P POINTS WITH COUNTABLY MANY CONSTELLATIONS

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ABSTRACT. If the continuum hypothesis (CH) is true, then for any P point ultrafilter D (on the set of natural numbers) there exist initial segments of the Rudin-Keisler ordering, restricted to (isomorphism classes of) P points which lie above D, of order type ω₁. In particular, if D is an RK-minimal ultrafilter, then we have (CH) that there exist P-points with countably many constellations.

0. Introduction. Our main result is that in the presence of the continuum hypothesis (henceforth denoted CH), there exist P point ultrafilters on ω with exactly ω₀ many constellations. Actually, we prove a somewhat stronger theorem about initial segments of the Rudin-Keisler (RK) ordering on the class of P points; in order to state this result, we begin with a few definitions. All ultrafilters here are nonprincipal ultrafilters on ω = {0, 1, 2, ...}. An ultrafilter D is a P point iff any function f: ω → ω is either constant or finite-to-one on a set in D. P points have been studied extensively, and we shall assume basic results about them and their RK ordering; good references are [B1 and Pu]. If F) is a P point, let < P D denote the RK ordering on (equivalence classes of) P points which lie above D in RK. An initial segment of < P D means a downward closed subset, and the initial segment determined by E is {F: D < F < E} (we use < to denote the RK ordering).

In his thesis [Ec], Eck showed (CH) that if D is any P point, then there exist P points E immediately above D in RK in the strong sense that any strict RK predecessor of E is a predecessor of D; we call such an E a strong immediate successor (s.i.s) of D. Iterating Eck’s theorem ω times yields the existence (CH), for any P point D, of initial segments of < P D of order type ω. Our main theorem is the existence (CH) of initial segments of < P D of order type ω₁; the bulk of the article is devoted to its proof.

In [B3], Blass proved the result just stated without the restriction to P points; that is, he showed (CH) that for any ultrafilter D, there exist initial segments of “RK above D” of order type ω₁. The proof involved reformulating the problem in model theoretic terms, and we shall take the same approach. Let N be the complete first order structure on ω (i.e. the language for N contains names for every finitary function and relation on ω). We use the term model to mean “nonstandard model of Th(N)”, and we use *f to indicate the interpretation of the function f: ω → ω in whichever model is under consideration. If D is an ultrafilter, then D-prod denotes

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585
the ultrapower of \( N \) by \( D \), and if \( f \) is a function from \( \omega \) to \( \omega \), then the corresponding element of the universe of \( D \)-prod (called the germ of \( f \)) is denoted \([ f ]_D\). In general, a model is isomorphic to an ultrapower iff it is finitely generated, which, due to the existence of pairing functions, is the same as saying that the model is generated by a single element in its universe.

There is an intimate relationship between the structure of an ultrapower \( D \)-prod and the RK ordering below \( D \); roughly, finitely generated submodels of \( D \)-prod correspond to RK predecessors of \( D \): the submodel generated by \([ f ]_D\) corresponds to the RK predecessor \( f(D) \) of \( D \) (the submodel is isomorphic to \( f(D) \)-prod by the map \( f^* : (D) \)-prod \rightarrow D \)-prod defined by \( f^*([g]_D) = [g \circ f]_D\)). The details for this construction are well known and can be found in [B2]. An ultrafilter \( D \) is a \( P \) point iff every (nonstandard) submodel of \( D \)-prod is cofinal in \( D \)-prod, and in general, we refer to models in which all nonstandard submodels are cofinal as “single-skied” (see [Pu] for a discussion of skies). Then a model \( \mathcal{A} \) of \( \text{Th}(N) \) which is single-skied and finitely generated is isomorphic to \( D \)-prod for some \( P \) point \( D \). Note that if \( f(E) = D \), then \( E \) is a s.i.s. of \( D \) iff the submodel \( \mathcal{E} \) of \( D \)-prod generated by \([ f ]_D\) (\( \mathcal{E} = f^*E \)-prod) is strictly maximal in \( D \)-prod (that is, every proper submodel of \( D \)-prod is a submodel of \( \mathcal{E} \)). Also, if \( \mathcal{D} \) is a strictly minimal extension of the model \( \mathcal{E} \) (that is, \( \mathcal{E} \) is strictly maximal in \( \mathcal{D} \)), then \( \mathcal{D} \) must be isomorphic to an ultrapower, since any element of \( \mathcal{D} - \mathcal{E} \) must generate \( \mathcal{D} \). The notation \( f^*X \) means the image of the set \( X \) under the function \( f \).

1. Extensions of countably generated models. The main result mentioned above will involve the construction of a sequence of \( P \) points \( \{ D_\alpha : \alpha < \omega_1 \} \) for any \( P \) point \( D \), with \( D_0 = D \), which form the desired initial segment of \( <_P,D \). At successor stages, we construct a \( P \) point \( D_{\alpha + 1} \) which is a s.i.s. of \( D_\alpha \) with a modified version of Eck’s technique; the modifications are included to make the limit stages go through. Model theoretically, the ultrapower \( D_{\alpha + 1} \)-prod is a strictly minimal, cofinal extension of (the embedded image of) \( D_\alpha \)-prod. The strategy at limit stages requires a few more definitions. Suppose \( \lambda \) is a limit ordinal and \( \{ D_\alpha : \alpha < \lambda \} \) is an RK-increasing sequence of ultrafilters. Call an ultrafilter \( E \) a strongly minimal upper bound (s.m.u.b.) for the given sequence if \( D_\alpha < E \) for all \( \alpha < \lambda \) and any strict RK predecessor of \( E \) is a predecessor of \( D_\alpha \) for some \( \alpha < \lambda \). Our construction will insure that \( D_\lambda \) is a \( P \) point and a s.m.u.b. for \( \{ D_\alpha : \alpha < \lambda \} \), and it is easy to see then that we will obtain our desired sequence. The actual construction of \( D_\lambda \) for countable limit ordinals \( \lambda \) involves an excursion into model theory, which we now describe. Suppose we have \( P \) points \( D_\alpha \) for \( \alpha < \lambda \) satisfying the description above. Let \( \alpha_1, \alpha_2, \ldots \) be a cofinal \( \omega \)-sequence in \( \lambda \), let \( E_i = D_{\alpha_i} \), and let \( p_i : \omega \rightarrow \omega \) such that \( p_i(E_{i+1}) = E_i \). Then \( p_i^* \) embeds \( E_i \)-prod into \( E_{i+1} \)-prod, and so we can form the direct limit \( \mathcal{A} \) of the system \( \{ E_i \)-prod, \( p_i^* \} : i = 1, 2, \ldots \); let \( \mathcal{E} \) be the canonical image of \( E_i \)-prod in \( \mathcal{A} \), so that \( \mathcal{A} \) is the union of the \( \mathcal{E} \). Now \( \mathcal{A} \) is a model of \( \text{Th}(N) \) and \( \mathcal{A} \) is single-skied since each of the \( E_i \) are \( P \) points and hence the models \( \mathcal{E} \) are mutually cofinal. Note that \( \mathcal{A} \) is not finitely generated and hence not isomorphic to an ultrapower.

Suppose that \( \mathcal{A} \) admits a strictly minimal, cofinal extension \( \mathcal{B} \). Then \( \mathcal{B} \) must be (isomorphic to) an ultrapower, and \( \mathcal{B} \) must be single-skied since any proper
submodel of \( \mathcal{B} \) is a submodel of, and hence cofinal in, \( \mathcal{A} \) and \( \mathcal{A} \) is cofinal in \( \mathcal{B} \). Thus \( \mathcal{B} \) is isomorphic to \( F\)-prod for some \( P \) point \( F \), and we set \( D_\lambda = F \). It is easy to check that \( D_\lambda \) is a s.m.u.b. for \( \{ D_\alpha : \alpha < \lambda \} \).

The discussion above shows that we can succeed at limit stages of our construction if we can find a strictly minimal, cofinal extension \( \mathcal{B} \) of the countably generated model \( \mathcal{A} \) which arises as a direct limit of previously constructed models. In [B3], Blass proved a characterization of those countably generated models of \( \text{Th}(N) \) which admit strictly minimal extensions. We shall require a number of modifications to that theorem, and what follows, through the proof of Theorem 2, is adapted from [B3].

Let \( \in' \) be the binary relation on \( \omega \) defined by \( m \in 'n \) iff \( 2^m \) occurs in the binary expansion of \( n \), so \( n \) codes the finite set \( \{ m : m \in 'n \} \). If \( \mathcal{A} \) is a model and \( a \in \mathcal{A} \), then let \( a(\mathcal{A}) = \{ b \in \mathcal{A} : a = (b \in 'a) \} \). If \( a \in \mathcal{A} < \mathcal{B} \), then \( a(\mathcal{A}) = a(\mathcal{B}) \cap \mathcal{A} \).

Assume that the set of finite sequences from \( \omega \) has been coded in some standard way, and let \( \langle , \ldots , \rangle \) denote the coding function. Let \( \text{Seq} \) be the set of codes, and for each \( x \in \text{Seq} \), \( lh(x) \) is the length of \( x \) and \( (x)_k \) is the \( k \)th component of \( x \) if \( k < lh(x) \) and \( (x)_k = 0 \) otherwise. Blass’s result is

**Theorem 1 (Blass [B3]) (CH).** Let \( \mathcal{A} \) be a countably generated model of \( \text{Th}(N) \). \( \mathcal{A} \) admits a strictly minimal extension if and only if for any sequence \( \{ a_i : i \in \omega \} \) with \( a_i(\mathcal{A}) \) nonempty and \( a_0(\mathcal{A}) \supseteq a_1(\mathcal{A}) \supseteq a_2(\mathcal{A}) \supseteq \cdots \), either

(i) \( \bigcap_{i \in \omega} a_i(\mathcal{A}) \neq N\emptyset \), or

(ii) for any \( b \in \mathcal{A} \), there is a \( c \in a_0(\mathcal{A}) \) and an \( f : \omega \to \omega \) such that \( *f(c) = b \).

**Remarks.** The “only if” direction does not use CH. If \( \mathcal{A} \) is finitely generated, and hence isomorphic to an ultrapower, then (i) always holds since ultrapowers are \( \aleph_1 \)-saturated [CK, p. 305], and so (CH) ultrapowers always admit strictly minimal extensions.

Our first modification takes care of insuring that the new model is a cofinal extension.

**Theorem 2 (CH).** Let \( \mathcal{A} \) be a single-skied, countably generated model of \( \text{Th}(N) \), and suppose \( \mathcal{A} \) satisfies the conditions of Theorem 1. Then \( \mathcal{A} \) admits a single-skied, strictly minimal extension.

**Proof.** Most of this proof is identical to the proof of Theorem 1 given in [B3]. First, if \( \mathcal{A} \) is finitely generated, and hence isomorphic to a \( P \) point ultrapower, then this theorem is simply Eck’s result that any \( P \) point has strong immediate successors which are \( P \) points. Assume therefore that \( \mathcal{A} \) is not finitely generated, and let \( \{ a_n : n = 1, 2, \ldots \} \) be a set which generates \( \mathcal{A} \); without loss of generality, the sequence \( \langle a_n : n = 1, 2, \ldots \rangle \) is not redundant, that is, \( a_{n+1} \) is not in the submodel of \( \mathcal{A} \) generated by \( *\langle a_1, a_2, \ldots, a_n \rangle \), and let \( g_n = *\langle a_1, \ldots, a_n \rangle \). Let \( \text{TR}_n : \text{Seq} \to \text{Seq} \) be the map which truncates sequences by removing all but the first \( n \) components (and leaves shorter sequences fixed). Then \( *\text{TR}_n(g_m) = g_n \) for all \( m > n \); note also that
$g_m$ is not in the submodel of $\mathcal{A}$ generated by $g_n$ if $n < m$ (by the nonredundancy of the $a_i$'s). Let $\mathcal{G}_n$ be the submodel generated by $g_n$.

Let $G_n$ be the type of $g_n$ in $\mathcal{A}$, that is $G_n = \{ X \subseteq \text{Seq}: \mathcal{A} \models \ast X(g_n) \}$. Then $G_n$ is an ultrafilter on $\text{Seq}$, in fact a $P$ point (since $\mathcal{A}$, and hence each of its submodels, is single-skied), and $\text{TR}_n(G_m) = G_n$ for $m > n$. $G_n$ concentrates on sequences of length $n$, that is $\{ x \subseteq \text{Seq}: \text{lh}(x) = n \} \subseteq G_n$, and the nonredundancy implies that $\text{TR}_n$ is not one-to-one on any set in $G_m$ for $m > n$.

To obtain a strictly minimal, single-skied extension of $\mathcal{A}$, it suffices to construct an ultrafilter $E$ on $\text{Seq}$ such that

1. for all $n \geq 1$, $\text{TR}_n(E) = G_n$,
2. for all $f: \text{Seq} \rightarrow \omega$, there is a set $A$ in $E$ such that either $f$ is one-to-one on $A$ or, for some $n$, $f$ is $\text{TR}_n$-fiberwise constant on $A$ (that is, $f$ is constant on sets of the form $A \cap \text{TR}_n^{-1}(i)$), and
3. for some set $B$ in $E$, $\text{TR}_1$ is finite-to-one on $B$.

Given such an ultrafilter $E$, we can embed $\mathcal{A}$ into $E$-prod by mapping $g_n$ to $[\text{TR}_n]_E$ (by (1)), and for simplicity we identify $A$ with its embedded image in $E$-prod. By (2), every element of $E$-prod either generates $E$-prod or is in the submodel $\mathcal{G}_n$ for some $n$, and so $E$-prod is a strictly minimal extension of $\mathcal{A}$. By (3), the submodel $\mathcal{G}_1$, and hence $\mathcal{A}$, is cofinal in $E$-prod. It follows that $E$-prod is single-skied since any proper submodel of $E$-prod is a submodel of (and hence cofinal in) $\mathcal{A}$, and $\mathcal{A}$ is cofinal in $E$-prod.

The existence proof for $E$ is a typical sort of inductive construction for ultrafilters on $\omega$. Call a subset $L$ of $\text{Seq}$ large if $\text{TR}_n''L \subseteq G_n$ for all $n$; otherwise $L$ is small. Any ultrafilter consisting entirely of large sets satisfies (1). Thus it suffices to construct a filter $F$ consisting of large sets, such that $F$ contains a set $B$ satisfying (3) and for each $f: \text{Seq} \rightarrow \omega$, $F$ contains a set $A$ satisfying (2). Then let $E$ be any ultrafilter extending $F$ and containing the complements of all small sets. To construct $F$, first order $\text{Seq}_\omega$ in an $\aleph_1$-sequence (by CH) and then inductively define large sets $L_\alpha$ for $\alpha < \aleph_1$ such that $L_0$ satisfies (3), $L_{\alpha + 1}$ works as $A$ in (2) for the $\alpha$th function $f$ and $L_\alpha - L_\beta$ is small for $\alpha \geq \beta$. It is easy to check that the finite union of small sets is small, and it follows that $\{ L_\alpha: \alpha < \aleph_1 \}$ generates a filter; let $F$ be this filter.

To construct $L_0$, first find a set $B_n$ in $G_n$ for $n \geq 1$ such that $B_n$ consists of sequences of length (exactly) $n$, $\text{TR}_1$ is finite-to-one on $B_n$ and $\text{TR}_1''B_n \subseteq \{ n, n + 1, n + 2, \ldots \}$. Such $B_n$ exist since the $G_n$'s are $P$ points (and $G_1$ is nonprincipal). Set $L_0 = \bigcup_{n \geq 1} B_n$, and then $\text{TR}_n''L_0$ includes $B_n$ for all $n$, so $L_0$ is large. For any $k \geq 1$, $\text{TR}_1\langle k \rangle$ is the union of $k$ finite sets, so $\text{TR}_1$ is finite-to-one on $L_0$.

The construction of $L_\lambda$ for limit $\lambda$ uses only that finite unions of small sets are small. The successor stages use the hypotheses of the theorem (that is, the conditions on $\mathcal{A}$ given in Theorem 1) to construct a large subset $A$ of $L$ which satisfies (2) for any large set $L$ and any $f: \text{Seq} \rightarrow \omega$. The details for both the limit and successor cases can be found in [B3, pp. 154–155].

By Theorem 2, we will succeed at limit stages in the construction of our sequence $\{ D_\alpha: \alpha < \aleph_1 \}$ if the associated (countably generated) model satisfies the conditions of Theorem 1. The model $\mathcal{A}$ arising in our construction has a special structure in that
those submodels of $\mathcal{A}$ which include (the embedded copy of) $D_0$-prod are linearly (in fact, well) ordered by inclusion. This makes it somewhat easier to satisfy the conditions of Theorem 1, as the next two lemmas show.

**Lemma 3 (CH).** Suppose $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \cdots$ is an ascending chain of countably generated models of Th($\mathbb{N}$) such that

(a) each $\mathcal{E}_n$ admits a strictly minimal extension,

(b) for any $b \in (\mathcal{E}_{n+1} - \mathcal{E}_n)$, the submodel generated by $b$ includes $\mathcal{E}_n$, and

(c) $\mathcal{A} = \bigcup_{n \geq 1} \mathcal{E}_n$ does not admit a strictly minimal extension.

Then there is an element $a \in \mathcal{A}$ and some $J \subseteq \omega$ such that $a(\mathcal{A}) \subseteq \mathcal{E}_J$ but $a \notin \mathcal{E}_J$.

**Proof.** First note that (b) says that for any $b \in \mathcal{E}_{n+1} - \mathcal{E}_n$ and any $c \in \mathcal{E}_n$, there is an $f: \omega \to \omega$ with $f(b) = c$. By Theorem 1, there is a sequence $\{a_i\}$ such that $a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{A})$, $a_i(\mathcal{A}) \neq \mathcal{A}$, $\bigcap_{i \geq 0} a_i(\mathcal{A}) = \emptyset$, and $a_0(\mathcal{A})$ does not generate $\mathcal{A}$ by standard unary functions. Then, for some $J$, $a_0(\mathcal{A}) \subseteq \mathcal{E}_J$, since otherwise, for arbitrarily large $k$, $a_0(\mathcal{A}) \cap (\mathcal{E}_{k+1} - \mathcal{E}_k)$ is nonempty, and then it follows from (b) that every element of $\mathcal{A}$ is obtainable from an element of $a_0(\mathcal{A})$ by a standard unary function, thus contradicting the choice of $\{a_i\}$. We have then that, for all $i$, $a_i(\mathcal{A}) \subseteq \mathcal{E}_J$, and if $a_i$ is an element of $\mathcal{E}_J$ then $a_i(\mathcal{E}_J) = a_i(\mathcal{A})$.

We now claim that for some $i$, $a_i$ is not an element of $\mathcal{E}_J$ (and then the proof is complete by setting $a = a_i$). Suppose not; then for all $i$, $a_i$ is in $\mathcal{E}_J$ and so $a_i$ is also in $\mathcal{E}_{J+1}$. Since $\mathcal{E}_{J+1}$ admits a strictly minimal extension, and since $a_i(\mathcal{E}_{J+1}) = a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{A}) = a_{i+1}(\mathcal{E}_{J+1})$, we have by Theorem 1 that either $a_0(\mathcal{E}_{J+1})$ generates $\mathcal{E}_{J+1}$ or $\bigcap_{i \geq 0} a_i(\mathcal{E}_{J+1}) = \emptyset$. The latter conclusion is impossible by the choice of $\{a_i\}$ and the fact that $a_i(\mathcal{A}) \subseteq \mathcal{E}_{J+1}$, and the former conclusion says that a subset of $\mathcal{E}_J$ generates $\mathcal{E}_{J+1}$, which is impossible since $\mathcal{E}_J$ is a proper submodel of $\mathcal{E}_{J+1}$.

Let $\mathcal{M}$ be a finitely generated model. We say that $\mathcal{M}$ is **element generated** iff for every generator $a$ of $\mathcal{M}$, there is a generator $b$ of $\mathcal{M}$ with $b \in a(\mathcal{M})$.

**Lemma 4 (CH).** Suppose that $\{\mathcal{E}_i: i = 1, 2, \ldots\}$ form an ascending chain of countably generated models, $\mathcal{A} = \bigcup_{i \geq 1} \mathcal{E}_i$, and that all the hypotheses of Lemma 3 are satisfied. Then there is a finitely generated submodel $\mathcal{M}$ of $\mathcal{A}$ which includes $\mathcal{E}_1$ and is not element generated.

**Proof.** Let $a$ and $\mathcal{E}_J$ satisfy the conclusion of Lemma 3, and let $\mathcal{M}$ be the submodel of $\mathcal{A}$ generated by $a$. Then $\mathcal{E}_J \subseteq \mathcal{E}_J \subseteq \mathcal{M}$ (the second inclusion follows from hypothesis (b) of Lemma 3), and $\mathcal{M}$ is not element generated since $a(\mathcal{M}) \subseteq \mathcal{E}_J$ and so $a(\mathcal{M})$ does not contain a generator of $\mathcal{M}$.

2. **Simple combinatorics.** Let $\langle A_1, A_2, \ldots, A_n \rangle$ be a sequence of nonempty sets. A complete set of distinct representatives (CDR) is the image of a one-to-one choice function on the set $\{A_1, \ldots, A_n\}$, that is, a sequence $\langle a_1, \ldots, a_n \rangle$ with $a_i \in A_i$ and $a_i \neq a_j$ if $i \neq j$. We will need the following simple combinatorial lemma in order to construct ultrapowers which are element generated.

**Lemma 5.** Let $\langle A_0, \ldots, A_{n-1} \rangle$ be a sequence of distinct, nonempty sets. Then there is a subsequence of length at least $\log_2(n)$ which admits a CDR.
Proof. The lemma is obvious by inspection for \( n < 5 \), so assume \( n \geq 5 \). Let 
\[
K = |\bigcup_{0 \leq i < n} A_i|.
\]
We prove the lemma by induction on \( K \). Since \( n > 4 \), we have \( K > 1 \), so assume the lemma for smaller \( K \), and fix an arbitrary \( x \in \bigcup A_i \). Reorder the \( A_i \) so that \( x \in A_i \) for \( i < p \) and \( x \notin A_i \) for \( i \geq p \), where \( p \) is the number of sets among the \( A_i \) which contain \( x \). Then \( 0 < p \leq n \).

Case 1: \( 0 < p < (n/2) \). In this case, \( \langle A_p, A_{p+1}, \ldots, A_{n-1} \rangle \) satisfies the induction hypothesis, so there is a subsequence \( \langle B_0, \ldots, B_{m-1} \rangle \) of \( \langle A_p, \ldots, A_{n-1} \rangle \) which admits a CDR \( \langle b_0, \ldots, b_{m-1} \rangle \), and \( m \geq \log_2(n-p) \geq \log_2(n/2) = \log_2(n) - 1 \). Then \( \langle x, b_0, \ldots, b_{m-1} \rangle \) is a CDR for \( \langle A_0, B_0, \ldots, B_{m-1} \rangle \), which has length \( m + 1 \geq \log_2(n) \).

Case 2: \( (n/2) \leq p \leq n \). For \( i < p \), let \( B_i = (A_i - \{x\}) \). By the induction hypothesis, there is a CDR \( \langle d_0, \ldots, d_{m-1} \rangle \) for some subsequence of the \( B_i \)'s with \( m \geq \log_2(p) \geq \log_2(n) - 1 \); without loss of generality, assume the \( A_i \) were ordered so that the subsequence of the \( B_i \)'s admitting this CDR is \( \langle B_0, \ldots, B_{m-1} \rangle \). If \( m \geq \log_2(n) \), we are done; otherwise, since \( n > 4 \) we have \( p \geq n/2 \geq \log_2(n) > m \). Thus \( \langle d_0, \ldots, d_{m-1}, x \rangle \) is a CDR for \( \langle A_0, \ldots, A_{m-1}, A_p \rangle \).

3. The successor case. The point of the theorem in this section is to insure the existence of a \( P \) point \( D_{a+1} \) which is a s.i.s. of \( D_a \) and such that \( D_{a+1} \)-prod is element generated. If \( E \) is an ultrafilter, then the generators of \( E \)-prod are the germs \([f]_E\) of one-to-one (mod \( E \)) functions \( f \), and so it follows that \( E \)-prod is element generated iff for any such one-to-one germ \( a = [f]_E \), there is a one-to-one (mod \( E \)) function \( g \) with \([g]_E \subseteq a(E \)-prod\), which means there is a set \( A \) in \( E \) such that for all \( x \) in \( A \), \( g(x) \equiv f(x) \).

The construction of \( D_{a+1} \) will be done within the framework of Theorem 2.2 of [Ro]. For the convenience of the reader we state this theorem below after supplying the requisite definitions. If \( X \) is a set and \( p \) is a function which is finite-to-one on \( X \), then the cardinality function of \( X \) with respect to \( p \), denoted \( C_{X,p} \), is defined by \( C_{X,p}(n) = |X \cap p^{-1}(n)| \). We will omit reference to \( p \) when there is no ambiguity. A (Dedekind) cut in an ultrapower \( D \)-prod is a partition of \( D \)-prod into convex sets \( S \) and \( L \) such that every element of \( S \) precedes every element of \( L \). A cut is fair if \( S \) and \( L \) are nonempty and \( L \) has no countable coinitial subset. If \( E \) is a \( P \) point and \( p : \omega \to \omega \) with \( p(E) = D \), then the cut in \( D \)-prod associated to \( p \) and \( E \) is defined by putting into \( L \) those \( D \)-germs \( [C_{X,p}]_D \) for \( X \in E \) (and all larger \( D \)-germs), and setting \( S = D \)-prod \( - L \) (see [B2] for a thorough discussion of Dedekind cuts in ultrapowers). Finally, a condition on \( X \) is simply a statement about \( X \).

Theorem 6 [Ro] (CH). Let \( D \) be a \( P \) point, \( \langle S, L \rangle \) a fair cut in \( D \)-prod such that \( S \) is closed under addition in \( D \)-prod, \( p \) the first projection from \( \omega^2 \) to \( \omega \) and \( \{C_i : i \in I\} \) a set of at most \( 2^{\aleph_0} \) conditions on subsets of \( \omega^2 \). Call a set \( X \subseteq \omega^2 \) large if it contains a subset \( Y \) on which \( p \) is finite-to-one and \( [C_Y]_D \) is in \( L \), and suppose that for any large \( X \) and condition \( C_i \), there is a large subset \( Y \) of \( X \) which satisfies \( C_i \).

Then there exist \( 2^{\aleph_1} \) many (pairwise nonisomorphic) \( P \) points \( E \) on \( \omega^2 \) with \( p(E) = D \) and associated cut \( \langle S, L \rangle \), such that for all \( i \) in \( I \), \( E \) contains a set satisfying condition \( C_i \).
PROOF. This is a special case of Theorem 2.2 in [Ro] obtained by taking the fiber measure of that theorem to be the cardinality function (see also [B2]).

**Theorem 7 (CH).** Let D be a P point, p the first projection from $\omega^2$ to $\omega$ and $[h]_D$ any element of D-prod. There exist $2^{N_1}$ P points E with $p(E) = D$ such that

1. E is a s.i.s. of D,
2. E-prod is element generated, and
3. $[h]_D$ is in the small part $S$ of the cut in D-prod associated to p and E.

**Proof.** Begin by defining functions $s_j : \omega \to \omega$ as follows:

\[
s_0(0) = h(0), \\
s_{i+1}(k) = 2^{s_i(k)}, \\
s_0(k+1) = \max \left\{ h(k+1), \sum_{j<k} s_j(j) \right\}.
\]

Define a cut $\langle S, L \rangle$ in D-prod by $a \in S$ iff $a \leq [s_j]_D$ for some $j$. Then $[h]_D \in S$ and $\langle S, L \rangle$ is a fair cut since the existence of a countable cofinal set in $S$ implies that there is no countable cofinal set in $L$ (by the $\aleph_1$-saturation of D-prod). It is easy to check that $S$ is closed under addition, multiplication and exponentiation. For each $f : \omega^2 \to \omega$, let $C_f$ be the following condition on Y: ($f$ is $p$-fiberwise constant on $Y$) or ($f$ is one-to-one on $Y$ and there is a $p$-fiberwise one-to-one function $g$ on $Y$ such that for all $y \in Y$, $g(y) \in f(y)$).

If we show that, for any $f$, any large set $X$ includes a large subset $Y$ satisfying $C_f$, then by Theorem 6 we will have shown the existence of $2^{N_1}$ many P points E with $p(E) = D$, associated cut $\langle S, L \rangle$, such that E contains sets satisfying $C_f$ for all $f$. Thus every $f$ is either $p$-fiberwise constant or (globally) one-to-one on a set in $S$; so every $[f]_E$ in E-prod either is in the embedded image (by $p^*$) of D-prod or is a generator of E-prod. It follows that E is a s.i.s. of D. To see that E-prod is element generated, let $a$ be a generator of E-prod; so $a = [f]_E$ for some function $f$ which is one-to-one on a set in $E$. Let $Y \in E$ satisfy $C_f$; then $f$ cannot be $p$-fiberwise constant on $Y$ (since $[f]_E$ is a generator of E-prod), so $Y$ satisfies the second part of $C_f$. Then there is a $p$-fiberwise one-to-one function $g$ on $X$ such that for all $y \in Y$, $g(y) \in f(y)$, and so $[g]_E$ is in a(E-prod). Now $[g]_E$ is not in $p^*''D$-prod since if it were, then $g$ would be $p$-fiberwise constant as well as $p$-fiberwise one-to-one on a set in $E$, which would imply that $p$ is one-to-one (mod E), contradicting the fairness of the associated cut. By the strict maximality of $p^*''$-prod, it follows then that $[g]_E$ is a generator of E-prod. Thus E-prod is element generated.

It remains to show that we can find a large subset satisfying any given condition $C_f$ for any large $X$. We can assume (by cutting down $X$ if necessary) that $p$ is finite-to-one on $X$. For each nonempty fiber $X_n = (X \cap p^{-1}(n))$, we can find a set $Z_n \subseteq X_n$ such that $f$ is constant on $Z_n$ or $f$ is one-to-one on $Z_n$ and $|Z_n|^2 \geq |X_n|$. Partition the fibers into sets $W_c$ and $W_1$, where $W_c$ consists of those $n$ such that $f$ is constant on $Z_n$ and $W_1$ consists of those $n$ such that $f$ is one-to-one on $Z_n$. One of these sets is in $D$ (else $X$ was not large). If $W_c$ is in $D$, then let $Y = \bigcup_{n \in W_c} Z_n$ and...
clearly $f$ is $p$-fiberwise constant on $Y$; for all $n$ in $W_c$, $(C_Y(n))^2 \geq C_X(n)$, and so $[C_Y]_D$ is in $L$ since $[C_X]_D$ is in $L$ and $S$ is closed under multiplication.

If, on the other hand, $W_1$ is in $D$, then we must cut down $Z = \bigcup_{n \in W_1} Z_n$ to a large $Y$ on which $f$ is one-to-one and for which we can find the desired function $g$. First define sets $B_i$ in $D$ ($i = 1, 2, 3, \ldots$) by

$$B_i = \{ m \in p''Z: |Z_m| \geq s_{i+1}(m) \} - \{0, 1, \ldots, m\}.$$  

Then $B_i$ is in $D$ since $Z$ is large (by the argument given above for $Y$) and hence, for all $j$, $C_Z(m)$ cannot be less than $s_j(m)$ for $D$-many $m$. Note also that $B_{i+1} \subseteq B_i$, and we may assume that $p''Z \subseteq B_0$ (throw away some fibers of $Z$ if necessary). Define $t(m) = \text{maximum } i \text{ such that } m \in B_i$ for $m$ in $B_0$. Then, for all $m \in B_0$, $t(m) < m$, $m \in B_{t(m)}$, and $|Z_m| \geq s_{t(m)+1}(m)$. Let $M$ be the minimal element of $B_0$, and let $Q_M$ be any subset of $Z_M$ with $|Q_M| = s_M(M)$. Note that $f$ is one-to-one on $Q_M$.

Let $m \in B_0$, let $H_m = \{ j \in B_0: j < m \}$, and suppose, as an induction hypothesis, that for all $j$ in $H_m$, we have defined $Q_j \subseteq Z_j$ such that $|Q_j| = s_{t(j)}(j)$ and $f$ is one-to-one on $(\bigcup_{j \in H_m} Q_j)$. Let $R_m = \{ x \in Z_m: f(x) = f(y) \text{ for some } y \in (\bigcup_{j \in H_m} Q_j) \}$. Since $f$ is one-to-one on $Z_m$, we have

$$|R_m| \leq \sum_{j \in H_m} |Q_j| = \sum_{j \in H_m} s_{t(j)}(j) \leq \sum_{j < m} s_j(j) \leq s_0(m),$$

and thus

$$|Z_m - R_m| \geq s_{t(m)+1}(m) - s_0(m) = 2^{s_{t(m)}(m)} - s_0(m) \geq s_{t(m)}(m).$$

Let $Q_m$ be any subset of $Z_m - R_m$ of size $s_{t(m)}(m)$; this completes the induction construction of the $Q_m$ for $m$ in $B_0$, for it is clear by the definition of $R_m$ that $f$ is one-to-one on $(\bigcup_{j \in H_m} Q_j) \cup Q_m$. Let $Q = \bigcup_{j \in H_m} Q_j$; then $f$ is one-to-one on $Q$ and for any $k$ in $\omega$, we have that for all $m$ in $B_k$, $|Q_m| = s_{t(m)}(m) \geq s_k(m)$, and so $[C_0]_D \geq [s_k]_D$. Thus $Q$ is large. The argument just given to make $f$ one-to-one on a large set is due to Eck [Ec].

If $f$ assumes the value 0 on $Q$, then remove that point from $Q$. Temporarily fix $m \in B_0 = p''Q$, and write $Q_m = (p^{-1}(m) \cap Q)$ as $\{a_1, a_2, \ldots, a_n\}$. Let $A_i = \{ t: t \in f(a_i) \}$; since $f$ is one-to-one on $Q$, $\{A_i: 1 \leq i \leq n \}$ is a collection of distinct nonempty sets, and so by Lemma 5, there is a subsequence $\langle B_1, \ldots, B_J \rangle$ of $\langle A_1, \ldots, A_n \rangle$ which admits a CDR $\langle u_1, \ldots, u_J \rangle$ with $J \geq \log_2(n)$. Reorder the $\{a_i\}$ so that $\langle u_1, \ldots, u_J \rangle$ is a CDR for $\langle A_1, \ldots, A_J \rangle$, and let $Y_m = \{a_1, \ldots, a_J\}$. Define $g$ on $Y_m$ by $g(a_i) = u_i$, so $g$ is one-to-one on $Y_m$.

Let $Y = \bigcup_{m \in \omega_0} Y_m$; clearly $Y$ satisfies condition $C_f$ and so we only need show that $Y$ is large. For all $m$ in $B_0$, $|Y_m| \geq \log_2(|Q_m|)$, whereupon the largeness of $Y$ follows from the largeness of $Q$ and the closure of $S$ under exponentiation.

4. The limit case. In order to apply Lemma 4 at limit stage $\lambda$ in the construction of our sequence of $P$ points, we need to know that (beyond some point in the sequence) each finitely generated submodel of the associated countably generated model $\mathcal{A}$ is element generated. The previous theorem insures this for finitely generated models.
arising at successor stages in the construction; the following theorem gives us this result for those arising at limit stages previous to \( \lambda \).

**Theorem 8 (CH).** Let \( \{ E_i : i = 1, 2, \ldots \} \) be an RK-increasing sequence of \( P \) points with \( p_i(E_{i+1}) = E_i \), and let \( \mathcal{A} \) be the direct limit of \( \langle E_i \text{-prod}, p_i^* \rangle \). For \( i \geq 2 \), let \( q_i = p_1 \circ p_2 \circ \cdots \circ p_{i-1} \) and let \( \langle S', L' \rangle \) be the cut in \( E_1 \text{-prod} \) associated to \( q_i \) and \( E_i \).

Assume that:

(a) \( S' \) admits a strictly minimal extension,

(b) \( S' \subseteq S'^{i+1} \) for all \( i \geq 2 \), and

(c) \( E_i \text{-prod} \) is element generated for all \( i \geq 1 \).

Then \( \mathcal{A} \) admits a single-skied, element generated, strictly minimal extension.

**Proof.** The proof is a modification of the proof of Theorem 2. Let \( \mathcal{E}_i \) be the canonical image of \( E_i \text{-prod} \) in \( \mathcal{A} \), and let \( a_i \) be a generator of \( \mathcal{E}_i \) in \( \mathcal{A} \). Let \( g_i \) and \( G_i \) be as in the proof of Theorem 2 (i.e. \( g_i = \tau(a_1, \ldots, a_i) \) and the ultrafilter \( G_i \) is the type of \( g_i \) in \( \mathcal{A} \)). Since the ultrafilters \( E_i \) form an increasing RK-sequence, it follows that \( g_i \) generates \( \mathcal{E}_i \) and hence \( G_i = E_i \); thus \( G_i \text{-prod} \) is element generated and if \( \langle S_i, L_i \rangle \) is the cut in \( G_i \text{-prod} \) associated to \( TR_1 \) and \( G_i \), then \( S_i \subseteq S_{i+1} \). Since \( \mathcal{A} \) admits a strictly minimal extension, the hypotheses of Theorem 1 are satisfied.

For each \( f : \text{Seq} \to \omega \), let \( C_f \) be the following condition on a subset \( X \) of \( \text{Seq} \): \( f \) is \( TR_n \)-fiberwise constant on \( X \) for some \( n \) or \( f \) is one-to-one on \( X \) and there is a \( TR_1 \)-fiberwise one-to-one function \( g \) on \( X \) such that \( g(x) = f(x) \) for all \( x \) in \( X \).

As in Theorem 2, we will construct a filter \( F \) on \( \text{Seq} \) consisting of large (in the sense of Theorem 2) sets on which \( TR_1 \) is finite-to-one as well as sets satisfying \( C_f \) for each \( f \); let \( E \) be an ultrafilter including \( F \) and the complements of all large sets. Let \( \mathcal{B} = E \text{-prod} \), and then \( \mathcal{B} \) is a single-skied, strictly minimal extension of \( \mathcal{A} \); in addition, \( \mathcal{B} \) is element generated, since the function \( g \) of condition \( C_f \) is \( TR_1 \) (and hence \( TR_n \) for all \( n \)) fiberwise one-to-one (mod \( E \)), and thus is not \( TR_n \)-fiberwise constant (mod \( E \)) for any \( n \), insuring (by strict minimality) that \( [g]_E \) is a generator of \( E \text{-prod} \).

List the conditions in an \( \mathcal{N}_1 \)-sequence. The proof proceeds exactly as for Theorem 2 (the definition of \( L_0 \) and the construction of \( L_\lambda \) for limit \( \lambda \) are identical), except that we must satisfy the stronger condition of the present theorem at successor stages. So assume that \( L \) is large and let \( C_f \) be the 8th condition. Since the hypotheses of Theorem 1 are satisfied by \( \mathcal{A} \), we can find a large \( A \subseteq L_\alpha \) such that \( f \) is one-to-one on \( A \), or, for some \( n, f \) is \( TR_n \)-fiberwise constant on \( A \). If the latter, then let \( L_\alpha+1 = A \). Assume therefore that \( f \) is one-to-one on \( A \); we must cut down \( A \) further so that we can define the desired function \( g \).

For \( n \geq 1 \), let \( R_n = TR_n ^" A \cap \{ x \in \text{Seq} : \text{lh}(x) = n \} \). Since \( A \) is large, \( R_n \) is in \( G_n \). Let \( B_1 = R_1 \), and for each \( x \) in \( B_1 \), let \( s(x, 1) \in A \) such that \( TR_1(s(x, n)) = x \).

Assume we have defined \( B_j \) for \( j < K \) such that \( B_j \subseteq R_j \), \( B_j \in G_j \), with \( TR_j ^" B_{j+1} \subseteq B_j \), and we have defined a function \( s \) such that \( s(x, j) \in A \) and \( TR_j(s(x, j)) = x \) for each \( x \) in \( B_j \).

Let

\[ B_K = (R_K \cap TR_K^1(B_{K-1})) - W, \]

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where \( W = \{ z \in R_K : z = \text{TR}_K(s(x, K - 1)) \text{ for some } x \in B_{K-1} \} \). It is easy to see that \( \text{TR}_{K-1} \) is one-to-one on \( W \) (two different \( z \)'s which map to \( x \) under \( \text{TR}_{K-1} \) can not both be truncations of \( s(x, K - 1) \)), and so \( W \) is not in the ultrafilter \( G_K \) (since \( \text{TR}_{K-1} \) maps \( G_K \) to \( G_{K-1} \) and is not an isomorphism). It follows that \( B_K \) is in \( G_K \), since both \( R_K \) and \( \text{TR}_{K-1}^{-1}(B_{K-1}) \) are in \( G_K \). For each \( x \in B_K \), let \( s(x, K) \) be an element of \( A \) such that \( \text{TR}_K(s(x, K)) = x \). This completes the inductive construction of the \( B_K \).

Let \( P = \{ s(x,n) : n \geq 1 \text{ and } x \in B_n \} \). Then \( P \subseteq A \), \( P \) is large, and by the construction we have that for all \( s \) in \( P \), there is a unique \( n \) and a unique \( x \) in \( B_n \) with \( s = s(x, n) \). Define functions \( f_n \) on \( B_n \) by \( f_n(x) = f(s(x, n)) \). Then \( f_n \) is one-to-one on \( B_n \) since \( f \) is one-to-one on \( A \). Since \( G_n \)-prod is element generated, we can find, for each \( n \), a set \( Z_n \) in \( G_n \), \( Z_n \subseteq B_n \), and a one-to-one function \( g_n \) on \( Z_n \) such that for all \( x \) in \( Z_n \), \( g_n(x) \in f_n(x) \).

Define \( g : P \to \omega \) by \( g(s(x, n)) = g_n(x) \). Then \( g \) is well defined, and for all \( s \) in \( P \), \( g(x) \in f(x) \). We need only cut down \( P \) to a large \( L_{\alpha+1} \) on which \( g \) is \( \text{TR}_1 \)-fiberwise one-to-one.

Since \( S_1 \subseteq S_{1+1} \), we can find functions \( h_i : \text{Seq}_1 \to \omega \) such that \( [h_i] \subseteq L_i - L_{i+1} \) (\( \text{Seq}_1 \) is the set of codes for sequences of length 1). Let \( P_1 = Z_1 \), and assume that we have defined sets \( P_i \subseteq Z_i \) for \( 1 \leq i < K \) such that

1. \( P_i \subseteq G_i \), and \( \text{TR}_{i-1}(P_i) \subseteq P_{i-1} \),
2. for all \( i \geq 2 \), \( |\text{TR}_i^{-1}(P_i) \cap P_j| \leq h_i(P) \) for all \( P \) in \( P_i \), and
3. \((\forall \langle p \rangle \in P_i)(\forall i,j < K)(\forall y,z \in (\text{TR}_1^{-1}(\langle p \rangle) \cap (\bigcup_{n < K} P_n)))(g_f(y) = g_j(z) \text{ iff } i = j \text{ and } y = z)\).

To define \( P_K \), first let \( V = \{ z \in Z_K \cap \text{TR}_K^{-1}(P_{K-1}) : (\exists j < K)(\exists y \in P_j)(g_k(z) = g_j(y) \text{ and } \text{TR}_1(y) = \text{TR}_1(z)) \} \).

If \( P \in P_i \), then

\[ |\text{TR}_i^{-1}(\langle p \rangle) \cap V| \leq 1 + \sum_{j < K} |P_j \cap \text{TR}_i^{-1}(\langle p \rangle)| \leq 1 + \sum_{j < K} h_j(\langle p \rangle) \]

(the first inequality follows since \( g_k \) is one-to-one on \( Z_K \); the second follows from (2)). Thus \( [C_V]_{G} \leq 1 + \sum_{j < K} [h_j]_{G} \), and since \( S_K \) is closed under addition (Theorem 2 of [B2]), it follows that \( [C_V]_{G} \subseteq S_K \) (as \( [h_j]_{G} \subseteq S_K \) for all \( j < K \)). Thus \( V \) is not in \( G_K \). Since \( [h_K]_{G} \subseteq L_K \), we can find \( W \) in \( G_K \) such that for all \( \langle p \rangle \) in \( P_1 \), \( C_W(\langle p \rangle) \leq h_K(\langle p \rangle) \).

Let

\[ P_K = (Z_K \cap W \cap \text{TR}_K^{-1}(P_{K-1})) - V. \]

Then \( P_K \) obviously satisfies (1) and (2), and it is easy to check, from the definition of \( V \), that \( P_K \) satisfies (3).

We have inductively defined \( \{ P_i : j = 1, 2, 3, \ldots \} \), and we set \( L = s''((\bigcup P_n \times \{ n \})) \). Then \( \text{TR}_K''L \) includes \( P_n \), and \( P_n \) is in \( G_n \), so \( L \) is large. If \( s \) and \( t \) are in \( L \) with \( g(s) = g(t) \) and \( \text{TR}_1(s) = \text{TR}_1(t) \), then there are unique \( x, y, m, n \) with \( x \in P_m \), \( y \in P_n \), \( s = s(x, m) \) and \( t = s(y, n) \) (by the construction of the set \( P \)). By the definition of \( g \), \( g_m(x) = g_n(y) \), and by (3) above, \( m = n \) and \( x = y \), and so \( s = t \). Thus \( g \) is \( \text{TR}_1 \)-fiberwise one-to-one on \( L \), and the proof is complete by setting \( L_{\alpha+1} = L \).
5. The finale. We now combine the previous results to produce our desired sequence of $P$ points.

**Theorem 9 (CH).** Let $D$ be a $P$ point. There exist initial segments of $<_P D$ of order type $\aleph_1$.

**Proof.** We construct a sequence of $P$ points $\{D_\alpha; \alpha < \aleph_1\}$ with $D_0 = D$ such that:

1. $D_{\alpha+1}$ is a strong immediate successor of $D_\alpha$,
2. for limit $\lambda < \aleph_1$, $D_\lambda$ is a strongly minimal upper bound for $\{D_\alpha; \alpha < \lambda\}$,
3. $D_\alpha$-prod is element generated for $\alpha \geq 1$, and
4. if $q_\alpha$ is the map from $D_\alpha$ to $D_1$, and $\langle S_\alpha, L_\alpha \rangle$ is the cut in $D_1$-prod associated to $q_\alpha$ and $D_\alpha$, then $S_\beta \subseteq S_\alpha$ whenever $\beta < \alpha$.

By (1) and (2), such a sequence forms an initial segment of $<_P D$.

If $\alpha = 1$, let $D_1$ be a s.i.s. of $D$ such that $D_1$-prod is element generated; we can find such a $D_1$ by Theorem 7.

If $\alpha = \beta + 1$, fix $g: \omega \to \omega$ such that $[g]_{D_\beta}$ is in the large part $L_\beta$ of the cut in $D_\beta$-prod associated to $D_\beta$ and $D_\beta$ (=$[g \circ q_\beta]_D$) is in the small part $S'$ of the cut in $D_\beta$-prod associated to $D$ and $D_\beta$ (since $D_\beta$ is the map from $D_\beta$ to $D_1$). If $\beta$ is any set in $D_\beta$, then we have that for all $n$ in some set $B$ in $D_\beta, g(q_\beta(n)) \leq (\beta \cap \beta^{-1}(n))$. It is easy then to check that for all $z$ in $q_\beta^{-1}(B)$, $A \cap q_\beta^{-1}(z) \geq g(z)$ ($q_\alpha = q_\beta \circ p$ is the map from $D_\alpha$ to $D_1$); since $q_\beta^{-1}(B) \subseteq D_1$, we have that $[C_\beta]_{D_\alpha} \supseteq [g]_{D_\beta}$, and so $[g]_{D_\beta}$ is in the small part $S_\alpha$ of the cut in $D_1$-prod associated to $q_\alpha$ and $D_\alpha$. Thus $[g]_{D_\beta}$ witnesses that $D_\alpha$ satisfies (4).

If $\alpha$ is a countable limit ordinal, let $\lambda = \alpha_1, \alpha_2, \ldots$ be an increasing $\omega$-sequence cofinal in $\alpha$, and let $E_i = D_{\alpha_i}$. Let $p_i$ be the map from $E_{i+1}$ to $E_i$, let $\mathcal{A}$ be the direct limit of $\{E_i$-prod, $p_i^*\}$, and let $\bar{\mathcal{A}}$ be the canonical image of $E_i$-prod in $\mathcal{A}$. We claim first that $\mathcal{A}$ admits a strictly minimal extension.

To prove this claim, first note that each $E_i$ admits a strictly minimal extension (since each $E_i$ is one of the previously constructed $D_\alpha$'s) and that any element of $\bar{\mathcal{A}}$ generates a submodel which includes $\bar{\mathcal{A}}$ (since the $D_\alpha$'s already constructed form an initial segment of $<_P D$). If $\mathcal{A}$ did not admit a strictly minimal extension, then by Lemma 4 there would be a finitely generated submodel $\mathcal{M}$ of $\mathcal{A}$ which includes $\bar{\mathcal{A}}$, but is not element generated. But $\mathcal{M}$ is the embedded image of one of the $D_\alpha$-prod (again since they form an initial segment) and so $\mathcal{M}$ must be element generated by the induction hypothesis. This contradiction proves the claim.

By the induction hypothesis, each of the $\bar{\mathcal{A}}$ is element generated and if $i < j$, then $S_{\alpha_i} \subseteq S_{\alpha_j}$. By Theorem 8, $\mathcal{A}$ admits a strictly minimal extension $\mathcal{B}$ which is single-skied and element generated. Let $D_\beta$ be an ultrafilter such that $D_\beta$-prod is isomorphic to $\beta$; then $D_\beta$ is a $P$ point and is a s.m.u.b. for $\{D_\alpha; \alpha < \lambda\}$. Finally, if $\alpha < \lambda$, it is easy to check that $S_{\alpha+1} \subseteq S_\lambda$ (since the map $q_\lambda$ factors through $q_{\alpha+1}$) and so $S_{\alpha} \subseteq S_\lambda$, showing that $D_\alpha$ satisfies (4).

This completes the inductive construction of $\{D_\alpha; \alpha < \aleph_1\}$, and with it, the proof of the theorem.
Corollary 10 (CH). There exist initial segments of the RK ordering, restricted to (isomorphism classes of) $P$ points, of order type $\mathfrak{S}_1$.

Proof. This is simply Theorem 9 with $D$ an RK-minimal ultrafilter. The existence of such $D$ follows from CH (see [Pu], for example); ultrafilters which are minimal in RK are also called selective or Ramsey, and they are all $P$ points.

Corollary 11 (CH). There exist $P$ points with countably many constellations; in fact, for any countable ordinal $\alpha$, there exist $P$ points $E$ such that the initial segment of RK determined by $E$ has order type $\alpha$.

Proof. Immediate by the previous corollary.

Corollary 12 (CH). For any $P$ point $D$, there exist initial segments $T$ of $<_P, D$ such that $T$ is a tree with $\mathfrak{S}_1$ levels, each node has $2^{\mathfrak{S}_1}$ immediate successors, and each countable increasing sequence in $T$ has a unique upper bound in $T$.

Proof. The proof is the same as the proof of Theorem 9, except at successor stages, use Theorem 7 to generate $2^{\mathfrak{S}_1}$ immediate successors.

6. Two questions. We conclude with two natural questions.

1. Given CH, is there an RK-increasing $\omega$-sequence of $P$ points which does not admit a s.m.u.b. which is a $P$ point? In other words, do we really need element generated models to prove the theorems here?

2. Can our result extend beyond $\mathfrak{S}_1$? That is, do there exist (CH) initial segments of $<_P, D$ of order type $\mathfrak{S}_1 + 1$ or perhaps $\mathfrak{S}_2$? In general, the successor cardinal of the continuum would be the best possible.

References


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