VARIANTS OF THE MAXIMAL DOUBLE HILBERT TRANSFORM

BY

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Abstract. We prove the boundedness on $L^p(T^2)$, $1 < p < \infty$, of two variants of the double Hilbert transform and maximal double Hilbert transform. They have an application to a problem of almost everywhere convergence of double Fourier series.

Introduction. In this paper we study two variants of the double Hilbert transform
\[
Df(x, y) = \int_{|y'| \leq \pi, |x'| \leq \pi} 1/|x' y'| f(x - x', y - y') \, dx' \, dy'
\]
and maximal double Hilbert transform
\[
\hat{D}f(x, y) = \sup_{\varepsilon, \delta > 0} \left| \int_{\varepsilon \leq |y'| \leq \pi, \varepsilon \leq |x'| \leq \pi} 1/|x' y'| f(x - x', y - y') \, dx' \, dy' \right|
\]
which are, roughly speaking, of the following kind. First we consider
\[
H_1 f(x, y) = \int_{(x', y') \in A} 1/|x' y'| f(x - x', y - y') \, dx' \, dy'
\]
and
\[
\hat{H}_1 f(x, y) = \sup_{\varepsilon, \delta > 0} \left| \int_{(x', y') \in A} 1/|x' y'| f(x - x', y - y') \, dx' \, dy' \right|
\]
where $A \subset \{(x', y'): |x'| \leq \pi, |y'| \leq \pi\} = T^2$ is a fixed region symmetrical with respect to the axes $x'$ and $y'$ but, except for this natural requirement, quite general. (The cut-off of the kernel $1/|x' y'|$, given by $\chi_A(x', y')$, is actually smoothly done. See §2 for the exact definition.) Secondly, we consider
\[
H_2 f(x, y) = \int_{(x', y') \in A_x} 1/|x' y'| f(x - x', y - y') \, dx' \, dy'
\]
and
\[
\hat{H}_2 f(x, y) = \sup_{\delta > 0} \left| \int_{(x', y') \in A_x} 1/|x' y'| f(x - x', y - y') \, dx' \, dy' \right|
\]
where, for every $x$, the domain of integration $A_x$ is symmetrical with respect to the axes and otherwise is quite general.
We shall prove that $H_1, \tilde{H}_1, H_2, \tilde{H}_2$ are bounded operators from $L^p(T^2)$ to itself, $1 < p < \infty$. Moreover, we shall give a pointwise estimate from above of $\tilde{H}_1$ and $\tilde{H}_2$ similar to the known one concerning $\tilde{D}$ (see [6, p. 218]). Namely, we are going to prove that

\[
\tilde{H}_1 f(x, y) \leq c \{ M_{x'} M_{y'} f(x, y) + M_{x'} \tilde{H}_{y'} f(x, y) \\
+ M_{y'} \tilde{H}_{x'} f(x, y) + M_{x'} M_{y'} (H_1 f)(x, y) \},
\]

(1)

\[
\tilde{H}_2 f(x, y) \leq c \{ M_{y'} \tilde{H}_{x'} f(x, y) + M_f (H_2 f)(x, y) \}
\]

(2)

where $M_{x'}$ is the Hardy-Littlewood maximal function acting on the $x'$ variable, $\tilde{H}$ denotes a variant of the maximal Hilbert transform (see §1). These results apply to a problem of almost everywhere convergence of double Fourier series [5], where it appears that $H_1$ and $\tilde{H}_2$ play the same central role that the maximal Hilbert transform plays in the proof of a.e. convergence of Fourier series of one variable [1, 2, 4].

Let us observe that $Z_1$ and $Z_2$ fall under the scope of Theorems 2 and 4 of [3]. The proof given in [3] of Theorem 4 uses complex interpolation and it is quite technical. Ours involves only elementary estimates; moreover, we are able to control $\tilde{H}_1$ from above by proving (1). This is most important for the mentioned application and it is not proved in [3].

The paper is structured as follows. In §1 we are concerned with the one-dimensional case and with the maximal Carleson operator. In §§2 and 3, respectively, we study $H_1, \tilde{H}_1$ and $H_2, \tilde{H}_2$. In §4 we consider an even more general operator $\tilde{H}_3$ (where the Sup is taken over all regions). We give a counterexample to show that $\tilde{H}_3$ is not a bounded operator. This sets a halt to our generalisations of the maximal double Hilbert transform. Finally, in §5 we say some more about the application we mentioned.

By $c$ we denote a constant not necessarily the same in all instances.

1. **The one-dimensional case.** There exists a $C^\infty$ function $\phi(x')$ supported on $\{|x'| \leq 2\pi\}$ such that if we write $\phi_k(x') = 2^k \phi(2^k x')$, then $1/x' = \sum_{k=0}^\infty \phi_k(x')$ for $|x'| \leq \pi$. Let $J$ be a fixed subset of the nonnegative integers $N$ and let us consider the operators

\[
H f(x, y) = \int \sum_{k \in J} \phi_k(x') f(x - x') \, dx'
\]

and

\[
\tilde{H} f(x) = \sup_{K>0} \left| \int \sum_{\substack{k \in J \\ k \leq K}} \phi_k(x') f(x - x') \, dx' \right|.
\]
\( \tilde{H} \) is clearly a variant of the maximal Hilbert transform. We have

**Lemma 1.** \( H \) and \( \tilde{H} \) are bounded operators on \( L_p(T) \), \( 1 < p < \infty \), with norm independent of \( J \). Moreover, the following inequality holds:

\[
\tilde{H}f(x) \leq c\{Mf(x) + M(Hf)(x)\}.
\]

**Remark.** This is exactly Lemma 3 of [2]. We are going to prove it for the reader’s convenience and for an inaccuracy that appears in [2]. Namely, one needs to use a smooth cut-off function like the following \( \theta(x') \) rather than a sharp one.

**Proof.** Clearly, \( \phi \) has the following properties:

1. \( \hat{\phi}(0) = 0 \),
2. \( |\hat{\phi}(\xi)| \leq c_M/|\xi|^M \) for \( |\xi| > 1 \) and for any integer \( M \geq 0 \),
3. \( |\hat{\phi}(\xi)| \leq c|\xi| \) for \( |\xi| \leq 1 \).

Since \( \hat{\phi}_k(\xi) \) is mainly supported on \( |\xi| \approx 2^k \), we have that

\[
|\hat{\phi}_k(\xi)| \leq \sum_{k=0}^{\infty} |\hat{\phi}_k(\xi)| \leq c
\]

independently of \( J \) and \( K \). We have

\[
m'_{k}(\xi) = \sum_{k \in J, k \leq K} \hat{\psi}_k(\xi), \quad \psi(x') = x'\phi(x') \quad \text{and} \quad \psi_k(x') = 2^k x'\phi(2^k x') = \psi(2^k x').
\]

Now \( \psi(x') \) is \( C^\infty \), compactly supported and, moreover,

4. \( |\hat{\psi}(\xi)| \leq c \) for every \( \xi \),
5. \( |\hat{\psi}(\xi)| \leq c_M/|\xi|^M \) for \( |\xi| > 1 \) and for any integer \( M \geq 0 \).

Therefore, for any dyadic interval \( I \subset \mathbb{R} \) we have that

\[
\int_I \left| m'_{k}(\xi) \right| d\xi \leq \int \sum_{k=0}^{\infty} |\hat{\psi}_k(\xi)| d\xi \leq c.
\]

By the Marcinkiewicz multiplier theorem, the operator

\[
H_k f(x) = \int \sum_{k \in J, k \leq K} \phi_k(x')f(x - x') dx'
\]

is bounded on \( L_p \) with norm independent of \( K \). By a standard argument there exists \( \lim_{K \to \infty} H_k f(x) = Hf(x) \) in \( L_p \) norm and \( \|Hf\|_p \leq c_p \|f\|_p, 1 < p < \infty \).

Now let \( \theta(x') \) be a positive \( C^\infty \) function supported on \( \{ |x'| \leq 1 \} \) and such that

\[
\int_{-1}^{1} \theta(x') dx' = 1.
\]

To prove equation (3) we are going to show that

\[
\left| \sum_{k \in J} \phi_k(x') - \theta_K \sum_{k \in J} \phi_k(x') \right| \leq c2^{-K}/(x')^2 + 2^{-2K},
\]

where \( \theta_K(x') = 2^K \theta(2^K x') \). If \( |x'| < 1002^K \) then \( \| \sum_{k \in J, k \leq K} \phi_k(x') \|_\infty \leq c2^K \) and

\[
\left\| \theta_K \sum_{k \in J} \phi_k(x') \right\|_\infty \leq \left\| \theta_K \cdot \sum_{k \in J} \hat{\phi}_k(\xi) \right\|_1 \leq c\|\theta_K\|_1 \leq c2^K.
\]
If, instead, $|x'| \geq 100^2 K$, then $\sum_{k \in J, k \leq K} \phi_k(x') = \sum_{k \in J, k \leq K} \phi_k(x')$, and so for a suitable $\tilde{x} = \tilde{x}(x'')$ we have that

$$\left| \sum_{k \in J, k \leq K} \phi_k(x') - \theta_K \ast \sum_{k \in J} \phi_k(x') \right| = \int \left| \sum_{k \in J, k \leq K} (\phi_k(x') - \phi_k(x' - x'')) \theta_K(x'') \right| dx''$$

$$\leq c \int \left| \phi'_K(\tilde{x}) \right| 2^{-K \theta_K(x'')} dx'' \leq c 2^{-K/(x')^2}.$$

Hence the claim is proved. Now if we write $P_K(x') = 2^{-K/(x')^2} + 2^{-2K}$ we have that

$$\left| \sum_{k \in J, k \leq K} \phi_k \ast f(x) \right| \leq c \left\{ \sup_K \theta_K \ast \sum_{k \in J} \phi_k \ast f(x) \right\} + \sup_K |P_K \ast f(x)|$$

$$\leq c \left\{ M(Hf)(x) + Mf(x) \right\}.$$

Therefore the lemma is proved.

The following are called Carleson operator and Carleson maximal operator:

$$Cf(x) = \int_{-\pi}^{\pi} \exp(iN(x)x')/x'f(x - x') dx',$$

$$\tilde{C}f(x) = \sup_{\varepsilon > 0, n \in \mathbb{Z}} \int_{\varepsilon < |x'| \leq \pi} \exp(iN(x)x')/x'f(x - x') dx',$$

where $N(x)$ is any measurable bounded integer valued function. We have

**Proposition 1.** The operators $C$ and $\tilde{C}$ are bounded from $L^p(T)$ to itself, $1 < p < \infty$, with norm independent of $N(x)$. Moreover, the following inequality holds: $\tilde{C}f(x) \leq c\{Mf(x) + M(Cf)(x)\}$.

**Proof.** $Cf(x)$ is pointwise dominated by the maximal partial sums operator $\sup_{n} \left| \int_{|x'| \leq \pi} \exp(inx')/x'f(x' - x') dx' \right|$ (also called Carleson operator) whose boundedness has been proved in [4]. As for $\tilde{C}f(x)$ one might go through Carleson and Hunt’s proof and see that it shows that $\tilde{C}$ is also bounded, or observe that (see [6, p. 218])

$$\tilde{C}f(x) \leq \sup_{\varepsilon > 0, n \in \mathbb{Z}} \int_{\varepsilon < |x'| \leq \pi} e^{inx'}/x'f(x' - x') dx'$$

$$\leq c \left\{ \sup_{n} \left( \int_{|x''| \leq \pi} e^{inx''}/x''f(x' - x'') dx'' \right) \right\} (x) \right\} \leq c \{Mf(x) + M(Cf)(x)\}.$$

This proves the desired inequality and concludes the proof.
2. The first variant: $\tilde{H}_1$. Let $B \subset N \times N$. We consider the operator
\[ H_1 f(x, y) = \int \int_{(k,h) \in B} \phi_k(x') \phi_h(y') f(x - x', y - y') \, dx' \, dy'. \]

We have the following

**Theorem 1.** $H_1$ is a bounded operator on $L_p(T^2)$, $1 < p < \infty$, with norm independent of $B$.

**Proof.** Since $\phi_k(\xi) \phi_h(\eta)$ is mainly supported on $\{|\xi| \sim 2^k, |\eta| \sim 2^h\}$, as in Lemma 1, one can show by the Marcinkiewicz multiplier theorem that the operator
\[ H^1_{k_0, h_0} f(x, y) = \int \int_{(k,h) \in B} \phi_k(x') \phi_h(y') f(x - x', y - y') \, dx' \, dy' \]

is bounded on $L_p$ with norm independent of $k_0, h_0$ and $B$. By a standard argument one can show that there exists $\lim_{k_0 \to \infty, h_0 \to \infty} H^1_{k_0, h_0} f(x, y) = H_1 f(x, y)$ in $L_p$ norm by checking it on a dense subset of functions $f$. It suffices to consider $f(x', y') = f_1(x') f_2(y')$, $f_1$ smooth. Then $H_1$ is bounded on $L_p$.

Now we have

**Theorem 2.** Let $J_1$ and $J_2$ be subsets of $N$. Let $B \subset J_1 \times J_2$ be a collection of pairs $(k, h)$ such that:

(i) For every $k$ the section $B_k = \{h \in J_2: (k, h) \in B\}$ is a truncation of $J_2$ possibly depending upon $k$.

(ii) For every $h$ the section $B_h = \{k \in J_1: (k, h) \in B\}$ is a truncation of $J_1$ possibly depending upon $h$.

Then the operator
\[ \tilde{H}_1 f(x, y) = \sup_{k_0, h_0} \left| \int \int_{(k,h) \in B} \phi_k(x') \phi_h(y') f(x - x', y - y') \, dx' \, dy' \right| \]

is bounded from $L_p(T^2)$, $1 < p < \infty$, to itself with norm independent of $B$. Moreover, if $\tilde{H}$ is defined as in the preceding section the following inequality holds:
\[ \tilde{H}_1 f(x, y) \leq c \{ M_{x'} M_{y'} f(x, y) + M_{x'} \tilde{H}_{y'} f(x, y) + M_{y'} \tilde{H}_{x'} f(x, y) + M_{x'} M_{y'} (H_1 f)(x, y) \}. \]

**Proof.** We consider the convolution kernel
\[ G_{k_0, h_0}(x', y') = \sum_{(k,h) \in B} \phi_k(x') \phi_h(y') - \theta_{k_0}(x') \theta_{h_0}(y') \sum_{(k,h) \in B} \phi_k(x') \phi_h(y'). \]

We claim that $|G_{k_0, h_0} f(x, y)|$ is dominated by the right-hand side of (4). To prove it we shall subdivide $T^2$ into four regions $R_i$, $i = 1, \ldots, 4$, and define $G^1_{k_0, h_0}(x', y') = G_{k_0, h_0}(x', y') \chi_{R_i}(x', y')$. The first region is defined as $R_1 = \{ |x'| \leq 1002^{-k_0}, |y'| \leq 1002^{-h_0} \}$. Now $|G^1_{k_0, h_0}(x', y')| \leq c^2 k_0 h_0$ as in the proof of Lemma 1. Hence
\[ |G^1_{k_0 h_0} * f(x, y)| \leq c M_{x'} M_{y'} f(x, y). \] The second region is defined as \( R_2 = \{|x'| < 1002^{-k_0}, |y'| > 1002^{-h_0}\}. \) We have that

\[
|G^2_{k_0 h_0} * f(x, y)| \leq c^{2k_0} \chi_{\{|x'| \leq 1002^{-k_0}\}}(x') \sum_{k=k_0-1}^{k_0} \sum_{h \in B_k \atop h \leq h_0} \phi_h(y') * f(x, y) \\
+ c^{2k_0} h_0 \chi_{\{|x'| \leq 1002^{-k_0}\}}(x') \chi_{\{|y'| \leq 1002^{-h_0}\}}(y') \\
\sum_{(k, h) \in B} \phi_k(x') \phi_h(y') * f(x, y) \\
\leq c \{M_{x'} \hat{H}_{y'} f(x, y) + M_{x'} M_{y'} (H_1 f)(x, y)\}.
\]

Similarly we define \( R_3 = \{|x'| > 1002^{-k_0}, |y'| < 1002^{-h_0}\} \) and we obtain

\[
|G^3_{k_0 h_0} * f(x, y)| \leq c \{M_{y'} \hat{H}_{x'} f(x, y) + M_{x'} M_{y'} (H_1 f)(x, y)\}.
\]

We are left with \( R_4 = \{|x'| \geq 1002^{-k_0}, |y'| \geq 1002^{-h_0}\}. \) Now

\[
|G^4_{k_0 h_0}(x', y')| = \theta_{k_0} \theta_{h_0} * \sum_{(k, h) \in B \atop k \leq k_0, h \leq h_0} \phi_k(x') \phi_h(y') - \sum_{(k, h) \in B \atop k \leq k_0, h \leq h_0} \phi_k(x') \phi_h(y') \\
= \iint \left\{ \sum_{(k, h) \in B \atop k \leq k_0, h \leq h_0} \phi_k(x' - x'') \phi_h(y' - y'') - \phi_k(x') \phi_h(y') \right\} \\
\times \theta_{k_0} (x'') \theta_{h_0} (y'') \, dx'' \, dy'' \\
\leq \iint \left\{ \sum_{(k, h) \in B \atop k \leq k_0, h \leq h_0} \phi_k(x' - x'') \{\phi_h(y' - y'') - \phi_h(y')\} \right\} \\
\times \theta_{k_0} (x'') \theta_{h_0} (y'') \, dx'' \, dy'' \\
+ \iint \sum_{(k, h) \in B \atop k \leq k_0, h \leq h_0} \phi_h(y') \{\phi_k(x' - x'') - \phi_k(x')\} \theta_{k_0} (x'') \theta_{h_0} (y'') \, dx'' \, dy'' \\
= \overline{G}^4_{k_0 h_0}(x', y') + \overline{G}^4_{k_0 h_0}(x', y').
\]
We observe that

$$|\tilde{G}_{k_0,h_0}^4(x',y')| = \left| \int \sum_{k \leq k_0} \{ \phi_k(x' - x'') - \phi_k(x') \} \theta_{k_0}(x'') \sum_{h \in B_k, h \leq h_0} \phi_h(y') \, dx'' \right|.$$ 

Hence

$$|\tilde{G}_{k_0,h_0}^4 * f(x,y)| \leq \int \sum_{k \leq k_0} \left( \int \phi_k(x' - x'') - \phi_k(x') \theta_{k_0}(x'') \, dx'' \right) \cdot \text{Sup}_{h_0} \left| \int \sum_{h \in B_k, h \leq h_0} \phi_h(y') f(x - x', y - y') \, dy' \right| \, dx' \leq c \int P_{k_0}(x') |\bar{H}_y f(x - x', y)| \, dx' \leq c M_x |\bar{H}_y f(x,y)|.$$ 

Now we write

$$\tilde{G}_{k_0,h_0}^4(x',y') = \sum_{(k,h) \in B} \{ \phi_h(y' - y'') - \phi_h(y') \} (\phi_k(x' - x'') - \phi_k(x')) \theta_{k_0}(x'') \sum_{h \leq h_0} \theta_{h_0}(y'') \, dx'' \, dy''$$

$$+ \int \sum_{(k,h) \in B} \{ \phi_h(y' - y'') - \phi_h(y') \} \phi_k(x') \theta_{k_0}(x'') \theta_{h_0}(y'') \, dx'' \, dy''$$

$$= \tilde{G}_{k_0,h_0}^4(x',y') + \tilde{G}_{k_0,h_0}^4(x',y').$$

We start by studying

$$|\tilde{G}_{k_0,h_0}^4(x',y')| \leq \int \sum_{h \leq h_0} \phi_h(y' - y'') - \phi_h(y') \theta_{h_0}(y'')$$

$$\cdot \sum_{k \leq k_0} \phi_k(x' - x'') - \phi_k(x') \theta_{k_0}(x'') \, dx'' \, dy''.$$ 

Therefore, $|\tilde{G}_{k_0,h_0}^4 * f(x,y)| \leq c \phi_{h_0}(y') \cdot P_{k_0}(x') \cdot f(x,y) \leq c M_x M_y f(x,y).$

We are left to study the action of

$$\tilde{G}_{k_0,h_0}^4(x',y') = \sum_{h \leq h_0} \{ \phi_h(y' - y'') - \phi_h(y') \} \theta_{h_0}(y'') \sum_{k \in B_h, k \leq k_0} \phi_k(x') \, dy''.$$ 

Now

$$|\tilde{G}_{k_0,h_0}^4 * f(x,y)| \leq c \phi_{h_0}(y') \text{Sup}_{k_0} \sum_{k \in B_h, k \leq k_0} \phi_k(x') \, f(x,y) \leq c M_y |\bar{H}_x f(x,y)|.$$
So the claim has been proved. Now observe that

\[
\int \int \sum_{(k,h) \in B} \phi_k(x')\phi_h(y') f(x-x', y-y') \, dx' \, dy' \leq |G_{ko\ho} \ast f(x,y)| + \left| \theta_{ko}(x')\theta_{ho}(y') \ast \sum_{(k,h) \in B} \phi_k(x')\phi_h(y') \ast f(x,y) \right|.
\]

The last term is dominated by \( cM_{x'}M_{y'}(H_1f)(x,y) \). This ends the proof.

3. The second variant: \( \hat{H}_2 \). For every \( x \) let \( B_x \) be a fixed subset of \( N \times N \) with the following property:

(ii') for every \( h \) there exists an integer \( r(x,h) \) such that \( B_{xh} = \{k \in N: (k,h) \in B_x\} = \{k \geq r(x,h)\} \).

We consider the operator

\[
H_2 f(x,y) = \int \int \sum_{(k,h) \in B_x} \phi_h(y')\phi_k(x') f(x-x', y-y') \, dx' \, dy'.
\]

The following theorem holds.

**THEOREM 3.** In the assumption (ii') there exists a constant \( c_p \) depending only upon \( p \) such that \( \|H_2f\|_p \leq c_p\|f\|_p \), \( 1 < p < \infty \).

**PROOF.** We are going to prove that the operator

\[
H_{ko\ho}^2 f(x,y) = \int \int \sum_{(k,h) \in B_x} \phi_h(y')\phi_k(x') f(x-x', y-y') \, dx' \, dy'
\]

that for simplicity we also denote by \( H_2 f(x,y) \), is bounded on \( L_p \) with bound depending only upon \( p \). For this purpose it is enough to consider those \( f \) which are smooth. The proof is based on (a) and (b) of §1. Let us denote by \( S \) the classical \( S \)-function acting on functions of one variable, the variable \( y \) in our case. By the Littlewood-Paley theorem we know that for a.e. \( x \) fixed

\[
\left\| \left( \sum_J |S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|_{L_p(dy)} \sim \|H_2 f(x,y)\|_{L_p(dy)}.
\]

We raise this relation to the \( p \)th power and integrate it with respect to \( x \), obtaining

\[
\left\| \left( \sum_J |S_J H_2 f(x,y)|^2 \right)^{1/2} \right\|_p^p \sim \|H_2 f(x,y)\|_{L_p(dydx)}^p.
\]

Therefore to prove the boundedness of \( H_2 \) it suffices to prove that

\[
(5) \quad \left\| \left( \sum_J S_J H_2 f(x,y) \right)^{1/2} \right\|_{L_p(dydx)} \leq c_p\|f\|_p.
\]
Let $\eta$ denote the dual variable of $y$. In estimating the left-hand side of (5) we consider the dyadic intervals $J$ of each half line separately. Assume from now on that $J$ belongs to $\{ \eta > 0 \}$. For a.e. $x$ fixed we have that

$$S_J H_2 f(x,y) = \int_J e^{i\eta y} \sum_{h \leq h_0} \sum_{k \geq r(x,h)} \phi_h(\eta) \phi_k(x') \ast f(x,y)(\eta) \, d\eta$$

$$= \int_J e^{i\eta y} \sum_{h \leq h_0} \left\{ \int_{2^h}^{2^{h+1}} (\hat{\phi}_h)'(t) \, dt + \hat{\phi}_h(2^{h+1}) \right\} \sum_{k \geq r(x,h)} \phi_k(x') \ast \hat{f}(x,\eta) \, d\eta$$

$$= S_J^{(1)} H_2 f(x,y) + S_J^{(2)} H_2 f(x,y).$$

Let $J = [2^h, 2^{h+1})$, $h \geq 0$. By exchanging the order of integration we have

$$|S_J^{(2)} H_2 f(x,y)| \leq \sum_{h \leq h_0} |\hat{\phi}_h(2^h)| \cdot \left| \sum_{k \geq r(x,h)} \phi_k(x') \ast S_J f(x,y) \right| \leq c H S_J f(x,y).$$

Next we turn to

$$S_J^{(1)} H_2 f(x,y) = \int_J \sum_{h \leq h_0} (\hat{\phi}_h)'(t) \left\{ \int_t^{2^{h+1}} e^{i\eta y} \sum_{k \geq r(x,h)} \phi_k(x') \ast \hat{f}(x,\eta) \, d\eta \right\} \, dt.$$

Now for every $t \in J$ let us denote by $S_t$ the multiplier transformation corresponding to the interval $[t, 2^{h+1})$. Using the relation $S_t S_J = S_t$, Schwarz inequality and (b) we obtain that

$$|S_J^{(1)} H_2 f(x,y)| \leq \left( \int_J \sum_h |(\hat{\phi}_h)'(t)| \left| \sum_{k \geq r(x,h)} \phi_k(x') \ast S_t S_J f(x,y) \right| \, dt \right)^{1/2}$$

$$\leq c \left( \int_J \sum_h |(\hat{\phi}_h)'(t)| \cdot |\tilde{H} S_t S_J f(x,y)|^2 \, dt \right)^{1/2}.$$
Now we observe that
\[
\left\| \left( \sum_j |S_j H_2 f(x, y)|^2 \right)^{1/2} \right\|_{L^p(dydx)}^p \leq \left\| \left( \sum_j |S_j^{(1)} H_2 f(x, y)|^2 \right)^{1/2} + \left( \sum_j |S_j^{(2)} H_2 f(x, y)|^2 \right)^{1/2} \right\|_{L^p(dydx)}^p.
\]

Now if we use (6) and (7) we obtain (5). So we proved that the operators $H_{k_0h_0}^2$ are uniformly bounded on $L^p$, $1 < p < \infty$. Then to show that there exists
\[
\lim_{k_0, h_0 \to \infty} H_{k_0h_0}^2 f(x, y) = H_2 f(x, y)
\]
in $L^p$ norm it is enough to apply $H_{k_0h_0}^2$ to $f(x', y') = f_1(x')f_2(y')$ where $f_i$ are smooth. Finally one can prove that $H_2$ is bounded on $L^p$.

Then we consider the operator
\[
\tilde{H}_2 f(x, y) = \sup_{h_0} \left| \int \int \sum_{(k, h) \in B_x} \phi_h(y') \phi_k(x') f(x - x', y - y') dx' dy' \right|.
\]

We have

**THEOREM 4.** If (ii') is satisfied then there exists a constant $c_p$ depending only upon $p$ such that $\|\tilde{H}_2 f\|_p \leq c_p \|f\|_p$, $1 < p < \infty$. Moreover, the following inequality holds:
\[
\tilde{H}_2 f(x, y) \leq c \{ M_{y'} \tilde{H}_x f(x, y) + M_y (H_2 f(x, y))(y) \}.
\]

**PROOF.** We observe that in the formula defining $H_2 f(x, y)$, once the convolution on the $x'$ variable has been performed, the operator acting on the $y'$ variable is a constant coefficients singular integral. Hence proceeding as in the proof of Theorem 2 one can show the stated inequality. As a consequence $\tilde{H}_2$ is bounded on $L^p$.

**4. A counterexample.** It would be useful for the applications to be able to use the operator
\[
\tilde{H}_3 f(x, y) = \sup_B \left| \int \int \sum_{(k, h) \in B} \phi_h(y') \phi_k(x') f(x - x', y - y') dx' dy' \right|,
\]
with $B$ an arbitrary subset of $N \times N$. Unfortunately such an operator is unbounded even on $L^2$ as it is shown by the following counterexample which has been pointed out to us by C. Fefferman. Consider the operator
\[
K f(x) = \sum_{k \in B_x} \phi_k \ast f(x).
\]

We are going to show how to define $B_x$ such that $K f(x) = \infty$ at every $x$ for a suitable $f$ belonging to $L^2(T)$. Let $f \sim \sum_{n=1}^\infty \{ \exp(i \lambda_n x) \}/n$, where the $\lambda_n$'s are sparse, for instance $\lambda_n = 2^{2^n}$. Observe that $\phi_{\lambda_n} \ast f(x) = c_0 \{ \exp(i \lambda_n x) \}/n + o(2^{-n})$,
where $c_0 = \hat{\phi}_{\lambda_n}(\lambda_n) = \hat{\phi}(1) \neq 0$ as we may assume. Let us subdivide $\{\theta: 0 < \theta \leq 2\pi\}$ into four disjoint quadrants $Q_1, \ldots, Q_4$, where $Q_1 = \{\theta: -\pi/4 < \theta \leq \pi/4\}$. For every $x$ fixed there exists at least one of the $Q_i$'s say $Q_1$, which contains infinitely many $\lambda_n x$, $n = 1, 2, \ldots$, and such that $\sum_{\lambda_n x \in Q_1} 1/n = \infty$. Define $B_x = \{k = \lambda_n: \lambda_n x \in Q_1\}$. We have that $|\text{Re}(\exp(ikx))|$ or $|\text{Im}(\exp(ikx))| > c$ for every $k \in B_x$. So $\sum_{k \in B_x} \phi_k * f(x) = \infty$.

5. An application. These results have an application to the study of the operator

$$Tf(x, y) = \int \int \exp \{i(N(x, y)x' + N^2(x, y)y')\}/x'y'f(x - x', y - y') \, dx' \, dy',$$

where $N(x, y)$ is a measurable bounded integer valued function. $Tf(x, y)$ is the maximal partial sums operator $\text{Sup}_N |S_{N,N^2}f(x, y)|$, where

$$S_{N,N^2}f(x, y) = \sum_{|n| \leq N \atop |m| \leq N^2} a_{nm} \cdot \exp \{i(nx + my)\}.$$

To prove a.e. convergence of $S_{N,N^2}f(x, y)$ one looks for an estimate of $Tf(x, y)$ which does not depend upon $N(x, y)$. In [5] we proved a uniform estimate in $L_p$ of $Tf(x, y)$ for $N(x, y) = |\lambda xy|$, $\lambda > 10^{10}$, using Theorems 1 and 2. Moreover, we gave an example of a family of functions $N(x, y)$ which cannot be handled by Theorems 1 and 2 and which leads instead to $H_2$ and $\hat{H}_2$. Let us finally observe that the boundedness of $H_1$ and $H_2$ is enough for the estimate of pairs of norm 1 while (1) and (2) are needed for the estimate of pairs of norm smaller than 1.

REFERENCES


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