DIRECTED SETS AND COFINAL TYPES

BY

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Abstract. We show that 1, ω, ω₁, ω × ω₁ and [ω₁]^<ω are the only cofinal types of directed sets of size ℵ₁, but that there exist many cofinal types of directed sets of size continuum.

A partially ordered set D is directed if every two elements of D have an upper bound in D. In this note we consider some basic problems concerning directed sets which have their origin in the theory of Moore-Smith convergence in topology [12, 3, 19, 9]. One such problem is to determine "all essential kind of directed sets" needed for defining the closure operator in a given class of spaces [3, p. 47]. Concerning this problem, the following important notion was introduced by J. Tukey [19]. Two directed sets D and E are cofinally similar if there is a partially ordered set C in which both can be embedded as cofinal subsets. He showed that this is an equivalence relation and that D and E are cofinally similar iff there is a convergent map from D into E and also a convergent map from E into D. The equivalence classes of this relation are called cofinal types. This concept has been extensively studied since then by various authors [4, 13, 7, 8]. Already, from the first introduction of this concept, it has been known that 1, ω, ω₁, ω × ω₁ and [ω₁]^<ω represent different cofinal types of directed sets of size ≤ ℵ₁, but no more than five such types were known. The main result of this paper shows that 1, ω, ω₁, ω × ω₁ and [ω₁]^<ω are the only cofinal types of convergence in spaces of character ≤ ℵ₁ which can be constructed without additional set-theoretic assumptions. On the other hand, we shall construct many different cofinal types of directed sets of size continuum. This gives a solution to Problem 1 of J. Isbell [7]. The paper also contains several results about the structure of the class of all cofinal types, as well as a result about decomposing arbitrary partially ordered sets into directed sets. The results of this note were proved in February-March 1982 and presented to the ASL in January 1983.

1. A decomposition theorem. In this section we show that an arbitrary partially ordered set can be decomposed into a number of its directed subsets depending on the sizes of its antichains. This result is connected with an unpublished problem of F. Galvin concerning the Dilworth decomposition theorem [6] and it generalizes a similar result of E. Milner and K. Prikry [11]. The transitivity condition of a partial ordering is not used in our proof, so we state our result so as to apply to an arbitrary

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binary relation rather than to a partial ordering itself. In this form the result also
subsumes a well-known partition relation for cardinals and it might have some
further applications.

Let \( \langle A, R \rangle \) be a fixed structure, where \( R \) is a binary relation on \( A \), and let \( \kappa, \lambda \) be
cardinals such that \( \kappa \geq \omega \) and \( \lambda \geq 3 \). We say that \( D \subseteq A \) is a \( \lambda \)-directed subset of
\( \langle A, R \rangle \) iff for all \( C \in [D]^{<\lambda} \) there is a \( d \in D \) such that for all \( c \in C \), \( c R d \) or \( c = d \).
If \( a, b \in A \), then \( a \parallel_{\kappa} b \) denotes the fact that \( a \neq b \) and \( (a, b) \notin R \) and \( (b, a) \notin R \).
We say that \( B \subseteq A \) is an antichain of \( \langle A, R \rangle \) iff \( a \parallel_{R} b \) for all \( a, b \in B \), \( a \neq b \). By
\( \kappa \ll \lambda \) we denote the fact that \( \kappa < \lambda \) and \( \rho^\kappa < \lambda \) for all \( \sigma < \kappa \) and \( \rho < \lambda \).

**Theorem 1.** Assume \( \lambda \) is regular and \( \kappa \ll \lambda \). Then any structure \( \langle A, R \rangle \) with no
antichain of size \( \kappa \) is the union of \( < \lambda \) of its \( \lambda \)-directed subsets.

**Proof.** Let \( \theta \) be a large enough regular cardinal and let \( M < H_\theta \) be such that \( \kappa, \lambda, \langle A, R \rangle \in M, M \cap \lambda \in \text{Ord}, [M]^{<\kappa} \subseteq M \) and \( M \) has size \( < \lambda \). We claim that
\[
A = \bigcup \{ D : D \subseteq M \text{ and } D \text{ is a } \lambda \text{-directed subset of } \langle A, R \rangle \}.
\]
Otherwise, pick a \( b \in A \) not in this union. By induction on \( \alpha < \kappa \) we define an
antichain \( \{ b_\alpha : \alpha < \kappa \} \subseteq A \cap M \) as follows. Assume \( \alpha < \kappa \) and \( \{ b_\beta : \beta < \alpha \} \subseteq A \cap M \) is an antichain of \( \langle A, R \rangle \) such that \( b_\beta \parallel \alpha \) for all \( \beta < \alpha \). Define \( D = \{ d \in A : b_\beta \parallel d \text{ for all } \beta < \alpha \} \). Then by our assumptions on \( M, D \in M \) and \( b \in D \). Hence \( D \) is not \( \lambda \)-directed. So there is a \( C \in [D]^{<\lambda} \) in \( M \) which has no upper bounds in \( D \). So
in particular \( b \) is not an upper bound of \( C \). Since \( C \subseteq M \) there must be a \( c \in C \) such
that \( c \parallel b \). So we can put \( b_n = c \). This completes the proof.

**Theorem 2.** Every structure \( \langle A, R \rangle \) with no antichain of size \( \kappa \) is the union of \( \leq \lambda^\kappa \)
of its \( \lambda^\kappa \)-directed subsets.

**Proof.** If \( \kappa \) is a regular cardinal the result follows from Theorem 1. So assume
\( \kappa > \text{cf} \kappa \) and fix \( \langle \kappa : \xi < \text{cf} \kappa \rangle \uparrow \kappa \). Pick a chain \( \{ M_\xi : \xi < \text{cf} \kappa \} \) of elementary
submodels of \( H_\theta \) such that \( \kappa, \lambda, \langle A, R \rangle \in M_\xi, M_\xi \cap \lambda^+ \in \text{Ord}, [M_\xi]^{<\kappa} \subseteq M_\xi, M_\xi \) has
size \( \lambda^\kappa \) and \( M_\xi \in M_{\xi+1} \). Working as in the proof of Theorem 1 one shows that
\[
A = \bigcup \{ D : D \subseteq \bigcup_{\xi < \text{cf} \kappa} M_\xi \text{ and } D \text{ is } \lambda^\kappa \text{-directed} \}.
\]
This will clearly finish the proof of Theorem 2.

Assume \( \lambda \) is regular and \( \kappa \ll \lambda \). Let \( [\lambda]^2 = K_0 \cup K_1 \) be a given partition. Define
\( R \subseteq \lambda \times \lambda \) by \( (\alpha, \beta) \in R \) iff \( \alpha < \beta \) and \( \{ \alpha, \beta \} \in K_0 \). If there is no \( B \subseteq [\lambda]^\kappa \) such
that \( [B]^2 \subseteq K_1 \), then by Theorem 1 there must be a \( C \subseteq [\lambda]^{\lambda^\kappa} \) such that \( [C]^2 \subseteq K_0 \).
Hence the well-known partition relation \( \lambda \rightarrow (\lambda, \kappa)^2 \) is a consequence of Theorem 1.

Assume \( \kappa \) and \( \lambda \) are as above and \( D \) is a directed set of cofinality \( \text{cf} \kappa \) of \( D \) equal to \( \lambda \).
Applying Theorem 1 to \( D \) it follows that \( D \) contains a cofinal subset which is the
union of \( \lambda \) chains of order type \( \lambda \). This has been proved first by R. Laver [10, p.
100] for the case \( \kappa = \aleph_0 \) and then by Milner and Prikry [11] in general.

Note that the proof of Theorem 1 also gives the following fact: If \( \kappa \) is a weakly
compact cardinal, then any structure \( \langle A, R \rangle \) with no antichain of size \( \kappa \) is the union
of \( < \kappa \) of its \( \kappa \)-directed subsets.
2. Cofinal types. We begin this section with a central notion in the theory of directed sets introduced by Tukey [19]. For two directed sets $D$ and $E$, we say that $D$ is cofinally finer than $E$, and write $D \succ E$, iff there exists a convergent map from $D$ into $E$, i.e., a function $f: D \to E$ such that for all $e \in E$ there is a $d \in D$ such that $f(c) \geq e$ for all $c \geq d$. It is clear that $\succ$ is a transitive relation on the class of all directed sets. The following useful fact is implicit in Tukey [19] and explicit in Schmidt [13].

**Proposition 1** [19, 13]. $D \succ E$ iff there is a map $g: E \to D$ which maps unbounded sets into unbounded sets.

**Proof.** Let $f: D \to E$ be convergent. Pick a function $g: E \to D$ such that $f(c) \geq e$ for all $c \geq g(e)$ in $D$. Then $g$ maps unbounded sets into unbounded sets.

Conversely, let $g: E \to D$ map unbounded sets into unbounded sets. Pick an $f: D \to E$ such that for all $d \in D$, $f(d)$ is an upper bound of $\{e \in E: g(e) \leq d\}$. Then $f$ converges.

A function $g: E \to D$ which maps unbounded sets into unbounded sets is called a Tukey function. The next result of Tukey [19] shows that the relation of cofinal similarity is an equivalence relation and that in fact it coincides with the relation $\equiv$ generated by the quasi-ordering $\succ$.

**Theorem 3** [19]. $D \succ E$ and $E \succ D$ iff there is a partially ordered set $C$ in which both $D$ and $E$ can be embedded as cofinal subsets.

**Proof.** Only the direct implication is nontrivial. So assume $D \succ E$ and $E \succ D$. Working as in the proof of Proposition 1, we can find functions $f: D \to E$ and $g: E \to D$ such that if $d \in D$ and $e \in E$, then $d \geq_D g(e)$ implies $f(d) \geq_E e$, and $e \geq_E f(d)$ implies $g(e) \geq_D d$. Assume $D \cap E = \emptyset$ and define $\leq_*$ on $D \cup E$ as follows: $\leq_*$ on $D$ is equal to $\leq_D$ and $\leq_*$ on $E$ is equal to $\leq_E$. For $d \in D$ and $e \in E$ we put $e \leq_* d$ if $g(e') \leq_D d$ for some $e' \geq_E e$, and $d \leq_* e$ if $f(d') \leq_E e$ for some $d' \geq_D d$. Then $\leq_*$ is a quasi-ordering on $D \cup E$. Let $\sim$ be the equivalence relation generated by $\leq_*$. Then $D$ and $E$ are isomorphic to cofinal subsets of $\langle D \cup E/\sim, \leq_* \rangle$ via mappings $d \to [d]$ and $e \to [e]$, respectively.

The next simple and useful result of Tukey [19] shows that the class of all cofinal types forms an upper semilattice.

**Proposition 2** [19]. If $n$ is finite, then $D_1 \times \cdots \times D_n$ is the least upper bound of $D_1, \ldots, D_n$.

**Proof.** Suppose $f_i: E \to D_i$ converges for $i = 1, \ldots, n$. Define $f: E \to D_1 \times \cdots \times D_n$ by $f(e) = (f_1(e), \ldots, f_n(e))$. Then $f$ is convergent.

In [7], Isbell showed that the upper semilattice of cofinal types is not a lattice and that Proposition 2 does not hold for infinite $n$.

Standard examples of directed sets are sets of the form $[\kappa]^{< \lambda}$ ordered by $\subseteq$. Note that by Proposition 1, $[\kappa]^{< \omega}$ is cofinally finer than any directed set of size $\leq \kappa$. The particular place of sets of the form $[\kappa]^{< \lambda}$ in the class of all directed sets has been studied by various authors [4, 13, 7]. Many natural questions about directed sets of
the form $[\kappa]^{<\lambda}$ have also been considered by various authors in connection with the well-known singular cardinals problem in set theory, and many of them are still open. For example, it can be proved that $[\omega_\alpha]^{\aleph_\alpha} = [\omega_\alpha + 1]^{\aleph_\alpha}$ holds iff there is a family $\mathcal{F} \subseteq \mathcal{P}(\omega_\alpha)$ of size $> \aleph_\omega$ such that $\{ f \cap X : f \in \mathcal{F} \}$ has size $\leq$ size of $X$ for all infinite $X \subseteq \omega_\alpha$ of size $< \aleph_\omega$, which is a well-known set-theoretical statement. We shall now give a natural generalization of sets of the form $[\kappa]^{<\lambda}$ in order to produce many nonequivalent directed sets. We shall restrict ourselves to the case $\lambda = \aleph_1$ since this suffices for all of our applications, but let us note that all our definitions and results have obvious generalizations to higher cardinals $\lambda$. By $\mathcal{D}_\kappa$ we shall denote the set of all cofinal types of directed sets of size $\leq \kappa$. Note that $\mathcal{D}_{\aleph_0} = \{1, \omega\}$, so the first nontrivial problem is to determine the possible structure of $\mathcal{D}_{\aleph_1}$.

Let $\lim(\omega)$ be the class of all ordinals of cofinality $\leq \omega$, and let $S \subseteq \lim(\omega)$ be a nonempty set. Then by $D(S)$ we denote the set of all countable sets $C \subseteq S$ such that $\sup(C \cap \alpha) \in C$ for all $\alpha$. We consider $D(S)$ as a directed set partially ordered by $\subseteq$. It is interesting to note that many known examples of cofinal types can be represented with $D(S)$ for some $S \subseteq \lim(\omega)$. For example,

$$D(1) = 1, \quad D(\omega) = \omega, \quad D(\omega_1) = \omega_1, \quad D(\{\alpha + 1 : \alpha < \omega_1\}) = [\omega_1]^{<\omega}, \quad \text{etc.}$$

**Lemma 1.** Let $\kappa > \omega$ be regular and let $S, S' \subseteq \lim(\omega) \cap \kappa$ be such that $S'$ is unbounded in $\kappa$ and $\lim(\omega) \setminus S'$ is stationary in $\kappa$. Then $D(S) \not\leq D(S')$ implies that $S \setminus S'$ is nonstationary in $\kappa$ and that $S$ is unbounded in $\kappa$.

**Proof.** Let $g : D(S') \to D(S)$ be a given function and let us assume that $S \setminus S'$ is stationary in $\kappa$. We shall show that $g$ is not a Tukey function. A very similar proof will show that $g$ is not Tukey if $S$ is bounded in $\kappa$.

Let $\theta$ be a large enough regular cardinal. Pick a countable $N < H_\theta$ such that $g, S, S' \subseteq N$ and $\sup(N \cap \kappa) \in S \setminus S'$. Let $F$ be the closure of $N \cap \kappa$, and let $\{\alpha_n : n < \omega\}$ be an enumeration of $F \setminus S$.

Since $N < H_\theta$ by induction on $n < \omega$ we can choose a sequence $S'_0 \supseteq S'_1 \supseteq \cdots$ of members of $N \cap [\kappa]^\omega$ and a sequence $\langle I_n, n < \omega \rangle$ of intervals of $\kappa$ such that

1. $\alpha_n \in I_n$ and $I_n \subseteq N$,
2. $g(\{\delta\}) \cap I_n = \emptyset$ for all $\delta \in S'_n$.

Now pick increasing $\delta_n \in N \cap S'_n (n < \omega)$ converging to $\sup(N \cap \kappa)$ and let

$$C = \bigcup_{n < \omega} g(\{\delta_n\}).$$

Then $C \subseteq S$ and $C$ is an upper bound of $\{g(\{\delta_n\}) : n < \omega\}$ in $D(S)$. But $\{\{\delta_n\} : n < \omega\}$ has no upper bounds in $D(S')$, so $g$ is not Tukey.

**Theorem 4.** Card $\mathcal{D}_{\aleph_\omega} \geq 2^\kappa$ for all regular $\kappa$.

**Proof.** Let $\mathcal{S}$ be a family of size $2^\kappa$ of subsets of $\lim(\omega) \cap \kappa$ such that $S \setminus S' \in \text{stat } \kappa$ for all distinct $S, S' \in \mathcal{S}$. Then by Lemma 1, $D(S) \not\leq D(S')$ for all $S, S' \in \mathcal{S}$ with $S \neq S'$.

**Corollary 5 (GCH).** Card $\mathcal{D}_\kappa = 2^\kappa$ for all regular $\kappa > \omega$. 

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Let us now consider the Cartesian product of two directed sets of the form $D(S)$.

**Lemma 2.** Let $\kappa > \omega$ be regular and let $S, S' \subseteq \lim(\omega) \cap \kappa$ be unbounded in $\kappa$. Then $D(S) \times D(S') \not\leq [\kappa]^{<\omega}$ iff $S \cap S'$ is nonstationary in $\kappa$.

**Proof.** Let $S \cap S'$ be stationary and let $g: [\kappa]^{<\omega} \to D(S) \times D(S')$ be given. Pick a countable $N < H_\theta$ such that $g, S, S' \in N$ and $\sup(N \cap \kappa) \in S \cap S'$. Working as in the proof of Lemma 1, we can find an increasing sequence $\{\delta_n: n < \omega\} \subseteq N \cap \kappa$ such that
\[
\bigcup_{n < \omega} \pi_1 \circ g(\{\delta_n\}) \subseteq S \quad \text{and} \quad \bigcup_{n < \omega} \pi_2 \circ g(\{\delta_n\}) \subseteq S',
\]
whence $g$ is not a Tukey function.

Conversely, assume $S \cap S'$ is nonstationary in $\kappa$. Pick a club $C$ such that $C \cap S \cap S' = \emptyset$. Pick sequences $\langle \alpha_\xi: \xi < \kappa \rangle$ and $\langle \alpha'_\xi: \xi < \kappa \rangle$ in $S$ and $S'$, respectively, such that for all $\xi < \eta < \kappa$ there is a $\gamma \in C$ such that $\alpha_\xi, \alpha'_\xi < \gamma < \alpha_\eta, \alpha'_\eta$. Let $X = \{\langle \alpha_\xi, \alpha'_\xi \rangle: \xi < \kappa\}$. Then $X$ is a subset of $D(S) \times D(S')$ of size $\kappa$, every infinite subset of which is unbounded in $D(S) \times D(S')$. Thus $D(S) \times D(S') \not\leq [\kappa]^{<\omega}$.

Since $D(S) \not\leq [\kappa]^{<\omega}$ iff $D(S) \times D(\lim(\omega) \cap \kappa) \not\leq [\kappa]^{<\omega}$, Lemma 2 gives the following

**Lemma 3.** Let $\kappa > \omega$ be regular and let $S \subseteq \lim(\omega) \cap \kappa$ be unbounded in $\kappa$. Then $D(S) \not\leq [\kappa]^{<\omega}$ iff $S$ is nonstationary in $\kappa$.

**Theorem 6.** For every regular cardinal $\kappa > \omega$ there exist directed sets $D$ and $E$ of size $\kappa^{\aleph_0}$ such that $D, E \not\leq [\kappa]^{<\omega}$ but $D \times E \not\leq [\kappa]^{<\omega}$.

The best previous result in the direction of Theorem 4 is due to Isbell [8] who showed that $\text{Card } \mathcal{D}_{\aleph_0} \geq 7$. Concerning Corollary 5, let us note that J. Stepräns [16] (see also [5]) has previously shown, using a forcing argument, that $\text{Card } \mathcal{D}_{\aleph_1} = 2^{\aleph_1}$ is consistent. In his model CH holds. Theorem 6 solves Problem 3 of Isbell [7]. Let us note that, in $L$, Theorems 4–6 also hold for all singular cardinals $\kappa$.

3. Martin's Axiom and cofinal types. In the last two sections of this paper we consider the problem about the possible size of the set $\mathcal{D}_{\aleph_1}$ of all cofinal types of directed sets of cardinality $\leq \aleph_1$. The following simple fact is implicit in Tukey [18] and more explicit in Isbell [7, 8].

**Proposition 3 [19, 7, 8].** Let $D$ be a directed set of size $\leq \aleph_1$. Then either $D \equiv 1$, or $D \equiv \omega$, or $D \equiv \omega_1$, or $[\omega_1]^{<\omega} \not\leq D \not\leq \omega \times \omega_1$.

**Proof.** We may assume $\text{cf } D = \aleph_1$ in which case $D \not\leq \omega_1$. If $D$ contains no unbounded countable subset, then clearly $D \equiv \omega_1$. Otherwise $D \not\geq \omega$ so, by Proposition 2, $D \not\geq \omega \times \omega_1$.

Note also the following consequence of a result mentioned in §1. If $D$ is a directed set of size $\leq \aleph_1$ with no infinite antichain, then either $D \equiv 1$, or $D \equiv \omega$, or $D \equiv \omega_1$, or $D \equiv \omega \times \omega_1$. 

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All relations between the basic five elements of $\mathcal{D}_\aleph_1$ are straightforward [19]: 1 is the minimal element and $[\omega_1]^{< \omega}$ is the maximal element of $\mathcal{D}_\aleph_1$; $\omega$ and $\omega_1$ are two immediate (incomparable) successors of 1 with join $\omega \times \omega_1$. The next result shows the strong effect of Martin’s Axiom on cofinal types of size $\aleph_1$.

**Theorem 7 (MAK).** Let $D$ be a directed set of size $\aleph_1$ in which every uncountable set contains a countable subset unbounded in $D$. Then $D \equiv [\omega_1]^{< \omega}$.

**Proof.** Clearly if $D = \aleph_1$ so we may and will assume that the domain of $D$ is equal to $\omega_1$ and that $\alpha < D \beta$ implies $\alpha < \beta$. Let us consider the following two cases:

**Case 1.** There is a strictly increasing continuous sequence $\langle N_\xi : \xi < \omega_1 \rangle$ of countable elementary submodels of $H_\aleph_2$ containing $D$ such that the following set is cofinal in $D$.

$$E = \{ d \in D : \text{if } \xi < \omega_1 \text{ and if } d \notin N_{\xi+1} \text{ then there is } c \in N_{\xi+1} \setminus N_\xi \text{ such that } c \leq D d \}.$$

Define $\mathcal{P}_1$ to be the set of all pairs $p = \langle A_p, B_p \rangle$, where $A_p$ and $B_p$ are finite subsets of $E$. We consider $\mathcal{P}_1$ as a partially ordered set ordered by $q < p$ iff $A_q \supseteq A_p$, $B_q \supseteq B_p$ and $a \not\leq_D b$ for all $a \in A_q \setminus A_p$ and $b \in B_p$.

**Claim 1.** $\mathcal{P}_1$ is a ccc poset.

**Proof.** Let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be a given sequence of elements of $\mathcal{P}_1$. We may assume that for some $n \geq 1$, $A_{p_\alpha}$ has size $n$ for all $\alpha$, and that $A_{p_\alpha} \cup B_{p_\alpha} < A_{p_\beta} \cup B_{p_\beta}$ for $\alpha < \beta$. For $\alpha < \omega_1$, let $b_\alpha$ be a fixed upper bound of $B_{p_\alpha}$ and let $\{a_1^\alpha, \ldots, a_n^\alpha\}$ be the increasing enumeration of $A_{p_\alpha}$. Define

$$X_1 = \{ a \in D : \{ \alpha < \omega_1 : a \leq a_1^\alpha \} \text{ is uncountable} \}.$$

Then by the choice of $E$, $X_1$ is an uncountable subset of $D$. So there is a countable set $C_1 \subseteq X_1$ with no upper bound in $D$. Pick a $c_1 \in C_1$ such that the sets

$$Z_1 = \{ \alpha < \omega_1 : c_1 \not\leq_D b_\alpha \} \quad \text{and} \quad Y_1 = \{ \alpha < \omega_1 : c_1 \leq_D a_1^\alpha \}$$

are both uncountable. Note that for each $\alpha \in Y_1$ and $\beta \in Z_1$ with $\alpha < \beta$, $a_1^\alpha \not\leq_D b$ for all $b \in B_{p_\beta}$. Define now

$$X_2 = \{ a \in D : \{ \alpha \in Y_1 : a \leq a_1^\alpha \} \text{ is uncountable} \}.$$

By the definition of $E$, $X_2$ is an uncountable subset of $D$ so there is a countable set $C_2 \subseteq X_2$ with no upper bound in $D$. Pick $c_2 \in C_2$ such that

$$Z_2 = \{ \alpha \in Z_1 : c_2 \not\leq_D b_\alpha \} \quad \text{and} \quad Y_2 = \{ \alpha \in Y_1 : c_1 \leq_D a_1^\alpha \}$$

are both uncountable. Iterating this procedure, we obtain sequences $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$ and $Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n$ of uncountable subsets of $\omega_1$ such that for all $\alpha \in Y_i$ and $\beta \in Z_i$ with $\alpha < \beta$, $a_1^\alpha \not\leq_D b$ for all $b \in B_{p_\beta}$. Thus if $\alpha \in Y_n$ and $\beta \in Z_n$ are such that $\alpha < \beta$, then $p_\alpha$ and $p_\beta$ are compatible in $\mathcal{P}_1$. This proves the claim.

Pick a filter $\mathcal{G}_1 \subseteq \mathcal{P}_1$ which intersects each of the dense sets $\{ p \in \mathcal{P}_1 : b \in B_p \text{ and } \max A_p \geq b \}$ where $b \in E$. Let $A = \bigcup \{ A_p : p \in \mathcal{G}_1 \}$. Then $A$ is an uncountable subset of $D$ such that each infinite subset of $A$ is unbounded in $D$. This shows that $D \equiv [\omega_1]^{< \omega}$. 

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Case II. An elementary chain of submodels as in Case I does not exist. So there must be a limit nonzero ordinal $\delta < \omega_1$ and a strictly increasing continuous sequence $\langle N_\xi : \xi < \delta \rangle$ of countable elementary submodels of $H_\kappa$, containing $D$ such that the following set is cofinal in $D$.

$$D' = \{ d \in D : \text{there is a } \xi < \delta \text{ such that } c \not\in_D d \text{ for all } c \in N_{\xi+1} \setminus N_\xi \}.$$ 

Assume $\delta$ is a minimal limit countable ordinal with this property. Then for every $b \in D$ and $\gamma < \delta$ there must be $d \geq_D b$ in $D'$ and $\xi \in (\gamma, \delta)$ such that $c \not\in_D d$ for all $c \in N_{\xi+1} \setminus N_\xi$.

Define $\mathcal{M}$ to be the set of all countable $M < H_\kappa$, containing $D$ such that for some limit nonzero $\delta_M < \omega_1$ and a strictly increasing continuous sequence $\langle M_\xi : \xi < \delta_M \rangle$ of countable elementary submodels of $H_\kappa$, the following conditions are satisfied:

(a) $M = \bigcup_{\xi < \delta_M} M_\xi$;

(b) $D_M = \{ d \in D : \text{there is a } \xi < \delta_M \text{ such that } c \not\in_M d \text{ for all } c \in M_{\xi+1} \setminus M_\xi \}$ is cofinal in $D$;

(c) for all $b \in D$ and $\gamma < \delta_M$ there exists $d \geq b$ in $D_M$ and $\xi \in (\gamma, \delta_M)$ such that $c \not\in_M d$ for all $c \in M_{\xi+1} \setminus M_\xi$.

Then $\mathcal{M}$ is a stationary subset of $[H_\kappa]^\aleph_0$ since clearly $\mathcal{M} \subseteq N_\delta$ and $N_\delta \cap H_\kappa \subseteq \mathcal{M}$.

Let $\mathcal{P}_2$ be the set of all pairs $p = \langle A_p, B_p \rangle$, where $A_p$ and $B_p$ are finite subsets of $D$. The ordering on $\mathcal{P}_2$ is defined by $q \preceq p$ iff $A_q \supseteq A_p$, $B_q \supseteq B_p$, and $a \not\in_D b$ for all $a \in A_q \setminus A_p$ and $b \in B_p$. Working as in the Case I, the following claim finishes our discussion of Case II and also the proof of Theorem 7.

**Claim 2.** $\mathcal{P}_2$ is a ccc poset.

**Proof.** Let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be a given sequence of elements of $\mathcal{P}_2$. Again we may assume that $A_{p_\alpha} \cup B_{p_\alpha} < A_{p_\beta} \cup B_{p_\beta}$ for $\alpha < \beta$. Since $\mathcal{M}$ is a stationary subset of $[H_\kappa]^\aleph_0$, there is an $M \in \mathcal{M}$ such that $\langle p_\alpha : \alpha < \omega_1 \rangle \subseteq M$. Let $\langle M_\xi : \xi < \delta_M \rangle$ be a decomposition of $M$ which satisfies (a)–(c). Pick a $\gamma \in \delta_M$ such that $\langle p_\alpha : \alpha < \omega_1 \rangle \subseteq M_\gamma$. Fix a $\beta < \omega_1$ such that $B_{p_\beta} \cap M = \emptyset$. By (c) we can find an upper bound $d$ of $B_{p_\beta}$ in $D_M$ such that for some $\xi \in (\gamma, \delta_M)$, $d \not\in_M d$ for all $c \in M_{\xi+1} \setminus M_\xi$. Note that $\langle A_{p_\alpha} : \alpha < \omega_1 \rangle \subseteq M_{\xi+1}$, so there is an $\alpha$ in $M_{\xi+1}$ such that $A_{p_\alpha} \cap M_\xi = \emptyset$. By the choice of $d$ and $\xi$ and the fact that $A_{p_\alpha} \subseteq M_{\xi+1} \setminus M_\xi$, we have that $a \not\in_D b$ for all $a \in A_{p_\alpha}$ and $b \in B_{p_\beta}$. So $p_\alpha$ and $p_\beta$ are compatible in $\mathcal{P}_2$. This completes the proof.

**Corollary 8 (MA).** Let $P$ be an uncountable partially ordered set with no uncountable antichain. Then $P$ contains an uncountable set $X$ such that every countable subset of $X$ has an upper bound in $P$.

**Proof.** Since $P$ is in particular a ccc poset it contains an uncountable directed subset $D$. Clearly $D \not\subseteq [\omega_1]^{<\omega}$, so by Theorem 7, there must be an uncountable $X \subseteq D$ such that every countable subset of $X$ has an upper bound in $D$.

Corollary 8 is a result announced in [18] and it is connected with an unpublished problem of Galvin. Let us note that K. Devlin and J. Stepräns [2, §5] have previously deduced the conclusion of Theorem 7 from a much stronger forcing axiom PFA. Theorem 7 is in a sense optimal since MA does not imply $\mathcal{D}_\kappa = \{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega} \}$. This can be proved using methods of [1].
4. Five cofinal types. In this section we prove the main result of this paper which
says that 1, \(\omega\), \(\omega_1\), \(\omega \times \omega_1\) and \([\omega_1]^{<\omega}\) are the only cofinal types of directed sets of
size \(\leq \aleph_1\) which can be constructed without additional set-theoretic assumptions.
This will be done using an iterated forcing construction and we assume the reader is
familiar with some basic facts about iterated forcing.

**Theorem 9.** If ZF is consistent, then so is ZFC plus MA plus
\[D_{\omega_1} = \{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}.\]

A partially ordered set \(P\) is proper [14] iff for all large enough regular cardinals \(\theta\)
and all countable \(N < H_\theta\) which contain \(P\), every condition from \(P \cap N\) can be
extended to a condition \(q\) which forces that \(\dot{G}_\theta \cap N\) intersect each dense subset of \(P\)
which is a member of \(N\). Such a condition \(q\) is called an \((N, P)\)-generic condition. A
basic result of Shelah [14] says that countable support iterations preserve properness.
This result will essentially be reproduced at the end of this section. Note that any
proper poset preserves \(\omega_1\).

Let \(D\) be a fixed directed set of size \(\aleph_1\) such that \(\omega \times \omega_1 \not\subseteq D\). We begin the proof
of Theorem 9 with a description of our basic poset \(\mathcal{P} = \mathcal{P}_D\) which forces \(D \equiv [\omega_1]^{<\omega}\).
Note that by Proposition 3 this will be a right step toward the model of Theorem 9.
We say that a subset \(E\) of \(D\) is \(\omega\)-bounded in \(D\) if every countable subset of \(E\) is
bounded in \(D\). Clearly \(\omega_1 \times \omega_1 \not\subseteq D\) is equivalent to the fact that \(D\) is not a countable
union of its \(\omega\)-bounded subsets. As usual we assume that the domain of \(D\) is equal to
\(\omega_1\) and that \(\alpha \prec \beta\) implies \(\alpha \prec \beta\).

If \(N\) is a countable elementary submodel of \(H_{\aleph_2}\), then by \(\widetilde{N}\) we denote the
transitive collapse of \(N\). In what follows, we shall add \(\{D\}\) as a predicate to \(H_{\aleph_2}\), so
all submodels of \(H_{\aleph_2}\) considered in this section are in fact submodels of this
expanded structure. Thus an isomorphism between two elementary submodels of
\(H_{\aleph_2}\), will have to map \(D\) into \(D\). Note that if \(M\) is a transitive closure of two
submodels \(N\) and \(N'\) of \(H_{\aleph_2}\), then the collapsing maps uniquely determine an
isomorphism between \(N\) and \(N'\) (which maps \(D\) into \(D\)). Let \(\mathcal{IM}\) denote the set of
all countable transitive sets. For \(M \in \mathcal{IM}\) let \(\mathcal{N}_M\) be the set of all \(N < H_{\aleph_2}\) with
transitive collapse equal to \(M\).

For \(F \in [\omega_1]^{<\omega}\), let \(\langle F \rangle\) denote the sequence from \(\langle \omega_1 \rangle^{<\omega}\) which enumerates \(F\) in
increasing order.

Finally, we are ready to define our poset \(\mathcal{P} = \mathcal{P}_D\) as the set of all triples
\(p = \langle A_p, B_p, \mathcal{N}_p \rangle\), where:

1. \(\mathcal{N}_p\) is a finite function with domain \(a \in \text{chain from } \mathcal{IM}\).
2. for all \(M \in \text{dom } \mathcal{N}_p\), \(\mathcal{N}_p(M)\) is a nonempty finite subset of \(\mathcal{N}_M\),
3. for all \(M \in M'\) in \(\text{dom } \mathcal{N}_p\) and all \(N \in \mathcal{N}_p(M)\) there is an \(N' \in \mathcal{N}_p(M')\) such
   that \(N \subseteq N'\),
4. \(A_p\) and \(B_p\) are finite subsets of \(\omega_1\) such that for all \(a \prec b\) in \(A_p\) there is an
   \(M \in \text{dom } \mathcal{N}_p\) such that \(a \in M\) but \(b \not\in M\),
5. if \(a \in A_p\) is not in \(M \in \text{dom } \mathcal{N}_p\), then \(a\) is not a member of any \(\omega\)-bounded
   subset of \(D\) lying in \(\cup \mathcal{N}_p(M)\).
For \( p, q \in \mathcal{P} \), we let \( q \leq p \) iff

\begin{enumerate}
  \item \( \text{dom } \mathcal{N}_p \supseteq \text{dom } \mathcal{N}_q \) and \( \mathcal{N}_q(M) \supseteq \mathcal{N}_p(M) \) for all \( M \in \text{dom } \mathcal{N}_p \),
  \item \( A_q \supseteq A_p, B_q \supseteq B_p \), and \( a \notin D b \) for all \( a \in A_q \setminus A_p \) and \( b \in B_p \).
\end{enumerate}

**Lemma 4.** \( \mathcal{P} \) is a proper poset.

**Proof.** Let \( \theta \) be a large enough regular cardinal and let \( N_\theta < H_\theta \) be countable such that \( p, \mathcal{P}, D \in N_\theta \). Define

\[ q = \left\{ A_p, B_p, \mathcal{N}_p \cup \left\{ \left( N_\theta \cap H_{\mathcal{N}_p}, \{ N_\theta \cap H_{\mathcal{N}_p} \} \right) \right\} \right\}. \]

Then \( q \in \mathcal{P} \) and \( q \leq p \). We shall prove that \( q \) is an \( (N_\theta, \mathcal{P}) \)-generic condition. So let \( \mathcal{D} \in N_\theta \) be a dense open subset of \( \mathcal{P} \) and let \( r \leq q \). We have to show that \( r \) is compatible with a member of \( \mathcal{D} \cap N_\theta \). Clearly, we may assume \( r \in \mathcal{D} \).

Define

\[ A_r = A_r \cap N_\theta, \quad B_r = B_r \cap N_\theta \]

and

\[ \mathcal{M}_r = \left\{ M \in \text{dom } \mathcal{N}_r; M \in N_\theta \cap H_{\mathcal{N}_r} \right\}. \]

Then \( A_r, B_r \) and \( \mathcal{M}_r \) are elements of \( N_\theta \). Let \( N_0, \ldots, N_k \) be a list of all members of \( \mathcal{N}_r(N_\theta \cap H_{\mathcal{N}_r}) \) with \( N_0 = N_\theta \cap H_{\mathcal{N}_r} \). For \( i \leq k \), let

\[ \pi_i: \left\langle N_i, \in, \{ D \} \right\rangle \rightarrow \left\langle N_0, \in, \{ D \} \right\rangle \]

be the induced isomorphism. Note that by (3), for all \( M \in \mathcal{M}_r \) and \( N \in \mathcal{N}_r(M) \) the set \( I(N) = \left\{ i \leq k; N \in N_i \right\} \) is nonempty. For \( M \in \mathcal{M}_r \), we define

\[ \mathcal{N}_r(M) = \left\{ \pi_i(N); N \in \mathcal{N}_r(M) \text{ and } i \in I(N) \right\}. \]

Then it is easily checked that \( \bar{r} = \langle A_r, B_r, \mathcal{M}_r \rangle \) is a member of \( \mathcal{P} \cap N_\theta \) and that \( \bar{r} \leq p \).

Let \( A = A_r \setminus A_r \). We may assume that \( A \) is nonempty since otherwise we are easily done. Let \( n \) be the size of \( A \) and define

\[ X = \{ x \in (\omega_1)^n; A_x < x = \langle A_y \setminus A_y \rangle \text{ for some } s \leq \bar{r} \text{ in } \mathcal{D} \}. \]

Define inductively \( X_0, \ldots, X_n \) such that \( X_0 = X \) and

\[ X_i = \{ x \in (\omega_1)^{n-i}; (\alpha \in (\omega_1)^{n-i}; x \prec (\alpha) \in X_{i-1} \} \text{ is not } \omega \text{-bounded in } D \} \]

for \( 0 < i \leq n \). Clearly, \( X_i \in N_\theta \cap H \) for all \( i \leq n \).

**Claim 3.** \( \langle \alpha \rangle \in X_n \).

**Proof.** Let \( \{ a_0, \ldots, a_{n-1} \} \) be the increasing enumeration of \( A \) and let \( x = \langle a_0, \ldots, a_{n-1} \rangle \). It suffices to show, by induction on \( i \leq n \), that \( x \uparrow (n - i) \in X_i \) for all \( i \leq n \). Since \( r \) satisfies (3) we can pick a chain \( N^0 \in \cdots \in N^{n-1} \) in \( \mathcal{U} \text{ range } \mathcal{N}_r \) such that

\begin{enumerate}
  \item \( N^0 = N_\theta \cap H_{\mathcal{N}_r} \),
  \item \( a_i \notin N^i \) for all \( i < n \),
  \item \( a_i \in N^{i+1} \) for all \( i < n - 1 \).
\end{enumerate}

The induction step from \( i - 1 \) to \( i \) follows from the facts that \( X_i, X_{i-1} \in N^{i-1}, x \uparrow (n - i) \in X_{i-1} \) and \( a_{n-i} \) is not in any \( \omega \)-bounded subset of \( D \) lying in \( N^{i-1} \). This finishes the proof.
Since \( \langle \cdot \rangle \in X_n \), the set \( E_0 = \{ e \in D : \langle e \rangle \in X_{n-1} \} \) is not \( \omega \)-bounded in \( D \). Clearly \( E_0 \subseteq N_0 \), so there must be an \( e_0 \in E_0 \cap N_0 \) such that \( e_0 \not\in b \) for all \( b \in B_0 \). Since \( \langle e_0 \rangle \in X_{n-1} \), the set \( E_1 = \{ e \in D : e_0 < e \} \) is not \( \omega \)-bounded in \( D \). Since clearly \( E_1 \subseteq N_0 \), there must be an \( e_1 \in E_1 \cap N_0 \) such that \( e_1 \not\in b \) for all \( b \in B_0 \). Continuing in this way, we obtain a sequence \( e_0 < e_1 \leq b \) for all \( i < n \) and \( b \in B_0 \). Pick an \( s \leq r \) in \( \mathcal{D} \cap N_0 \) such that \( A_i < \{ e_0, \ldots, e_{n-1} \} = A_i \setminus A_0 \). Define \( s = (A_0, A_1, \mathcal{N}_s) \) as follows:

1. \( A_s = A_0 \cup A_1 \) and \( B_s = B_0 \cup B_1 \),
2. \( \text{dom } \mathcal{N}_s = \text{dom } \mathcal{N}_r \cup \text{dom } \mathcal{N}_r' \),
3. \( \mathcal{N}_s(M) = \mathcal{N}_r(M) \) for \( M \in \text{dom } \mathcal{N}_r \cap \text{dom } \mathcal{N}_r' \),
4. \( \mathcal{N}_s(M) = \mathcal{N}_r(M) \cup \{ \pi_i^{-1}(N) : i < k \land N \in \mathcal{N}_r'(M) \} \) for \( M \in \text{dom } \mathcal{N}_r' \).

Claim 4: \( s \in \mathcal{P} \) and \( s \leq r, s \).

PROOF. Clearly \( s \) satisfies (1) and (4), (2), (3) and (5) follow from the corresponding conditions of \( r \) and \( s \) and the choice of the \( \pi_i \)'s. The fact that \( s \) is the greatest lower bound of \( r \) and \( s \) follows from the definitions of \( r \) and \( s \) using the isomorphisms \( \pi_i \)'s. The condition (7) for \( s \leq r \) follows from the choice of the set \( \{ e_0, \ldots, e_{n-1} \} = A_s \setminus A_r \). This completes the proof of Claim 4 and also the proof that \( q \) is an \( (N_0, \mathcal{P}) \)-generic condition.

Lemma 5. \( \mathcal{D}_b = \{ p \in \mathcal{P} : b \in B_p \text{ and } \max A_p > b \} \) is a dense open subset of \( \mathcal{P} \) for all \( b \in D \).

PROOF. Let \( q \in \mathcal{P} \) be given. Pick countable \( N < H_\kappa \) such that \( b, q \in N \). Since \( D \) is not a countable union of its \( \omega \)-bounded subsets there is an \( a \in D \setminus N \) which is not in any \( \omega \)-bounded subset of \( D \) lying in \( N \). Define \( A_p = A_q \cup \{ a \} \), \( B_p = B_q \cup \{ b \} \) and \( \mathcal{N}_p = \mathcal{N}_q \cup \{ \langle N, \{ N \} \rangle \} \). Then \( p = (A_p, B_p, \mathcal{N}_p) \) is a member of \( \mathcal{D}_b \) which extends \( q \).

Suppose \( p \) and \( q \) are two conditions from \( \mathcal{P} \) such that \( A_p = A_q, B_p = B_q \) and \( \text{dom } \mathcal{N}_p = \text{dom } \mathcal{N}_q \). Define \( r = (A_r, B_r, \mathcal{N}_r) \) as follows:

1. \( A_r = A_p \) and \( B_r = B_p \),
2. \( \text{dom } \mathcal{N}_r = \text{dom } \mathcal{N}_p \cup \mathcal{N}_q \), and
3. \( \mathcal{N}_r(M) = \mathcal{N}_p(M) \cup \mathcal{N}_q(M) \) for \( M \in \text{dom } \mathcal{N}_r \).

Then it is easily seen that \( r \) is the greatest lower bound of \( p \) and \( q \) in \( \mathcal{P} \), so in particular \( p \) and \( q \) are compatible in \( \mathcal{P} \). Hence if CH holds, then \( \mathcal{P} \) satisfies the \( \mathfrak{S}_2 \)-chain condition. Actually, the \( \mathfrak{S}_2 \)-chain condition of \( \mathcal{P} \) is strong enough to be preserved under countable support iterations of any length \( \leq \omega_2 \). This can be proved via standard arguments using certain canonical names for reals which code models from \( \text{dom } \mathcal{N}_p \) for \( p \in \mathcal{P} \). While this traditional method works, it is far less general and elegant than the following scheme of Shelah [14, VIII, §2] which our poset \( \mathcal{P} \) satisfies and which will be reproduced here in some detail.

A poset \( \mathcal{P} \) satisfies the \( \mathfrak{S}_2 \)-isomorphism condition (\( \mathfrak{S}_2 \)-ic) iff the following holds for all large enough regular cardinals \( \theta \) where a well-ordering \(< \) of \( H_\theta \) is added as a predicate: If \( \alpha < \beta < \omega_2 \), if \( \alpha \in N_\alpha < H_\theta \) and \( \beta \in N_\beta < H_\theta \) are countable such that...
$2 \in N_\alpha \cap N_\beta$, $N_\alpha \cap \omega_2 \subseteq \beta$, $N_\alpha \cap \alpha = N_\beta \cap \beta$, if $p \in N_\alpha$ and if $\pi: N_\alpha \to N_\beta$ is an isomorphism such that $\pi(\alpha) = \pi(\beta)$ and $\pi: N_\alpha \cap N_\beta = \text{id}$, then there is an $(N_\alpha, \mathcal{D})$-generic condition $q \leq p$, $\pi(p)$ such that $q \Vdash \pi'' \dot{G}_\mathcal{D} \cap N_\alpha = \dot{G}_\mathcal{D} \cap N_\beta$.

Roughly speaking, this condition is saying (among other things) that if $p$ and $\pi(p)$ are two conditions with isomorphic countable “histories” which use only ordinals $< \omega_2$, then they are compatible in $\mathcal{D}$. So if CH holds, then any $\kappa$-ic poset satisfies the $\kappa$-cc and preserves $\omega_1$. It is also clear that any proper poset of size $\aleph_1$ satisfies $\kappa$-ic. Note that the condition $q$ is also $(N_\beta, \mathcal{D})$-generic, and that $q$ forces $\pi$ to naturally extend to an isomorphism of $N_\alpha[\dot{G}_\mathcal{D}]$ and $N_\beta[\dot{G}_\mathcal{D}]$.

**Lemma 6.** $\mathcal{D}$ satisfies the $\kappa$-ic.

**Proof.** Let $\alpha$, $\beta$, $N_\alpha$, $N_\beta$, $\pi$ and $p$ satisfy the hypothesis of $\kappa$-ic. Since $N_\alpha$ and $N_\beta$ have the same reals and ordinals $< \omega_1$, it follows that $A_p = A_{\pi(p)}$, $B_p = B_{\pi(p)}$ and $\text{dom } \mathcal{N}_p = \text{dom } \mathcal{N}_{\pi(p)}$.

Let $r = p \land \pi(p)$, and define $q \in \mathcal{D}$ by

(a) $A_q = A_r$, $B_q = B_r$,
(b) $\mathcal{N}_q = \mathcal{N}_r \cup \{(N_\alpha \cap H_{\kappa_2}, \{N_\alpha \cap H_{\kappa_2}, N_\beta \cap H_{\kappa_2}\})\}$.

Then $q$ satisfies the conclusion of $\kappa$-ic. The proof that $q$ is an $(N_\alpha, \mathcal{D})$-generic condition is almost the same as the proof of Lemma 4. The fact that $q \Vdash \pi'' \dot{G}_\mathcal{D} \cap N_\alpha = \dot{G}_\mathcal{D} \cap N_\beta$ follows easily from the compatibility conditions of the poset $\mathcal{D} = \mathcal{D}_\mathcal{D}$. This finishes the proof.

Let $\langle \mathcal{D}_\xi : \xi < \omega_2 \rangle$ be a countable support iteration of some posets of the form $\mathcal{D}_\mathcal{D}$ and some ccc posets of size $\aleph_1$. Using a standard diagonalization argument [15], in order to prove that $\langle \mathcal{D}_\xi : \xi < \omega_2 \rangle$ can be chosen in such a way that $\mathcal{D}_\omega_1$ forces the conclusion of Theorem 9, it suffices to show that $\mathcal{D}_\omega_1$ satisfies the $\omega_1$-ccc. This is a consequence of a general result of Shelah [14, VIII, 2.4]. In order to make this paper self-contained we sketch the argument from [14].

**Lemma 7.** For all $\xi < \omega_2$, $\mathcal{D}_\xi$ satisfies the $\kappa$-ic.

**Proof.** Let $\xi$ be a fixed ordinal $< \omega_2$ and let $\alpha$, $\beta$, $N_\alpha$, $N_\beta$, $\mathcal{D}_\xi$ and $\pi$ satisfy the hypothesis of $\kappa$-ic. Note that $N_\alpha \cap \xi = N_\beta \cap \xi$. By induction on $\xi < \xi$ in $\alpha$, we shall prove the following stronger result:

For all $\zeta < \xi$ in $N_\alpha$, all $p \in N_\alpha \cap \mathcal{D}_\xi$ and all $q_\zeta \in \mathcal{D}_\xi$ with $q_\zeta \leq p \upharpoonright \zeta$, $\pi(p) \upharpoonright \zeta$, if $q_\zeta$ is $(N_\alpha, \mathcal{D}_\xi)$-generic and $q_\zeta \Vdash \pi'' \dot{G}_\mathcal{D}$, then there is an $(N_\alpha, \mathcal{D}_\xi)$-generic $q_\xi \leq p \upharpoonright \xi$, $\pi(p) \upharpoonright \xi$ such that $q_\zeta \upharpoonright \zeta = q_\xi$ and $q_\zeta \Vdash \pi'' \dot{G}_\xi \cap N_\alpha = \dot{G}_\xi \cap N_\beta$.

The case $\xi = \zeta + 1$ is straightforward, so we assume $\xi$ is a limit ordinal. Pick an increasing sequence $\zeta = \zeta_0 < \cdots < \zeta_n < \cdots$ in $\xi \cap N_\alpha$ cofinal with $\xi \cap N_\alpha$. Let $\langle \mathcal{D}_\eta : \eta < \omega \rangle$ be a list of all dense open subsets of $\mathcal{D}_\xi$ which are members of $N_\alpha$. Let
$q^0 = q_\xi$, and let $\mathcal{A}_0$ be a maximal antichain in $\{ r \uparrow \xi_1 : r \in \mathcal{D}_0 \text{ and } r \leq p \}$ such that $\mathcal{A}_0 \subseteq N_\alpha$. For $r \in \mathcal{A}_0$, we let $r^*$ denote the $\prec$-minimal element of $\mathcal{D}_0$ such that $r^* \leq p$ and $r^* \uparrow \xi_1 = r$. For each $r \in \mathcal{A}_0 \cap N_\alpha$, we fix $q^1(r) \leq r^* \uparrow \xi_1$, $\pi(r^*) \uparrow \xi_1$ which end-extends $q^0$ and satisfies $(i_{\xi_1})$. Define $q^1 \in \mathcal{P}_{\xi_1}$ to be equal to $q^0$ on $\xi_0$ and to $q^1(r)$ on $[\xi_0, \xi_1)$ if $r \in \mathcal{A}_0 \cap N_\alpha \cap \dot{G}_{\xi_0}$. Now for each $r \in \mathcal{A}_0 \cap N_\alpha$, fix, in $N_\alpha$, a maximal antichain $\mathcal{A}'_1$ in $\{ s \uparrow \xi_2 : s \in \mathcal{D}_1 \text{ and } s \leq r^* \}$. For $s \in \mathcal{A}'_1$, let $s^*$ be the $\prec$-minimal element of $\mathcal{D}_1$ such that $s^* \leq r^*$ and $s^* \uparrow \xi_2 = s$. For $r \in \mathcal{A}_0 \cap N_\alpha$ and $s \in \mathcal{A}'_1 \cap N_\alpha$, fix $q^2(s) \leq s^* \uparrow \xi_2$, $\pi(s^*) \uparrow \xi_2$ which end-extends $q^1(r)$ and satisfies $(i_{\xi_2})$. Define $q^2 \in \mathcal{P}_{\xi_2}$ to be equal to $q^1$ on $\xi_1$ and to $q^2(s)$ on $[\xi_1, \xi_2)$ if $s$ is in $\mathcal{A}'_1 \cap N_\alpha \cap \dot{G}_{\xi_1}$ for some $r \in \mathcal{A}_0 \cap N_\alpha$. Iterating this procedure, we obtain an end-extending sequence $q^n \in \mathcal{P}_{\xi_n} (n < \omega)$ whose union $q_\xi$ satisfies the conclusion of $(i_{\xi})$. This completes the proof of Lemma 7 and also the proof of Theorem 9.

The proof of Theorem 9 can also be given via an axiomatic approach [2, 14], but this is only a matter of taste since the proof remains essentially the same. However, if we are willing to use an inaccessible cardinal, then the poset $\mathcal{P}_D$ can be made simpler and there is no need in proving any chain condition for $\mathcal{P}_D$ at all. Namely, in this case, as a side condition in $p \in \mathcal{P}_D$ we use just an $\in$-chain of submodels of $H_{\aleph_\alpha}$. Some information concerning this approach can be found in [18].

In [17] we have stated the following partition property of $\omega_1$ as a strengthening of a similar partition relation considered in the same paper:

For every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^\aleph_1$ such that $[A]^2 \subseteq K_0$ or else there exist $(\{ A_\alpha \} \cap \omega_1 \text{ disjoint finite subsets of } \omega_1$;

(i) $\omega_1 \setminus \bigcup_n A_n$ is countable;

(ii) $\mathcal{B}_n$ is a family of $\aleph_1$, disjoint finite subsets of $\omega_1$;

(iii) $(\{ \alpha \} \cap \omega_1) \cap K_1 = \emptyset$ for all $\alpha \in A_n$ and $F \in \mathcal{B}_n$ with $\alpha < \min F$.

It should be clear that the poset $\mathcal{P}_D$ can easily be modified so as to give a poset for $(\ast)$, so the model of Theorem 9 can also satisfy $(\ast)$. In [17] we have mentioned that it is impossible to strengthen $(\ast)$ by demanding $\omega_1$ to be a countable union of $0$-homogeneous sets if there are no $A_n$'s and $\mathcal{B}_n$'s satisfying (i)–(iii). Namely, let

$$T = \{ s \in (2)^{<\omega_1} : s^{-1}(1) \text{ is finite} \}$$

and define $[T]^2 = K_0 \cup K_1$ by: $\{ s, t \} \in K_1$ iff $s \subset t$.

References


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