies). Then a model \( \mathcal{A} \) of \( \text{Th}(N) \) which is single-skied and finitely generated is isomorphic to \( D \)-prod for some \( P \) point \( D \). Note that if \( f(E) = D \), then \( E \) is a s.i.s. of \( D \) iff the submodel \( \mathcal{E} \) of \( D \)-prod generated by \([f]_D \) (\( \mathcal{E} = f^*E \)-prod) is strictly maximal in \( D \)-prod (that is, every proper submodel of \( D \)-prod is a submodel of \( \mathcal{E} \)). Also, if \( \mathcal{D} \) is a strictly minimal extension of the model \( \mathcal{E} \) (that is, \( \mathcal{E} \) is strictly maximal in \( \mathcal{D} \)), then \( \mathcal{D} \) must be isomorphic to an ultrapower, since any element of \( \mathcal{D} - \mathcal{E} \) must generate \( \mathcal{D} \). The notation \( f''X \) means the image of the set \( X \) under the function \( f \).

1. Extensions of countably generated models. The main result mentioned above will involve the construction of a sequence of \( P \) points \( \{D_\alpha: \alpha < \omega_1\} \) for any \( P \) point \( D \), with \( D_0 = D \), which form the desired initial segment of \( <_{P,D} \). At successor stages, we construct a \( P \) point \( D_{\alpha+1} \) which is a s.i.s. of \( D_\alpha \) with a modified version of Eck’s technique; the modifications are included to make the limit stages go through. Model theoretically, the ultrapower \( D_{\alpha+1} \)-prod is a strictly minimal, cofinal extension of (the embedded image of) \( D_\alpha \)-prod. The strategy at limit stages requires a few more definitions. Suppose \( \lambda \) is a limit ordinal and \( \{D_\alpha: \alpha < \lambda\} \) is an RK-increasing sequence of ultrapowers. Call an ultrafilter \( E \) a strongly minimal upper bound \( (s.m.u.b.) \) for the given sequence if \( D_\alpha < E \) for all \( \alpha < \lambda \) and any strict RK predecessor of \( E \) is a predecessor of \( D_\alpha \) for some \( \alpha < \lambda \). Our construction will insure that \( D_\lambda \) is a \( P \) point and a s.m.u.b. for \( \{D_\alpha: \alpha < \lambda\} \), and it is easy to see then that we will obtain our desired sequence. The actual construction of \( D_\lambda \) for countable limit ordinals \( \lambda \) involves an excursion into model theory, which we now describe. Suppose we have \( P \) points \( D_\alpha \) for \( \alpha < \lambda \) satisfying the description above. Let \( \alpha_1, \alpha_2, \ldots \) be a cofinal \( \omega \)-sequence in \( \lambda \), let \( E_i = D_{\alpha_i} \) and let \( p_i: \omega \rightarrow \omega \) such that \( p_i(E_{i+1}) = E_i \). Then \( p_i^* \) embeds \( E_i \)-prod into \( E_{i+1} \)-prod, and so we can form the direct limit \( \mathcal{A} \) of the system \( \{E_i \)-prod, \( p_i^*: i = 1, 2, \ldots \} \); let \( \mathcal{E} \) be the canonical image of \( E_i \)-prod in \( \mathcal{A} \), so that \( \mathcal{A} \) is the union of the \( \mathcal{E}_i \). Now \( \mathcal{A} \) is a model of \( \text{Th}(N) \) and \( \mathcal{A} \) is single-skied since each of the \( E_i \) are \( P \) points and hence the models \( \mathcal{E}_i \) are mutually cofinal. Note that \( \mathcal{A} \) is not finitely generated and hence not isomorphic to an ultrapower.

Suppose that \( \mathcal{A} \) admits a strictly minimal, cofinal extension \( \mathcal{B} \). Then \( \mathcal{B} \) must be (isomorphic to) an ultrapower, and \( \mathcal{B} \) must be single-skied since any proper
submodel of $\mathcal{B}$ is a submodel of, and hence cofinal in, $\mathcal{I}$ and $\mathcal{I}$ is cofinal in $\mathcal{B}$. Thus $\mathcal{B}$ is isomorphic to $F$-prod for some $P$ point $F$, and we set $D_\lambda = F$. It is easy to check that $D_\lambda$ is a s.m.u.b. for $\{ D_\alpha : \alpha < \lambda \}$.

The discussion above shows that we can succeed at limit stages of our construction if we can find a strictly minimal, cofinal extension $\mathcal{A}$ of the countably generated model $\mathcal{I}$ which arises as a direct limit of previously constructed models. In [B3], Blass proved a characterization of those countably generated models of $\text{Th}(N)$ which admit strictly minimal extensions. We shall require a number of modifications to that theorem, and what follows, through the proof of Theorem 2, is adapted from [B3].

Let $G'$ be the binary relation on $\omega$ defined by $m \mathrel{G'} n$ iff $2^m$ occurs in the binary expansion of $n$, so $n$ codes the finite set $\{ m : m \in \omega \}$. If $\mathcal{A}$ is a model and $a \in \mathcal{A}$, then let $a(\mathcal{A}) = \{ b \in \mathcal{A} : \mathcal{A} \models (b \mathrel{G'} a) \}$. If $a \in \mathcal{A} < \mathcal{B}$, then $a(\mathcal{A}) = a(\mathcal{B}) \cap \mathcal{A}$. Assume that the set of finite sequences from $\omega$ has been coded in some standard way, and let $\langle \ldots \rangle$ denote the coding function. Let $\text{Seq}$ be the set of codes, and for each $x$ in $\text{Seq}$, $lh(x)$ is the length of $x$ and $(x)_k$ is the $k$th component of $x$ if $k < lh(x)$ and $(x)_k = 0$ otherwise. Blass's result is

**Theorem 1 (Blass [B3]) (CH).** Let $\mathcal{A}$ be a countably generated model of $\text{Th}(N)$. $\mathcal{A}$ admits a strictly minimal extension if and only if for any sequence $\{ a_i : i \in \omega \}$ with $a_i(\mathcal{A})$ nonempty and $a_0(\mathcal{A}) \supseteq a_1(\mathcal{A}) \supseteq a_2(\mathcal{A}) \supseteq \cdots$, either

(i) $\bigcap_{i \in \omega} a_i(\mathcal{A}) \neq \emptyset$, or 
(ii) for any $b$ in $\mathcal{A}$, there is a $c$ in $a_0(\mathcal{A})$ and an $f : \omega \to \omega$ such that $f(c) = b$.

**Remarks.** The “only if” direction does not use CH. If $\mathcal{A}$ is finitely generated, and hence isomorphic to an ultrapower, then (i) always holds since ultrapowers are $\aleph_1$-saturated [CK, p. 305], and so (CH) ultrapowers always admit strictly minimal extensions.

Our first modification takes care of insuring that the new model is a cofinal extension.

**Theorem 2 (CH).** Let $\mathcal{A}$ be a single-skied, countably generated model of $\text{Th}(N)$, and suppose $\mathcal{A}$ satisfies the conditions of Theorem 1. Then $\mathcal{A}$ admits a single-skied, strictly minimal extension.

**Proof.** Most of this proof is identical to the proof of Theorem 1 given in [B3]. First, if $\mathcal{A}$ is finitely generated, and hence isomorphic to a $P$ point ultrapower, then this theorem is simply Eck's result that any $P$ point has strong immediate successors which are $P$ points. Assume therefore that $\mathcal{A}$ is not finitely generated, and let $\{ a_n : n = 1, 2, \ldots \}$ be a set which generates $\mathcal{A}$; without loss of generality, the sequence $\langle a_n : n = 1, 2, \ldots \rangle$ is not redundant, that is, $a_{n+1}$ is not in the submodel of $\mathcal{A}$ generated by $\langle a_1, a_2, \ldots, a_n \rangle$, and let $g_n = \langle a_1, \ldots, a_n \rangle$. Let $\text{TR}_n : \text{Seq} \to \text{Seq}$ be the map which truncates sequences by removing all but the first $n$ components (and leaves shorter sequences fixed). Then $\text{TR}_n(g_m) = g_n$ for all $m > n$; note also that
Let \( g_m \) be not in the submodel of \( \mathcal{A} \) generated by \( g_n \) if \( n < m \) (by the nonredundancy of the \( a_i \)'s). Let \( \mathcal{G}_n \) be the submodel generated by \( g_n \).

Let \( G_n \) be the type of \( g_n \) in \( \mathcal{A} \), that is \( G_n = \{ X \subseteq \text{Seq}: \mathcal{A} \models \star X(g_n) \} \). Then \( G_n \) is an ultrafilter on \( \text{Seq} \); in fact a \( P \) point (since \( \mathcal{A} \), and hence each of its submodels, is single-skied), and \( \text{TR}_n(G_m) = G_n \) for \( m > n \). \( G_n \) concentrates on sequences of length \( n \), that is \( \{ x \subseteq \text{Seq}: \text{lh}(x) = n \} \in G_n \), and the nonredundancy implies that \( \text{TR}_n \) is not one-to-one on any set in \( G_m \) for \( m > n \).

To obtain a strictly minimal, single-skied extension of \( \mathcal{A} \), it suffices to construct an ultrafilter \( E \) on \( \text{Seq} \) such that

1. for all \( n \geq 1 \), \( \text{TR}_n(E) = G_n \),
2. for all \( f: \text{Seq} \rightarrow \omega \), there is a set \( A \) in \( E \) such that either \( f \) is one-to-one on \( A \) or, for some \( n, f \) is \( \text{TR}_n \)-fiberwise constant on \( A \) (that is, \( f \) is constant on sets of the form \( A \cap \text{TR}_1^{-1}(i) \)), and
3. for some set \( B \) in \( E \), \( \text{TR}_1(B) \) is finite-to-one on \( B \).

Given such an ultrafilter \( E \), we can embed \( \mathcal{A} \) into \( E \)-prod by mapping \( g_n \) to \([\text{TR}_nE] \) (by (1)), and for simplicity we identify \( A \) with its embedded image in \( E \)-prod. By (2), every element of \( E \)-prod either generates \( E \)-prod or is in the submodel \( \mathcal{G}_n \) for some \( n \), and so \( E \)-prod is a strictly minimal extension of \( \mathcal{A} \). By (3), the submodel \( \mathcal{G}_1 \), and hence \( \mathcal{A} \), is cofinal in \( E \)-prod. It follows that \( E \)-prod is single-skied since any proper submodel of \( E \)-prod is a submodel of (and hence cofinal in) \( \mathcal{A} \), and \( \mathcal{A} \) is cofinal in \( E \)-prod.

The existence proof for \( E \) is a typical sort of inductive construction for ultrafilters on \( \omega \). Call a subset \( L \) of \( \text{Seq} \) large if \( \text{TR}_n L \subseteq G_n \) for all \( n \); otherwise \( L \) is small. Any ultrafilter consisting entirely of large sets satisfies (1). Thus it suffices to construct a filter \( F \) consisting of large sets, such that \( F \) contains a set \( B \) satisfying (3) and for each \( f: \text{Seq} \rightarrow \omega \), \( F \) contains a set \( A \) satisfying (2). Then let \( E \) be any ultrafilter extending \( F \) and containing the complements of all small sets. To construct \( F \), first order \( \text{Seq} \omega \) in an \( \aleph_1 \)-sequence (by CH) and then inductively define large sets \( L_\alpha \) for \( \alpha < \aleph_1 \) such that \( L_0 \) satisfies (3), \( L_{\alpha + 1} \) works as \( A \) in (2) for the \( \alpha \)th function \( f \) and \( L_\alpha - L_\beta \) is small for \( \alpha > \beta \). It is easy to check that the finite union of small sets is small, and it follows that \( \{ L_\alpha: \alpha < \aleph_1 \} \) generates a filter; let \( F \) be this filter.

To construct \( L_0 \), first find a set \( B_n \) in \( G_n \) for \( n \geq 1 \) such that \( B_n \) consists of sequences of length (exactly) \( n \), \( \text{TR}_1 B_n \) is finite-to-one on \( B_n \) and \( \text{TR}_1'' B_n \subseteq \{ n, n + 1, n + 2, \ldots \} \). Such \( B_n \) exist since the \( G_n \)'s are \( P \) points (and \( G_1 \) is nonprincipal). Set \( L_0 = \bigcup_{n \geq 1} B_n \), and then \( \text{TR}_1'' L_0 \) includes \( B_n \) for all \( n \), so \( L_0 \) is large. For any \( k \geq 1 \), \( \text{TR}_1'(k) \) is the union of \( k \) finite sets, so \( \text{TR}_1 \) is finite-to-one on \( L_0 \).

The construction of \( L_\lambda \) for limit \( \lambda \) uses only that finite unions of small sets are small. The successor stages use the hypotheses of the theorem (that is, the conditions on \( \mathcal{A} \) given in Theorem 1) to construct a large subset \( A \) of \( L \) which satisfies (2) for any large set \( L \) and any \( f: \text{Seq} \rightarrow \omega \). The details for both the limit and successor cases can be found in [B3, pp. 154–155].

By Theorem 2, we will succeed at limit stages in the construction of our sequence \( \{ D_\alpha: \alpha < \aleph_1 \} \) if the associated (countably generated) model satisfies the conditions of Theorem 1. The model \( \mathcal{A} \) arising in our construction has a special structure in that
those submodels of \( \mathcal{A} \) which include (the embedded copy of) \( D_0 \)-prod are linearly (in fact, well) ordered by inclusion. This makes it somewhat easier to satisfy the conditions of Theorem 1, as the next two lemmas show.

**Lemma 3 (CH).** Suppose \( \mathcal{E}_1 \leq \mathcal{E}_2 \leq \mathcal{E}_3 \leq \cdots \) is an ascending chain of countably generated models of \( \text{Th}(N) \) such that

(a) each \( \mathcal{E}_n \) admits a strictly minimal extension,
(b) for any \( b \in (\mathcal{E}_{n+1} - \mathcal{E}_n) \), the submodel generated by \( b \) includes \( \mathcal{E}_n \), and
(c) \( \mathcal{A} = \bigcup_{n \geq 1} \mathcal{E}_n \) does not admit a strictly minimal extension.

Then there is an element \( a \in \mathcal{A} \) and some \( J \in \omega \) such that \( a(\mathcal{A}) \subseteq \mathcal{E}_j \) but \( a \notin \mathcal{E}_j \).

**Proof.** First note that (b) says that for any \( b \in \mathcal{E}_{n+1} - \mathcal{E}_n \) and any \( c \in \mathcal{E}_n \), there is an \( f: \omega \to \omega \) with \( *f(b) = c \). By Theorem 1, there is a sequence \( \{a_i\} \) such that \( a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{A}) \), \( a_i(\mathcal{A}) \neq \emptyset \), \( \bigcap_{i>0} a_i(\mathcal{A}) = \emptyset \), and \( a_0(\mathcal{A}) \) does not generate \( \mathcal{A} \) by standard unary functions. Then, for some \( J \), \( a_0(\mathcal{A}) \subseteq \mathcal{E}_j \), since otherwise, for arbitrarily large \( k \), \( a_0(\mathcal{A}) \cap (\mathcal{E}_{k+1} - \mathcal{E}_k) \) is nonempty, and then it follows from (b) that every element of \( \mathcal{A} \) is obtainable from an element of \( a_0(\mathcal{A}) \) by a standard unary function, thus contradicting the choice of \( \{a_i\} \). We have then that, for all \( i \), \( a_i(\mathcal{A}) \subseteq \mathcal{E}_j \), and if \( a_i \) is an element of \( \mathcal{E}_j \) then \( a_i(\mathcal{E}_j) = a_i(\mathcal{A}) \).

We now claim that for some \( i \), \( a_i \) is not an element of \( \mathcal{E}_j \) (and then the proof is complete by setting \( a = a_i \)). Suppose not; then for all \( i \), \( a_i \) is in \( \mathcal{E}_j \) and so \( a_i \) is also in \( \mathcal{E}_{j+1} \). Since \( \mathcal{E}_{j+1} \) admits a strictly minimal extension, and since \( a_i(\mathcal{E}_{j+1}) = a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{E}_{j+1}) = a_{i+1}(\mathcal{A}) \), we have by Theorem 1 that either \( a_0(\mathcal{E}_{j+1}) \) generates \( \mathcal{E}_{j+1} \) or \( \bigcap_{i>0} a_i(\mathcal{E}_{j+1}) = \emptyset \). The latter conclusion is impossible by the choice of \( \{a_i\} \) and the fact that \( a_i(\mathcal{A}) \subseteq \mathcal{E}_{j+1} \), and the former conclusion says that a subset of \( \mathcal{E}_j \) generates \( \mathcal{E}_{j+1} \), which is impossible since \( \mathcal{E}_j \) is a proper submodel of \( \mathcal{E}_{j+1} \).

Let \( \mathcal{M} \) be a finitely generated model. We say that \( \mathcal{M} \) is element generated iff for every generator \( a \) of \( \mathcal{M} \), there is a generator \( b \) of \( \mathcal{M} \) with \( b \in a(\mathcal{M}) \).

**Lemma 4 (CH).** Suppose that \( \{\mathcal{E}_i: i = 1, 2, \ldots\} \) form an ascending chain of countably generated models, \( \mathcal{A} = \bigcup_{i \geq 1} \mathcal{E}_i \), and that all the hypotheses of Lemma 3 are satisfied. Then there is a finitely generated submodel \( \mathcal{M} \) of \( \mathcal{A} \) which includes \( \mathcal{E}_1 \) and is not element generated.

**Proof.** Let \( a \) and \( \mathcal{E}_j \) satisfy the conclusion of Lemma 3, and let \( \mathcal{M} \) be the submodel of \( \mathcal{A} \) generated by \( a \). Then \( \mathcal{E}_1 \subseteq \mathcal{E}_j \subseteq \mathcal{M} \) (the second inclusion follows from hypothesis (b) of Lemma 3), and \( \mathcal{M} \) is not element generated since \( a(\mathcal{M}) \subseteq \mathcal{E}_j \) and so \( a(\mathcal{M}) \) does not contain a generator of \( \mathcal{M} \).

2. Simple combinatorics. Let \( \langle A_1, A_2, \ldots, A_n \rangle \) be a sequence of nonempty sets. A complete set of distinct representatives (CDR) is the image of a one-to-one choice function on the set \( \{A_1, \ldots, A_n\} \), that is, a sequence \( \langle a_1, \ldots, a_n \rangle \) with \( a_i \in A_i \) and \( a_i \neq a_j \) if \( i \neq j \). We will need the following simple combinatorial lemma in order to construct ultrapowers which are element generated.

**Lemma 5.** Let \( \langle A_0, \ldots, A_{n-1} \rangle \) be a sequence of distinct, nonempty sets. Then there is a subsequence of length at least \( \log_2(n) \) which admits a CDR.
Proof. The lemma is obvious by inspection for \( n < 5 \), so assume \( n \geq 5 \). Let \( K = \left| \bigcup_{0 \leq i < n} A_i \right| \). We prove the lemma by induction on \( K \). Since \( n > 4 \), we have \( K > 1 \), so assume the lemma for smaller \( K \), and fix an arbitrary \( x \in \bigcup A_i \). Reorder the \( A_i \) so that \( x \in A_i \) for \( i < p \) and \( x \notin A_i \) for \( i \geq p \), where \( p \) is the number of sets among the \( A_i \) which contain \( x \). Then \( 0 < p \leq n \).

Case 1: \( 0 < p < \left( \frac{n}{2} \right) \). In this case, \( \langle A_p, A_{p+1}, \ldots, A_{n-1} \rangle \) satisfies the induction hypothesis, so there is a subsequence \( \langle B_0, \ldots, B_{m-1} \rangle \) of \( \langle A_p, \ldots, A_{n-1} \rangle \) which admits a CDR \( \langle b_0, \ldots, b_{m-1} \rangle \), and \( m \geq \log_2(n-p) \geq \log_2(n/2) = \log_2(n) - 1 \).

Case 2: \( \left( \frac{n}{2} \right) \leq p \leq n \). For \( i < p \), let \( B_i = (A_i \setminus \{x\}) \). By the induction hypothesis, there is a CDR \( \langle d_0, \ldots, d_{m-1} \rangle \) for some subsequence of the \( B_i \)'s with \( m \geq \log_2(p) \geq \log_2(n) - 1 \); without loss of generality, assume the \( A_i \) were ordered so that the subsequence of the \( B_i \)'s admitting this CDR is \( \langle B_0, \ldots, B_{m-1} \rangle \). If \( m \geq \log_2(n) \), we are done; otherwise, since \( n > 4 \) we have \( p \geq n/2 \geq \log_2(n) > m \). Thus \( \langle d_0, \ldots, d_{m-1}, x \rangle \) is a CDR for \( \langle A_0, B_0, \ldots, B_{m-1} \rangle \), which has length \( m + 1 \geq \log_2(n) \).

3. The successor case. The point of the theorem in this section is to insure the existence of a \( P \) point \( D_{a+1} \) which is a s.i.s. of \( D_a \) and such that \( D_{a+1} \)-prod is element generated. If \( E \) is an ultrafilter, then the generators of \( E \)-prod are the germs \( [f]_E \) of one-to-one (mod \( E \)) functions \( f \), and so it follows that \( E \)-prod is element generated iff for any such one-to-one germ \( a = [f]_E \), there is a one-to-one (mod \( E \)) function \( g \) with \( [g]_E \in a(E \)-prod), which means there is a set \( A \) in \( E \) such that for all \( x \) in \( A \),

\[
g(x) = f(x).
\]

The construction of \( D_{a+1} \) will be done within the framework of Theorem 2.2 of [Ro]. For the convenience of the reader we state this theorem below after supplying the requisite definitions. If \( X \) is a set and \( p \) is a function which is finite-to-one on \( X \), then the cardinality function of \( X \) with respect to \( p \), denoted \( C_{X,p} \), is defined by

\[
C_{X,p}(n) = |X \cap p^{-1}(n)|.
\]

We will omit reference to \( p \) when there is no ambiguity. A (Dedekind) cut in an ultrapower \( D \)-prod is a partition of \( D \)-prod into convex sets \( S \) and \( L \) such that every element of \( S \) precedes every element of \( L \). A cut is fair if \( S \) and \( L \) are nonempty and \( L \) has no countable coinitial subset. If \( E \) is a \( P \) point and \( p: \omega \to \omega \) with \( p(E) = D \), then the cut in \( D \)-prod associated to \( p \) and \( E \) is defined by putting into \( L \) those \( D \)-germs \( \langle C_{X,p} \rangle_D \) for \( X \in E \) (and all larger \( D \)-germs), and setting \( S = D \)-prod \(- L \) (see [B2] for a thorough discussion of Dedekind cuts in ultrapowers). Finally, a condition on \( X \) is simply a statement about \( X \).

Theorem 6 [Ro] (CH). Let \( D \) be a \( P \) point, \( \langle S, L \rangle \) a fair cut in \( D \)-prod such that \( S \) is closed under addition in \( D \)-prod, \( p \) the first projection from \( \omega^2 \) to \( \omega \) and \( \{C_i: i \in I\} \) a set of at most \( 2^\aleph_0 \) conditions on subsets of \( \omega^2 \). Call a set \( X \subseteq \omega^2 \) large if it contains a subset \( Y \) on which \( p \) is finite-to-one and \( \langle C_Y \rangle_D \) is in \( L \), and suppose that for any large \( X \) and condition \( C_i \), there is a large subset \( Y \) of \( X \) which satisfies \( C_i \).

Then there exist \( 2^\aleph_1 \) many (pairwise nonisomorphic) \( P \) points \( E \) on \( \omega^2 \) with \( p(E) = D \) and associated cut \( \langle S, L \rangle \), such that for all \( i \) in \( I \), \( E \) contains a set satisfying condition \( C_i \).
PROOF. This is a special case of Theorem 2.2 in [Ro] obtained by taking the fiber
measure of that theorem to be the cardinality function (see also [B2]).

THEOREM 7 (CH). Let D be a P point, p the first projection from $\omega^2 \rightarrow \omega$ and $[h]_D$
any element of $D$-prod. There exist $2^{\aleph_1}$ P points $E$ with $p(E) = D$ such that
(1) $E$ is a s.i.s. of $D$,
(2) $E$-prod is element generated, and
(3) $[h]_D$ is in the small part $S$ of the cut in $D$-prod associated to $p$ and $E$.

PROOF. Begin by defining functions $s_j: \omega \rightarrow \omega$ as follows;

$$s_0(0) = h(0),$$
$$s_{i+1}(k) = 2^{s_i(k)},$$
$$s_0(k + 1) = \text{maximum of } \left\{ h(k + 1), \sum_{j < k} s_j(j) \right\}.$$

Define a cut $\langle S, L \rangle$ in $D$-prod by $a \in S$ iff $a \leq [s_j]_D$ for some $j$. Then $[h]_D \in S$
and $\langle S, L \rangle$ is a fair cut since the existence of a countable cofinal set in $S$ implies
that there is no countable cofinal set in $L$ (by the $\aleph_1$-saturation of $D$-prod). It is
easy to check that $S$ is closed under addition, multiplication and exponentiation. For
each $f: \omega^2 \rightarrow \omega$, let $C_f$ be the following condition on $Y$: ($f$ is $p$-fiberwise constant on
$Y$) or ($f$ is one-to-one on $Y$ and there is a $p$-fiberwise one-to-one function $g$ on $Y$
such that for all $y \in Y, g(y) \equiv f(y)$).

If we show that, for any $f$, any large set $X$ includes a large subset $Y$ satisfying $C_f$,
then by Theorem 6 we will have shown the existence of $2^{\aleph_1}$ many $P$ points $E$ with
$p(E) = D$, associated cut $\langle S, L \rangle$, such that $E$ contains sets satisfying $C_f$ for all $f$.
Thus every $f$ is either $p$-fiberwise constant or (globally) one-to-one on a set in $E$; so
every $[f]_E$ in $E$-prod either is in the embedded image (by $p^*$) of $D$-prod or is a
generator of $E$-prod. It follows that $E$ is a s.i.s. of $D$. To see that $E$-prod is element
generated, let $a$ be a generator of $E$-prod; so $a = [f]_E$ for some function $f$ which is
one-to-one on a set in $E$. Let $Y \in E$ satisfy $C_f$; then $f$ cannot be $p$-fiberwise constant
on $Y$ (since $[f]_E$ is a generator of $E$-prod), so $Y$ satisfies the second part of $C_f$. Then
there is a $p$-fiberwise one-to-one function $g$ on $X$ such that for all $y \in Y, g(y) \equiv f(y)$,
and so $[g]_E$ is in $a(E$-prod). Now $[g]_E$ is not in $p^{**}D$-prod since if it were, then $g$
would be $p$-fiberwise constant as well as $p$-fiberwise one-to-one on a set in $E$, which
would imply that $p$ is one-to-one (mod $E$), contradicting the fairness of the associ-
ated cut. By the strict maximality of $p^{**}$-prod, it follows then that $[g]_E$ is a
generator of $E$-prod. Thus $E$-prod is element generated.

It remains to show that we can find a large subset satisfying any given condition
$C_f$ for any large $X$. We can assume (by cutting down $X$ if necessary) that $p$
is finite-to-one on $X$. For each nonempty fiber $X_n = (X \cap p^{-1}\{n\})$, we can find a set
$Z_n \subseteq X_n$ such that $f$ is constant on $Z_n$ or $f$ is one-to-one on $Z_n$ and $|Z_n|^2 \geq |X_n|$.
Partition the fibers into sets $W_c$ and $W_1$, where $W_c$ consists of those $n$ such that $f$
is constant on $Z_n$ and $W_1$ consists of those $n$ such that $f$ is one-to-one on $Z_n$. One of
these sets is in $D$ (else $X$ was not large). If $W_c$ is in $D$, then let $Y = \bigcup_{n \in W_c} Z_n$ and
clearly \( f \) is \( p \)-fiberwise constant on \( Y \); for all \( n \) in \( W_c \), \((C_Y(n))^2 \geq C_X(n)\), and so \([C_Y]_D\) is in \( L \) since \([C_X]_D\) is in \( L \) and \( S \) is closed under multiplication.

If, on the other hand, \( W_1 \) is in \( D \), then we must cut down \( Z = \bigcup_{n \in W_1} Z_n \) to a large \( Y \) on which \( f \) is one-to-one and for which we can find the desired function \( g \). First define sets \( B_i \) in \( D \) (\( i = 1, 2, 3, \ldots \)) by

\[
B_i = \{ m \in p''Z : |Z_m| \geq s_{i+1}(m) \} - \{0, 1, \ldots, m \}.
\]

Then \( B_i \) is in \( D \) since \( Z \) is large (by the argument given above for \( Y \)) and hence, for all \( j \), \( C_Z(m) \) cannot be less than \( s_j(m) \) for \( D \)-many \( m \). Note also that \( B_{i+1} \subseteq B_i \), and we may assume that \( p''Z \subseteq B_0 \) (throw away some fibers of \( Z \) if necessary).

Define \( t(m) = \text{maximum} \ i \text{ such that} \ m \in B_i \) for \( m \) in \( B_0 \). Then, for all \( m \in B_0 \), \( t(m) < m \), \( m \in B_{t(m)} \), and \( |Z_m| \geq s_{t(m)+1}(m) \). Let \( M \) be the minimal element of \( B_0 \), and let \( Q_M \) be any subset of \( Z_M \) with \( |Q_M| = s_{t(M)}(M) \). Note that \( f \) is one-to-one on \( Q_M \).

Let \( m \in B_0 \), let \( H_m = \{ j \in B_0 : j < m \} \), and suppose, as an induction hypothesis, that for all \( j \) in \( H_m \), we have defined \( Q_j \subseteq Z_j \) such that \( |Q_j| = s_{t(j)}(j) \) and \( f \) is one-to-one on \( \bigcup_{j \in H_m} Q_j \). Let \( R_m = \{ x \in Z_m : f(x) = f(y) \text{ for some } y \text{ in } \bigcup_{j \in H_m} Q_j \} \). Since \( f \) is one-to-one on \( Z_m \), we have

\[
|Z_m| - |R_m| \geq \sum_{j \in H_m} s_{t(j)}(j) - |R_m| = \sum_{j < m} s_{t(j)}(j) - |R_m| \geq s_0(m).
\]

and thus

\[
|Z_m - R_m| \geq s_{t(m)+1}(m) - s_0(m) = s_{t(m)}(m) - s_0(m) \geq s_t(m)(m).
\]

Let \( Q_m \) be any subset of \( Z_m - R_m \) of size \( s_{t(m)}(m) \); this completes the induction construction of the \( Q_m \) for \( m \) in \( B_0 \), for it is clear by the definition of \( R_m \) that \( f \) is one-to-one on \( \bigcup_{j \in H_m} Q_j \cup Q_m \). Let \( Q = \bigcup_{j \in H_m} Q_j \); then \( f \) is one-to-one on \( Q \) and for any \( k \) in \( \omega \), we have that for all \( m \) in \( B_k \), \( |Q_m| = s_{t(m)}(m) \geq s_k(m) \), and so \([C_0]_D \geq [s_k]_D \). Thus \( Q \) is large. The argument just given to make \( f \) one-to-one on a large set is due to Eck [Ec].

If \( f \) assumes the value 0 on \( Q \), then remove that point from \( Q \). Temporarily fix \( m \in B_0 = p''Q \), and write \( Q_m = (p^{-1}(\{m\}) \cap Q) \) as \( \{a_1, a_2, \ldots, a_n\} \). Let \( A_i = \{ t : t \in f(a_i) \} \); since \( f \) is one-to-one on \( Q \), \( \{A_i : 1 \leq i \leq n\} \) is a collection of distinct nonempty sets, and so by Lemma 5, there is a subsequence \( \langle B_1, \ldots, B_j \rangle \) of \( \langle A_1, \ldots, A_n \rangle \) which admits a CDR \( \langle u_1, \ldots, u_j \rangle \) with \( J \geq \log_2(n) \). Reorder the \( \{a_i\} \) so that \( \langle u_1, \ldots, u_j \rangle \) is a CDR for \( \langle A_1, \ldots, A_j \rangle \), and let \( Y_m = \{a_1, \ldots, a_j\} \). Define \( g \) on \( Y_m \) by \( g(a_j) = u_j \), so \( g \) is one-to-one on \( Y_m \).

Let \( Y = \bigcup_{m \in B_0} Y_m \); clearly \( Y \) satisfies condition \( C_f \) and so we only need show that \( Y \) is large. For all \( m \) in \( B_0 \), \( |Y_m| \geq \log_2(|Q_m|) \), whereupon the largeness of \( Y \) follows from the largeness of \( Q \) and the closure of \( S \) under exponentiation.

4. The limit case. In order to apply Lemma 4 at limit stage \( \lambda \) in the construction of our sequence of \( P \) points, we need to know that (beyond some point in the sequence) each finitely generated submodel of the associated countably generated model \( \mathcal{A} \) is element generated. The previous theorem insures this for finitely generated models.
arising at successor stages in the construction; the following theorem gives us this result for those arising at limit stages previous to $\lambda$.

**THEOREM 8 (CH).** Let \( \{ E_i : i = 1, 2, \ldots \} \) be an RK-increasing sequence of $P$ points with \( p_i(E_{i+1}) = E_i \), and let $\mathcal{A}$ be the direct limit of \( \langle E_i \text{-prod}, p_i^* \rangle \). For $i \geq 2$, let $q_i = p_1 \circ p_2 \circ \cdots \circ p_{i-1}$ and let $\langle S^i, L^i \rangle$ be the cut in $E_1 \text{-prod}$ associated to $q_i$ and $E_i$. Assume that:

(a) $S^i \subseteq S^{i+1}$ for all $i \geq 2$, and

(b) $E_i \text{-prod}$ is element generated for all $i \geq 1$.

Then $\mathcal{A}$ admits a single-skied, element generated, strictly minimal extension.

**Proof.** The proof is a modification of the proof of Theorem 2. Let $E_i^\prime$ be the canonical image of $E_i \text{-prod}$ in $\mathcal{A}$, and let $a_i$ be a generator of $E_i^\prime$ in $\mathcal{A}$. Let $g_i$ and $G_i$ be as in the proof of Theorem 2 (i.e. $g_i = \langle a_1, \ldots, a_i \rangle$ and the ultrafilter $G_i$ is the type of $g_i$ in $\mathcal{A}$). Since the ultrafilters $E_i$ form an increasing RK-sequence, it follows that $g_i$ generates $E_i^\prime$ and hence $G_i = E_i^\prime$; thus $G_i \text{-prod}$ is element generated and if $\langle S_i, L_i \rangle$ is the cut in $G_i \text{-prod}$ associated to $S_i$ and $G_i$, then $S_i \subseteq S_{i+1}$. Since $\mathcal{A}$ admits a strictly minimal extension, the hypotheses of Theorem 1 are satisfied.

For each $f : \text{Seq} \to \omega$, let $C_f$ be the following condition on a subset $X$ of $\text{Seq}$: $(f$ is TR$_n$-fiberwise constant on $X$ for some $n)$ or $(f$ is one-to-one on $X$ and there is a TR$_1$-fiberwise one-to-one function $g$ on $X$ such that $g(x) = f(x)$ for all $x$ in $X$).

As in Theorem 2, we will construct a filter $F$ on $\text{Seq}$ consisting of large (in the sense of Theorem 2) sets such that TR$_1$ is finite-to-one as well as sets satisfying $C_f$ for each $f$; let $E$ be an ultrafilter including $F$ and the complements of all large sets. Let $\mathcal{B} = E$-prod, and then $\mathcal{B}$ is a single-skied, strictly minimal extension of (the embedded image of) $\mathcal{A}$; in addition, $\mathcal{B}$ is element generated, since the function $g$ of condition $C_f$ is TR$_1$ (and hence TR$_n$ for all $n$) fiberwise one-to-one (mod $E$), and thus is not TR$_n$-fiberwise constant (mod $E$) for any $n$, insuring (by strict minimality) that $[g]_E$ is a generator of $E$-prod.

List the conditions in an $\aleph_1$-sequence. The proof proceeds exactly as for Theorem 2 (the definition of $L_0$ and the construction of $L_\lambda$ for limit $\lambda$ are identical), except that we must satisfy the stronger condition of the present theorem at successor stages. So assume that $L$ is large and let $C_f$ be the $\alpha$th condition. Since the hypotheses of Theorem 1 are satisfied by $\mathcal{A}$, we can find a large $A \subseteq L_\alpha$ such that $f$ is one-to-one on $A$, or, for some $n$, $f$ is TR$_n$-fiberwise constant on $A$. If the latter, then let $L_{\alpha+1} = A$. Assume therefore that $f$ is one-to-one on $A$; we must cut down $A$ further so that we can define the desired function $g$.

For $n \geq 1$, let $R_n = \text{TR}_n "A \cap \{ x \in \text{Seq} : \text{lh}(x) = n \}$. Since $A$ is large, $R_n$ is in $G_n$. Let $B_1 = R_1$, and for each $x$ in $B_1$, let $s(x, 1) \in A$ such that TR$_1(s(x, n)) = x$. Assume we have defined $B_j$ for $j < K$ such that $B_j \subseteq R_j$, $B_j \in G_j$, with TR$_j "B_{j+1} \subseteq B_j$, and we have defined a function $s$ such that $s(x, j) \in A$ and TR$_j(s(x, j)) = x$ for each $x$ in $B_j$. Let

$$B_K = (R_K \cap \text{TR}^{-1}_{K-1}(B_{K-1})) - W,$$
where \( W = \{ z \in R_K: z = \text{TR}_K(s(x, K - 1)) \} \) for some \( x \in B_{K-1} \). It is easy to see that \( \text{TR}_{K-1} \) is one-to-one on \( W \) (two different \( z \)'s which map to \( x \) under \( \text{TR}_{K-1} \) can not both be truncations of \( s(x, K - 1) \)), and so \( W \) is not in the ultrafilter \( G_K \) (since \( \text{TR}_{K-1} \) maps \( G_K \) to \( G_{K-1} \) and is not an isomorphism). It follows that \( B_K \) is in \( G_K \), since both \( R_K \) and \( \text{TR}_{K-1}^{-1}(B_{K-1}) \) are in \( G_K \). For each \( x \in B_K \), let \( s(x, K) \) be an element of \( A \) such that \( \text{TR}_K(s(x, K)) = x \). This completes the inductive construction of the \( B_K \).

Let \( P = \{ s(x, n): n \geq 1 \) and \( x \in B_n \} \). Then \( P \subseteq A \), \( P \) is large, and by the construction we have that for all \( s \) in \( P \), there is a unique \( n \) and a unique \( x \) in \( B_n \) with \( s = s(x, n) \). Define functions \( f_n \) on \( B_n \) by \( f_n(x) = f(s(x, n)) \). Then \( f_n \) is one-to-one on \( B_n \) since \( f \) is one-to-one on \( A \). Since \( G_n \)-prod is element generated, we can find, for each \( n \), a set \( Z_n \) in \( G_n \), \( Z_n \subseteq B_n \), and a one-to-one function \( g_n \) on \( Z_n \) such that for all \( x \) in \( Z_n \), \( g_n(x) \in f_n(x) \).

Define \( g: P \to \omega \) by \( g(s(x, n)) = g_n(x) \). Then \( g \) is well defined, and for all \( s \) in \( P \), \( g(x) \subseteq f(x) \). We need only cut down \( P \) to a large \( L_{\alpha+1} \) on which \( g \) is \( \text{TR}_1 \)-fiberwise one-to-one.

Since \( S_i \subseteq S_{i+1} \), we can find functions \( h_i: \text{Seq}_1 \to \omega \) such that \( [h_i] \subseteq L_i - L_{i+1} \) (\( \text{Seq}_1 \) is the set of codes for sequences of length 1). Let \( P_1 = Z_1 \), and assume that we have defined sets \( P_i \subseteq Z_i \) for \( 1 \leq i < K \) such that

1. \( P_i \subseteq G_i \), and \( \text{TR}_{i-1}(P_i) \subseteq P_{i-1} \),
2. for all \( i \geq 2, [\text{TR}_i^{-1}(p) \cap P_i] \subseteq h_i(p) \) for all \( p \) in \( P_i \), and
3. \( (\forall \langle p \rangle \in P_i)(\forall i, j < K)(\forall y, z \in (\text{TR}_i^{-1}(p) \cap (\bigcup_{n<K} P_n))) (g_i(y) = g_j(z) \iff i = j \) and \( y = z \).

To define \( P_K \), first let \( V = \{ z \in Z_K \cap \text{TR}_K^{-1}(P_{K-1}): (\exists j < K)(\exists y \in P_j)(g_j(z) = g_j(y) \) and \( \text{TR}_j(y) = \text{TR}_j(z) \} \).

If \( p \in P_i \), then

\[
|\text{TR}_i^{-1}(p) \cap V| \leq 1 + \sum_{j<K} |P_i \cap \text{TR}_i^{-1}(p)| \leq 1 + \sum_{j<K} h_j(\langle p \rangle)
\]

(the first inequality follows since \( g_K \) is one-to-one on \( Z_K \); the second follows from (2)). Thus \( [C_V]_G \leq 1 + \sum_{j<K} [h_j]_G \), and since \( S_K \) is closed under addition (Theorem 2 of \([B2]\)), it follows that \([C_V]_G \subseteq S_K \) (as \([h_j]_G \subseteq S_K \) for all \( j < K \)). Thus \( V \) is not in \( G_K \). Since \([h_K]_G \subseteq L_K \), we can find \( W \) in \( G_K \) such that for all \( \langle p \rangle \) in \( P_1 \), \( C_W(\langle p \rangle) \leq h_K(\langle p \rangle) \).

Let

\[
P_K = (Z_K \cap W \cap \text{TR}_K^{-1}(P_{K-1})) - V.
\]

Then \( P_K \) obviously satisfies (1) and (2), and it is easy to check, from the definition of \( V \), that \( P_K \) satisfies (3).

We have inductively defined \( \{ P_j: j = 1, 2, 3 \ldots \} \), and we set \( L = s''(\bigcup(P_n \times \{ n \})) \). Then \( \text{TR}_n''L \) includes \( P_n \), and \( P_n \) is in \( G_n \), so \( L \) is large. If \( s \) and \( t \) are in \( L \) with \( g(s) = g(t) \) and \( \text{TR}_1(s) = \text{TR}_1(t) \), then there are unique \( x, y, m, n \) with \( x \in P_m, y \in P_n, s = s(x, m) \) and \( t = s(y, n) \) (by the construction of the set \( P \) ). By the definition of \( g \), \( g_m(x) = g_n(y) \), and by (3) above, \( m = n \) and \( x = y \), and so \( s = t \). Thus \( g \) is \( \text{TR}_1 \)-fiberwise one-to-one on \( L \), and the proof is complete by setting \( L_{\alpha+1} = L \).
5. The finale. We now combine the previous results to produce our desired sequence of $P$ points.

**Theorem 9 (CH).** Let $D$ be a $P$ point. There exist initial segments of $\leq_{P,D}$ of order type $\mathbb{N}_1$.

**Proof.** We construct a sequence of $P$ points $\{D_\alpha; \alpha < \mathbb{N}_1\}$ with $D_0 = D$ such that:

1. $D_{\alpha+1}$ is a strong immediate successor of $D_\alpha$,
2. for limit $\lambda < \mathbb{N}_1$, $D_\lambda$ is a strongly minimal upper bound for $\{D_\alpha; \alpha < \lambda\}$,
3. $D_\alpha$-prod is element generated for $\alpha \geq 1$, and
4. if $q_\alpha$ is the map from $D_\alpha$ to $D_1$, and $\langle S_\alpha, L_\alpha \rangle$ is the cut in $D_1$-prod associated to $q_\alpha$ and $D_1$, then $S_\beta \subseteq S_\alpha$ whenever $\beta < \alpha$.

By (1) and (2), such a sequence forms an initial segment of $\leq_{P,D}$.

If $\alpha = 1$, let $D_1$ be a s.i.s. of $D$ such that $D_1$-prod is element generated; we can find such a $D_1$ by Theorem 7.

If $\alpha = \beta + 1$, fix $g: \omega \to \omega$ such that $[g]_{D_\beta}$ is in the large part $L_\beta$ of the cut in $D_1$-prod associated to $q_\beta$ and $D_\beta$. Use Theorem 7 to find an element generated $P$ point $D$ with $p(D_\alpha) = D_\beta$ such that $D_\alpha$ is a s.i.s. of $D_\beta$ and such that $q_\beta([g]_D)$ (where $[g]_D$ is in the small part $S^\prime$ of the cut in $D_\alpha$-prod associated to $p$ and $D_\alpha$ (the cut from $D_\beta$ to $D_1$). If $A$ is any set in $D_\alpha$, then we have that for all $n$ in some set $B$ in $D_\beta$, $g(q_\beta(n)) \leq [A \cap p^{-1}(\{n\})]$. It is easy to check that for all $z$ in $q_\beta''B$, $|A \cap q_\alpha^{-1}(z)| \geq g(z)$, and $p$ is the map from $D_\alpha$ to $D_1$; since $q_\beta''B$ is in $D_1$, we have that $[C]_{D_\beta} > [g]_{D_\beta}$, and so $[g]_{D_\beta}$ is in the small part $S_\alpha$ of the cut in $D_1$-prod associated to $q_\alpha$ and $D_\alpha$. Thus $[g]_{D_\beta}$ witnesses that $D_\alpha$ satisfies (4).

If $\lambda$ is a countable limit ordinal, let $1 = \alpha_1, \alpha_2, \ldots$ be an increasing $\omega$-sequence cofinal in $\lambda$, and let $E_i = D_{\alpha_i}$. Let $p_i$ be the map from $E_{i+1}$ to $E_i$, let $A$ be the direct limit of $\{E_i\}$, and let $d$ be the canonical image of $E_1$-prod in $A$. We claim first that $A$ admits a strictly minimal extension.

To prove this claim, first note that each $d_i$ admits a strictly minimal extension (since each $E_i$ is one of the previously constructed $D_\alpha$'s) and that any element of $E_{i+1} - E_i$ generates a submodel which includes $E_i$ (since the $D_\alpha$'s already constructed form an initial segment of $\leq_{P,D}$). If $A$ did not admit a strictly minimal extension, then by Lemma 4 there would be a finitely generated submodel $M$ of $A$ which includes $E_1$ but is not element generated. But $M$ is the embedded image of one of the $D_\alpha$-prod (again since they form an initial segment) and so $M$ must be element generated by the induction hypothesis. This contradiction proves the claim.

By the induction hypothesis, each of the $d_i$ is element generated and if $i < j$, then $S_{\alpha_i} \subseteq S_{\alpha_j}$. By Theorem 8, $A$ admits a strictly minimal extension $\mathcal{A}$ which is single-skied and element generated. Let $D_\lambda$ be an ultrafilter such that $D_\lambda$-prod is isomorphic to $\beta$; then $D_\lambda$ is a $P$ point and is a s.m.u.b. for $\{D_\alpha; \alpha < \lambda\}$. Finally, if $\alpha < \lambda$, it is easy to check that $S_{\alpha+1} \subseteq S_\lambda$ (since the map $q_\lambda$ factors through $q_{\alpha+1}$) and so $S_{\alpha} \subseteq S_\lambda$, showing that $D_\lambda$ satisfies (4).

This completes the inductive construction of $\{D_\alpha; \alpha < \mathbb{N}_1\}$, and with it, the proof of the theorem.
Corollary 10 (CH). There exist initial segments of the RK ordering, restricted to (isomorphism classes of) P points, of order type $\mathfrak{S}_1$.

Proof. This is simply Theorem 9 with $D$ an RK-minimal ultrafilter. The existence of such $D$ follows from CH (see [Pu], for example); ultrafilters which are minimal in RK are also called selective or Ramsey, and they are all P points.

Corollary 11 (CH). There exist P points with countably many constellations; in fact, for any countable ordinal $\alpha$, there exist P points $E$ such that the initial segment of RK determined by $E$ has order type $\alpha$.

Proof. Immediate by the previous corollary.

Corollary 12 (CH). For any P point $D$, there exist initial segments $T$ of $<_P, D$ such that $T$ is a tree with $\mathfrak{S}_1$ levels, each node has $2^{|\mathfrak{S}_1|}$ immediate successors, and each countable increasing sequence in $T$ has a unique upper bound in $T$.

Proof. The proof is the same as the proof of Theorem 9, except at successor stages, use Theorem 7 to generate $2^{|\mathfrak{S}_1|}$ immediate successors.

6. Two questions. We conclude with two natural questions.

1. Given CH, is there an RK-increasing $\omega$-sequence of P points which does not admit a s.m.u.b. which is a P point? In other words, do we really need element generated models to prove the theorems here?

2. Can our result extend beyond $\mathfrak{S}_1$? That is, do there exist (CH) initial segments of $<_P, D$ of order type $\mathfrak{S}_1 + 1$ or perhaps $\mathfrak{S}_2$? In general, the successor cardinal of the continuum would be the best possible.

References


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