DESCRIPTIVE COMPLEXITY OF FUNCTION SPACES

BY

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ABSTRACT. In this paper we show that \( C_\pi(X) \), the set of continuous, real-valued functions on \( X \) topologized by the pointwise convergence topology, can have arbitrarily high Borel or projective complexity in \( \mathbb{R}^X \) even when \( X \) is a countable regular space with a unique limit point. In addition we show how to construct countable regular spaces \( X \) for which \( C_\pi(X) \) lies nowhere in the projective hierarchy of the complete separable metric space \( \mathbb{R}^X \).

1. Introduction. Let \( C_\pi(X) \) be the set of continuous, real-valued functions on a space \( X \) and topologize \( C_\pi(X) \) as a subspace of the full product \( \mathbb{R}^X \). In [DGLvM] it is shown that if \( X \) is completely regular, then \( C_\pi(X) \) cannot be a \( G_\delta \), \( F_\sigma \)- or \( G_\delta \sigma \)-subset of \( \mathbb{R}^X \) unless \( X \) is discrete and that for any countable metrizable space \( X \), \( C_\pi(X) \) will be an \( F_\sigma \delta \)-subset of \( \mathbb{R}^X \). In the terminology of [KM and K], \( C_\pi(X) \) cannot have multiplicative class 1 and cannot have additive class 1 or 2, but may have multiplicative class 2.

In this paper we study the descriptive complexity of \( C_\pi(X) \) in \( \mathbb{R}^X \) when \( X \) is countable (so that \( \mathbb{R}^X \) is a complete separable metric space). Our main results can be summarized as follows.

THEOREM. (a) Given any \( \alpha < \omega_1 \), there is a countable regular space \( X \) such that \( C_\pi(X) \) is a Borel subset of \( \mathbb{R}^X \) having additive class \( \beta \), where \( \alpha \leq \beta \leq 3 + \alpha + 2 \) (§§2 and 3).

(b) Given any \( n \geq 1 \) there is a countable regular space \( Y \) such that \( C_\pi(Y) \in \mathcal{L}_n(\mathbb{R}^Y) - \mathcal{L}_{n-1}(\mathbb{R}^Y) \), where \( \mathcal{L}_n(\mathbb{R}^Y) \) is the family of projective sets of class \( n \) in the complete separable metric space \( \mathbb{R}^Y \) (§4).

(c) There is a countable regular space \( Z \) such that \( C_\pi(Z) \notin \bigcup \{ \mathcal{L}_n(\mathbb{R}^Z) : 0 \leq n < \omega \} \) (§§4 and 5).

The spaces \( X, Y \) and \( Z \) in the above Theorem can be obtained from a single general construction which associates with each subset \( S \subset 2^\omega \) a certain countable regular space \( \Sigma_S \) having a unique nonisolated point. The descriptive complexity of \( S \) in \( 2^\omega \) determines the complexity of \( C_\pi(\Sigma_S) \) in \( \mathbb{R}^{\Sigma_S} \). To describe \( \Sigma_S \) precisely, we begin by letting \( T_n = 2^n \) be the set of functions from \( \{0, 1, \ldots, n - 1\} \) into \( \{0, 1\} \), i.e., the set of ordered \( n \)-tuples of 0's and 1's. Let \( T = \bigcup \{ T_n | n \geq 1 \} \) and partially order \( T \) by function extension. A branch of \( T \) is a maximal linearly ordered subset of \( T \), i.e., a linearly ordered subset \( B \subset T \) having \( \text{card}(B \cap T_n) = 1 \) for each \( n \geq 1 \). Observe that if \( B \) and \( \hat{B} \) are distinct branches of \( T \), then \( B \cap \hat{B} \) must be a finite set.
Given \( x \in 2^\omega \), the set \( B_x = \{ \langle x(0) \rangle, \langle x(0), x(1) \rangle, \langle x(0), x(1), x(2) \rangle, \ldots \} \) is a branch of \( T \). Conversely, each branch \( B \) of \( T \) has the form \( B = B_x \) for a unique \( x \in 2^\omega \).

Let \( \mathcal{B} = \{ B \mid B \text{ is a branch of } T \} \).

Let \( \mathcal{P}(T) = \{ A \mid A \subset T \} \) and topologize \( \mathcal{P}(T) \) using open sets of the form \( [Y, N] = \{ A \in \mathcal{P}(T) \mid Y \subset A \subset T - N \} \), where \( Y \) and \( N \) are arbitrary finite subsets of \( T \). The resulting space is compact and metrizable, and is homeomorphic to the product space \( 2^T \) under the mapping which identifies each subset \( A \in \mathcal{P}(T) \) with its characteristic function \( \chi_A \). The mapping \( x \to B_x \) is easily seen to be a homeomorphism of \( 2^\omega \) into \( \mathcal{P}(T) \) whose image is exactly the set \( \mathcal{B} \) defined above.

For each subset \( S \subset 2^\omega \), the collection \( \{ T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) \mid n \geq 1, x_i \in S \text{ and } F \subset T \text{ is finite set} \} \) is a filter base. Let \( p_S \) be the filter generated by that filter base. Let \( \infty \) be any point not in \( T \cup 2^\omega \) and let \( \Sigma_S = T \cup \{ \infty \} \). Topologize \( \Sigma_S \) by isolating each point of \( T \) and by using the family \( \{ \mathcal{P} \cup \{ \infty \} \mid \mathcal{P} \in p_S \} \) as a neighborhood base at \( \infty \). The space \( \Sigma_S \) is countable, regular and (since \( p_S \) is a free filter) is \( T_1 \). The spaces mentioned in the above Theorem are all of the form \( \Sigma_S \) for various subsets \( S \) of \( 2^\omega \).

However, even though the function spaces \( C_\pi(\Sigma_S) \) for \( S \subset 2^\omega \) provide enough pathology to prove our Theorem, they are all well behaved in some senses. In §5 we prove that each \( C_\pi(\Sigma_S) \) is a Baire Property subset of \( R^{\Sigma_S} \) and is meager in \( R^{\Sigma_S} \) (equivalently, \( C_\pi(\Sigma_S) \) is not a Baire space) and we exhibit a countable regular space \( X \) with a unique nonisolated point such that \( C_\pi(X) \) is a second category subset of \( R^X \) (equivalently, \( C_\pi(X) \) is a Baire space), is not a Baire Property subset of \( R^X \), and is not a Borel, analytic or co-analytic subset of \( R^X \) (see Example 5.5).

The standard references for descriptive theory in complete separable metric spaces are [K and KM]. Our topological terminology is consistent with [E] and [Ox2] is a good source for properties of Baire spaces. The authors wish to thank Jean Calbrix and Fons van Engelen for their comments on an earlier version of this paper.

2. A lower bound for the complexity of \( C_\pi(\Sigma_S) \).

2.1 THEOREM. Let \( S \subset 2^\omega \) and let \( \Sigma = \Sigma_S \). Then \( C_\pi(\Sigma) \) contains a relatively closed subset which is homeomorphic to \( S \).

PROOF. Recall that in \( \Sigma_S \), the point \( \infty \) has a neighborhood base consisting of all sets of the form \( \{ \infty \} \cup (T - (B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup F)) \), where \( x_i \in S \) and \( F \subset T \) is finite. For each \( x \in 2^\omega \) define a function \( f_x : \Sigma \to R \) by \( f_x(\infty) = 0, \ f_x(t) = 0 \) if \( t \in T - B_x \) and \( f_x(t) = 1 \) if \( t \in B_x \). Define \( \lambda : 2^\omega \to R^{\Sigma} \) by \( \lambda(x) = f_x \). Clearly \( \lambda \) is 1-1 and continuous, so that \( \lambda \) embeds \( 2^\omega \) as a closed subspace of \( R^{\Sigma} \). Furthermore \( \lambda(x) \in C_\pi(\Sigma) \) whenever \( x \in S \) because for such an \( x \), the function \( f_x \) is constant on the neighborhood \( \{ \infty \} \cup (T - B_x) \) of \( \infty \). Conversely, if \( f_x \in C_\pi(\Sigma) \) for some \( x \in 2^\omega \), then \( f_x^{-1}([0, \ 1]) \) must be a neighborhood of \( \infty \) so that for some \( x_1, \ldots, x_n \in S \) and some finite \( F \), the set \( f_x^{-1}([0, \ 1]) = \{ \infty \} \cup (T - B_x) \) must contain the basic neighborhood \( \{ \infty \} \cup (T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)) \). But then \( B_x \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F \) so that \( B_{x_i} \cap B_x \) is infinite for some \( i \) and hence \( B_x = B_{x_i} \), i.e., \( x = x_i \in S \). Therefore \( \lambda[S] = C_\pi(\Sigma) \cap \lambda[2^\omega] \) showing that \( \lambda[S] \) is a relatively closed subset of \( C_\pi(\Sigma) \). □

2.2 COROLLARY. If \( S \) is not a Borel subset of \( 2^\omega \) (resp., if \( S \) is not a projective subset of \( 2^\omega \)), then \( C_\pi(\Sigma_S) \) is not a Borel subset (resp. a projective subset) of \( R^{\Sigma_S} \).
PROOF. Write \( \Sigma = \Sigma_S \). In the complete separable metric space \( \mathbb{R}^\Sigma \), a relatively closed subset of a Borel (resp., projective) set is again a Borel (resp., projective) set in \( \mathbb{R}^\Sigma \) and it is known that homeomorphisms preserve Borel (resp., projective) sets [K, Chapter 3, §35, IV, Corollary 1 and Chapter 3, §38, VII, Theorem 1] contrary to our assumption that \( S \) is not Borel (resp., projective) in \( 2^\omega \). □

2.3 COROLLARY. There is a countable regular space \( X \) such that \( C_\pi(X) \) is not a Borel subset of \( \mathbb{R}^X \).

PROOF. Let \( S \) be a non-Borel subset of \( 2^\omega \) and let \( X = \Sigma_S \). Now apply 2.2. □

3. An upper bound for the Borel complexity of \( C_\pi(\Sigma_S) \). In §2 we proved that \( C_\pi(\Sigma_S) \) always contains a closed subspace homeomorphic to \( S \) so that if \( S \) is not a Borel set, then neither is \( C_\pi(\Sigma_S) \). In this section we study the situation where \( S \) is a Borel subset of \( 2^\omega \) and we prove

3.1 THEOREM. Let \( S \) be a Borel subset of \( 2^\omega \) having additive class \( \alpha \geq 1 \) and let \( \Sigma = \Sigma_S \). Then \( C_\pi(\Sigma) \) is a Borel subset of \( \mathbb{R}^\Sigma \) of class \( \beta \), where \( \alpha \leq \beta \leq 3 + \alpha + 2 \).

PROOF. Following the notation of §1, we let \( p = p_S \) be the filter on \( T \) generated by all sets of the form \( T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F) \), where \( x_j \in S \) for \( 1 \leq j \leq n \) and \( F \) is any finite subset of \( T \).

For each \( m \geq 1 \), define \( \psi_m: \mathbb{R}^\Sigma \rightarrow \mathcal{P}(T) \) by \( \psi_m(f) = \{ t \in T | |f(\infty) - f(t)| \geq 1/m \} \). In Lemma 3.2 we show that \( \psi_m \) is a Borel mapping of class 1. Next, define a set \( \mathcal{D} \subset \mathcal{P}(T) \) by \( \mathcal{D} = \{ A \in \mathcal{P}(T) | A \cap P = \emptyset \ \text{for some} \ P \in p \} \). In Lemma 3.6 we prove that \( \mathcal{D} \) is a Borel subset of \( \mathcal{P}(T) \) of additive class \( \leq 2 + \alpha \) so that \( \psi_m^{-1}[\mathcal{D}] \) is a Borel set of additive class \( \leq 3 + \alpha \). Because a function \( f \in \mathbb{R}^\Sigma \) is continuous if and only if \( \{ t \in T | |f(\infty) - f(t)| < 1/m \} \) belongs to \( p \) for each \( m \), we have \( C_\pi(\Sigma) = \bigcap \{ \psi_m^{-1}[\mathcal{D}] | |m| \geq 1 \} \) showing that \( C_\pi(\Sigma) \) is a Borel set of additive class \( \beta \leq (3 + \alpha + 2) \).

From §2, a closed subspace of \( C_\pi(\Sigma) \) is homeomorphic to \( S \), so the additive class of \( C_\pi(\Sigma) \) cannot be smaller than the additive class of \( S \) and we obtain \( \alpha \leq \beta \). □

All that remains is to prove some lemmas.

3.2 LEMMA. Each \( \psi_m \) is a Borel map of class 1.

PROOF. It is enough to show that \( \psi_m^{-1}[[Y,N]] \) is an \( F_\sigma \)-subset of \( \mathbb{R}^\Sigma \) for each basic open set \([Y,N]\) in \( \mathcal{P}(T) \). Now

\[
\psi_m^{-1}[[Y,N]] = \{ f \in \mathbb{R}^\Sigma | \begin{array}{c} Y \subset \{ t \in T | |f(t) - f(\infty)| \geq 1/m \} \\ \cap \{ f \in \mathbb{R}^\Sigma | \{ t \in T | |f(t) - f(\infty)| \geq 1/m \} \subset T - N \} \}
\]

The first of those two sets is closed and, since \( N \) is finite, the second is open. Hence their intersection is an \( F_\sigma \)-set, as claimed. □

3.3 LEMMA. The set \( \mathcal{A} = \{ A \in \mathcal{P}(T) | \text{for some} \ B_1, \ldots, B_n \in \mathcal{B} \text{ and some finite } F \subset T, \ A \subset B_1 \cup \cdots \cup B_n \cup F \} \) is a \( \sigma \)-compact subset of \( \mathcal{P}(T) \).

PROOF. For a fixed finite \( F \subset T \) and a fixed \( n \), let \( \mathcal{A}(F,n) = \{ (A,B_1,\ldots,B_n) | B_i \in \mathcal{B} \text{ and } A \subset B_1 \cup \cdots \cup B_n \cup F \} \). Then \( \mathcal{A}(F,n) \) is a closed subset of the compact space \( \mathcal{P}(T) \times \mathcal{B}^n \). Let \( \pi_n: \mathcal{P}(T) \times \mathcal{B}^n \rightarrow \mathcal{P}(T) \) denote first coordinate projection. Then \( \mathcal{A} = \bigcup \{ \pi_n(\mathcal{A}(F,n)) | n \geq 1 \text{ and } F \subset T \text{ is finite} \} \) so that \( \mathcal{A} \) is a \( \sigma \)-compact subspace of \( \mathcal{P}(T) \) as claimed. □
3.4 NOTATION. Recall that $\mathcal{B}$ is the set of all branches of $T$, topologized as a subspace of the compact metric space $\mathcal{P}(T)$. Being the continuous image of $2^\omega$ under the map $\mu(x) = B_x$, $\mathcal{B}$ is compact. For $n \geq 1$, let $\Phi_n = \{K \subset \mathcal{B} | \text{card}(K) = n\}$ and let $\Phi = \bigcup \{\Phi_n | n \geq 0\}$. Topologize $\Phi$ with the Vietoris topology, i.e., by using all subsets of $\Phi$ of the forms $\{K \in \Phi | K \subset U\}$ and $\{K \in \Phi | K \cap V \neq \emptyset\}$ as a subbase where $U$ and $V$ are arbitrary open subsets of $\mathcal{B}$. Then $\Phi$ is a $\sigma$-compact metrizable space [KM, p. 392]. Recall that each branch of $T$ is of the form $B_x$ for some $x \in 2^\omega$ and let $\Phi_S = \{K \in \Phi | K \subset \{B_x | x \in S\}\} = \{K | K$ is a finite subset of $\{B_x | x \in S\}\}.

3.5 LEMMA. With $\mathcal{A}$ as in 3.3, for each $A \in \mathcal{A}$ let $i(A) = \{B \in \mathcal{B} | B \cap A$ is infinite$\}$. Then $i: \mathcal{A} \to \Phi$ is a Borel mapping of class 2.

PROOF. Fix $A \in \mathcal{A}$ and choose branches $B_1, \ldots, B_n$ and a finite set $F$ with $A \subset B_1 \cup \cdots \cup B_n \cup F$. If $B$ is any branch of $T$ such that $A \cap B$ is infinite, then $B \cap B_k$ is infinite for some $k = 1, 2, \ldots, n$ so that $B$ is one of the branches $B_1, \ldots, B_n$. Hence $i(A)$ is finite so $i(A) \in \Phi$. (If $A$ is finite, then $i(A) = \emptyset \in \Phi$.)

(a) Fix an open subset $U$ of $\mathcal{B}$ and consider $i^{-1}\{\{K \in \Phi | K \subset U\}\} = \{A \in \mathcal{A} | i(A) \subset U\}$. Because $U$ is an open subset of the compact metric space $\mathcal{B}$, $U$ is $\sigma$-compact. According to 3.3, so is $\mathcal{A}$, and we conclude that the product space $\mathcal{A} \times U^n$ is $\sigma$-compact for each $n \geq 1$, where $U^n$ is the product of $n$ copies of $U$. Fix $n \geq 1$ and fix a finite set $F \subset T$. Then the set $C(n, F) = \{(A, B_1, \ldots, B_n) \in \mathcal{A} \times U^n | A \subset B_1 \cup \cdots \cup B_n \cup F\}$ is closed in $\mathcal{A} \times U^n$, so $C(n, F)$ is $\sigma$-compact. Let $\pi_n: \mathcal{A} \times U^n \to \mathcal{A}$ be first coordinate projection. Then $i^{-1}\{\{K \in \Phi | K \subset U\}\} = \bigcup \{\pi_n[C(n, F)] | n \geq 1\} \supset A$ and $F \subset T$ is finite so $i^{-1}\{\{K \in \Phi | K \subset U\}\}$ is a $\sigma$-compact subset of $\mathcal{A}$ (and therefore a $G_{\sigma}$-subset of $\mathcal{A}$).

(b) Next consider $i^{-1}\{\{K \in \Phi | K \cap V \neq \emptyset\}\}$, where $V$ is a compact, open subset of $\mathcal{B}$. Then $\mathcal{B} - V$ is open and $\{K \in \Phi | K \cap V \neq \emptyset\} = \Phi - \{K \in \Phi | K \subset \mathcal{B} - V\}$. Hence $i^{-1}\{\{K \in \Phi | K \cap V \neq \emptyset\}\} = \Phi - i^{-1}\{\{K \in \Phi | K \subset \mathcal{B} - V\}\}$ which is a $G_\delta$-subset in light of (a).

(c) Finally, consider $i^{-1}\{\{K \in \Phi | K \cap U \neq \emptyset\}\}$, where $U$ is an arbitrary open subset of $\mathcal{B}$. There is a sequence $\langle V_n \rangle$ of compact, open subsets of $\mathcal{B}$ having $U = \bigcup \{V_n | n \geq 1\}$ so that $i^{-1}\{\{K \in \Phi | K \cap U \neq \emptyset\}\} = \bigcup \{i^{-1}\{\{K \in \Phi | K \cap V_n \neq \emptyset\}\} | n \geq 1\}$ which is a $G_{\delta\sigma}$-set in $\mathcal{A}$ because of (b).

(d) Since sets of the form $\{K \in \Phi | K \subset U\}$ and $\{K \in \Phi | K \cap U \neq \emptyset\}$ form a subbase for the separable metric space $\Phi$, it follows that $i$ is a Borel mapping of class 2. □

3.6 LEMMA. With $\Phi_S$ as defined in 3.4, $\Phi_S$ is a Borel subset of $\Phi$ whose additive class is $\alpha$ (= the additive class of $\mathcal{S}$).

PROOF. For $n \geq 1$, define $\theta_n: (2^\omega)^n \to \Phi$ by $\theta_n(x_1, x_2, \ldots, x_n) = \{B_{x_1}, B_{x_2}, \ldots, B_{x_n}\}$. Then $\theta_n$ is continuous. Let $G_n = \{(x_1, \ldots, x_n) \in (2^\omega)^n | x_j \neq x_k \text{ whenever } 1 \leq j < k \leq n\}$. Then $G_n$ is open and given $(x_1, \ldots, x_n) \in G_n$ there is an open neighborhood $N$ of $(x_1, \ldots, x_n)$ in $G_n$ and an open neighborhood $\Phi'$ of $\theta_n(x_1, \ldots, x_n)$ in $\Phi$ such that $\theta_n$ maps $N$ homeomorphically onto $\Phi' \cap \Phi_n$. (We say that $\theta_n$ is a local homeomorphism from $G_n$ onto $\Phi_n$.)

Now consider the subspace $S$ of $2^\omega$. Clearly $\theta_n|G_n \cap S^n = \Phi_n \cap \Phi_S$ so $\theta_n$ is a local homeomorphism from $G_n \cap S^n$ onto $\Phi_n \cap \Phi_S$. Because $S$ is of additive class $\alpha$, so is $S^n$ [K, p. 346]. Hence so is $G_n \cap S^n$ as is each relatively open subset of $G_n \cap S^n$.
(Recall that since $\alpha \geq 1$, each open subset of $G_n$ is of additive class $\alpha$.) Therefore, the metric space $\Phi_S \cap \Phi_n$ admits an open cover by sets of additive class $\alpha$ so that $\Phi_S \cap \Phi_n$ has additive class $\alpha$ [K, p. 358]. Because $\Phi_S = \{\emptyset\} \cup (\cup\{\Phi_S \cap \Phi_n | n \geq 1\})$, $\Phi_S$ also has additive class $\alpha$, as claimed. □

3.7 Lemma. Let $\mathcal{D} = \{A \in \mathcal{P}(T) | \text{ some } P \in \mathcal{P} \text{ has } P \cap A = \emptyset\}$. Then $\mathcal{D}$ is of additive class $2 + \alpha$.

Proof. With $i$ as in 3.5, we claim that $\mathcal{D} = i^{-1}[\Phi_S]$. For let $A \in \mathcal{D}$. Choose $P \in \mathcal{P}$ with $P \cap A = \emptyset$. Then $P$ contains some set $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$, where $x_j \in S$, so $A \subseteq B_{x_1} \cup \cdots \cup B_{x_n} \cup F$. Hence $A \in \mathcal{A}$ so that $i(A)$ is defined. As noted in the proof of 3.5, since $A \subseteq B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, $i(A) \subseteq \{B_{x_1}, \ldots, B_{x_n}\}$ showing that $i(A) \in \Phi_S$. Conversely, suppose $A \in i^{-1}[\Phi_S]$. Then either there are points $x_1, \ldots, x_n \in S$ with $i(A) = \{B_{x_1}, \ldots, B_{x_n}\}$ or else $i(A) = \emptyset$ in which case $A$ is finite. Consider the first possibility. If the set $A - (B_{x_1} \cup \cdots \cup B_{x_n})$ were infinite, some other branch of $T$ would have an infinite intersection with $A$ which is impossible, so the set $F = A - (B_{x_1} \cup \cdots \cup B_{x_n})$ is finite and we have $A \subset B_{x_1} \cup \cdots \cup B_{x_n} \cup F$, so that $A$ is disjoint from $T - (B_{x_1} \cup \cdots \cup B_{x_n} \cup F)$ which belongs to the filter $\mathcal{P}$ so that $A \in \mathcal{D}$. The case where $A$ is finite is easy because then the set $P_0 = T - A$ belongs to $\mathcal{P}$ so that $A \in \mathcal{D}$.

Because $i$ is a Borel map of class $2$ and because by 3.6 the set $\Phi_S$ has additive class $\alpha$ (where $\alpha$ is the additive class of $S$), $i^{-1}[\Phi_S]$ has additive class $2 + \alpha$, as claimed. □

4. The projective hierarchy. Recall the definition of the projective classes in a complete separable metric space $Z$ [K, Chapter 3, §38]:

\[ \mathcal{L}_0(Z) = \{A : A \text{ is a Borel subset of } Z\}, \]

\[ \mathcal{L}_{n+1}(Z) = \left\{ \begin{array}{ll} \{f[A] : A \in \mathcal{L}_n(Z) \text{ and } f : A \to Z \text{ is continuous}\} & \text{if } n \text{ is even}, \\
\{Z - A : A \in \mathcal{L}_n(Z)\} & \text{if } n \text{ is odd}. \end{array} \right. \]

Thus, $\mathcal{L}_1(Z)$ is the family of analytic sets in $Z$, $\mathcal{L}_2(Z)$ is the family of co-analytic sets in $Z$, etc. The techniques of §§2 and 3 can be used to prove an analogue of 3.1 for projective sets. In our proof we will invoke theorems which are ordinarily stated for mappings into complete metric spaces [K, §38, III, Propositions 2 and 5, and VII, Theorem 1], applying those results to mappings into the $\sigma$-compact metric space $\Phi$ defined in 3.4. Extending the proofs given in [K] to cover this situation is easily done.

4.1 Theorem. Suppose $S \in \mathcal{L}_r(2^\omega)$ for some $r \geq 1$. Let $\Sigma = \Sigma_S$. Then $C_\pi(\Sigma) \in \mathcal{L}_r(\mathcal{R}^\Sigma)$. Furthermore, if $S \notin \mathcal{L}_{r-1}(2^\omega)$, then $C_\pi(\Sigma) \notin \mathcal{L}_{r-1}(\mathcal{R}^\Sigma)$.

Proof. Define $\psi_m : \mathcal{R}^\Sigma \to \mathcal{P}(T)$ and $\mathcal{D} \subset \mathcal{P}(T)$ as in 3.1. Suppose we know that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$. Then by [K, §38, III, Proposition 5], $\psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathcal{R}^\Sigma)$ for each $m$ so that by [K, §38, III, Proposition 3] we would have $C_\pi(\Sigma) = \bigcap_{m=1}^\infty \psi_m^{-1}[\mathcal{D}] \in \mathcal{L}_r(\mathcal{R}^\Sigma)$ as claimed. Thus it will be enough to show that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$.

To prove that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$, we define the $\sigma$-compact set $\mathcal{A} \subset \mathcal{P}(T)$ as in 3.3, the $\sigma$-compact metric space $\Phi$ as in 3.4, the Borel measurable mapping $i : \mathcal{A} \to \Phi$ as in (3.5), and the set $\Phi_S$ as in 3.4. As in the proof of 3.7, $\mathcal{D} = \mathcal{A} \cap i^{-1}[\Phi_S]$. If we knew that $\Phi_S \in \mathcal{L}_r(\Phi)$, it would follow from [K, §38, III, Proposition 5] that $i^{-1}[\Phi_S] \in \mathcal{L}_r(\mathcal{A})$. Since $\mathcal{A}$ is $\sigma$-compact and hence in $\mathcal{L}_r(\mathcal{P}(T))$, it would follow
that $\mathcal{D} \in \mathcal{L}_r(\mathcal{P}(T))$ [K, §38, III, Proposition 2]. Therefore it will be enough to show that $\Phi_s \in \mathcal{L}_r(\Phi)$. Define function $\theta_n : (2^\omega)^n \to \Phi$ as in 3.6. According to [K, §38, III, Proposition 1], $s^n \in \mathcal{L}_r((2^\omega)^n)$. Because each open subset $H$ of $(2^\omega)^n$ also belongs to $\mathcal{L}_r((2^\omega)^n)$ we see that $H \cap G_n \cap s^n \in \mathcal{L}_r((2^\omega)^n)$ whenever $H$ is open in $(2^\omega)^n$. But $\theta_n$ is known to be a local homeomorphism of $G_n \cap s^n$ onto the separable metric space $\Phi_s \cap \Phi_n$ so there is a sequence $H_1, H_2, \ldots$ of subsets of $G_n$ such that for each $k$, $\theta_n$ maps $H_k \cap G_n \cap s^n$ homeomorphically onto a relatively open subset of $\Phi_n \cap \Phi_s$ and such that $\Phi_n \cap \Phi_s = \bigcup \{\theta_n[H_k \cap G_n \cap s^n]|k \geq 1\}$. Because $H_k \cap G_n \cap s^n \in \mathcal{L}_r((2^\omega)^n)$ for each $k$, it follows from [K, §38, VII, Theorem 1] that $\theta_n[H_k \cap G_n \cap s^n] \in \mathcal{L}_r(\Phi)$. But then $\Phi_n \cap \Phi_s$, being a countable union of members of $\mathcal{L}_r(\Phi)$, also belongs to $\mathcal{L}_r(\Phi)$. For the same reason, the set $\Phi_s = \bigcup \{\Phi_s \cap \Phi_n|n \geq 1\}$ also belongs to $\mathcal{L}_r(\Phi)$ as claimed.

Finally suppose $S \notin \mathcal{L}_{r-1}(2^\omega)$. According to 2.1, there is a (relatively) closed subspace $S^* = C_\pi(\Sigma) \cap D$, where $D$ is some closed subset in $R^\Sigma$. If $C_\pi(\Sigma) \in \mathcal{L}_{r-1}(R^\Sigma)$, then $S^* = C_\pi(\Sigma) \cap D$ would also belong to $\mathcal{L}_{r-1}(R^\Sigma)$. According to [K, §38, VII, Theorem 1], we would then have $S \in \mathcal{L}_{r-1}(2^\omega)$ because $S$ is homeomorphic to $S^*$, which is impossible. □

4.2 COROLLARY. For each $n \geq 1$ there is a countable regular space $X_n$ such that $C_\pi(X_n) \in \mathcal{L}_n(R^{X_n}) - \mathcal{L}_{n-1}(R^{X_n})$ and there is a countable regular space $Y$ such that $C_\pi(Y) \notin \bigcup \{\mathcal{L}_n(R^Y)|n \geq 1\}$.

PROOF. Fix $n$. By [K, §38, VI, Theorem 1] there is a set $S_n \subset 2^\omega$ having $S_n \in \mathcal{L}_n(2^\omega) - \mathcal{L}_{n-1}(2^\omega)$. Let $X_n = \Sigma_{S_n}$. To obtain the space $Y$, choose any $S \subset 2^\omega$ with $S \notin \bigcup \{\mathcal{L}_n(2^\omega)|n \geq 1\}$ [K, §38, VI, Remark 1] and let $Y = \Sigma_S$. Because $C_\pi(Y)$ contains a closed subset homeomorphic to $S$, $C_\pi(Y) \notin \bigcup \{\mathcal{L}_n(R^Y)|n \geq 1\}$. □

5. Baire category and Baire Property subsets of $R^X$. For any space $Z$, $\mathcal{B}(Z)$ is the $\sigma$-algebra generated by the open sets and the first category subsets of $Z$. Members of $\mathcal{B}(Z)$ are called Baire Property subsets of $Z$ [Ox2, p. 19]. For a space $X$ with a unique limit point (such as the spaces $\Sigma_S$ for $S \subset 2^\omega$ constructed in §1) it is easy to characterize which function spaces $C_\pi(X)$ belong to $\mathcal{B}(R^X)$.

5.1 THEOREM. Suppose $X$ is a countable space with a unique limit point $\infty$ and let $p$ be the trace on $X - \{\infty\}$ of the neighborhood filter of $\infty$. Then the following are equivalent:

(a) $C_\pi(X)$ is a first category subset of $R^X$;
(b) $C_\pi(X) \in \mathcal{B}(R^X)$;
(c) there is an array

\[
\begin{array}{cccc}
A(1,1) & A(1,2) & A(1,3) & \cdots \\
A(2,1) & A(2,2) & A(2,3) & \cdots \\
A(3,1) & A(3,2) & A(3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

satisfying

(i) each $A(m,n)$ is a finite subset of $X - \{\infty\}$;
(ii) each row $A(m,1), A(m,2), A(m,3), \ldots$ is a pairwise disjoint sequence;
(iii) for every sequence $k(1), k(2), \ldots$ and every $U \in p$, $U \cap (\bigcup \{A(m,k(m))|m \geq 1\}) \neq \emptyset$. 

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PROOF. The equivalence of (a) and (c) follows from [LM, Theorems 6.3 and 5.1] and obviously (a) implies (b). We prove that (b) implies (a). Suppose \( C_\pi(X) \in \mathcal{B}_1(R^X) \). To simplify notation, we will identify the countably many isolated points of \( X \) with elements of \( \omega \) and we will write \( X = \omega \cup \{ \infty \} \). Define a function \( \nu: R^X \rightarrow R^\omega \times R \) by the rule that \( \nu(f) = (f^*, f(\infty)) \), where \( f^* \in R^\omega \) is given by \( f^*(n) = f(n) - f(\infty) \). Then \( \nu \) is a homeomorphism of \( R^X \) onto \( R^\omega \times R \) and \( \nu[C_\pi(X)] = C_0 \times R \), where \( C_0 = \{ g \in R^\omega \mid \text{for each } \varepsilon > 0 \text{ there is a neighborhood } U \text{ of } \infty \text{ having } g[U \cap \omega] < | - \varepsilon, \varepsilon | \} \). Since \( C_\pi(X) \in \mathcal{B}_1(R^X) \), \( C_0 \times R \in \mathcal{B}_1(R^\omega \times R) \).

It is easily seen that \( C_0 \) is a tailset in \( R^\omega \), i.e. that if \( g \in C_0 \) and if the equality \( h(n) = g(n) \) holds except for finitely many values of \( n \), then \( h \in C_0 \). We now need a slight variation of a result due to Oxtoby [Oxi]; the proof is only trivially different from Oxtoby’s argument.

5.2 LEMMA. Let \( C \) be a tailset in \( R^\omega \) and suppose that \( C \times R \in \mathcal{B}_1(R^\omega \times R) \). Then either \( C \times R \) is a first category subset of \( R^\omega \times R \) or else \( C \times R \) contains a dense \( G_\delta \)-subset of \( R^\omega \times R \).

Given 5.2, either \( C_0 \times R \) is a first category subset of \( R^\omega \times R \), in which case \( C_\pi(X) \) is also a first category subset of \( R^X \), or else \( C_0 \times R \) contains a dense \( G_\delta \)-subset of \( R^\omega \times R \), in which case \( C_\pi(X) \) contains a dense \( G_\delta \) in \( R^X \). But the latter situation occurs if and only if \( X \) is a discrete space [DGLvM, Theorem 1] so that \( C_\pi(X) \) must be a first category subset of \( R^X \), as claimed. \( \square \)

5.3 REMARK. The reason for creating a variant of Oxtoby’s theorem as in 5.2 is that one cannot deduce \( C_0 \in \mathcal{B}_1(R^\omega) \) from \( C_0 \times R \in \mathcal{B}_1(R^\omega \times R) \).

5.4 COROLLARY. For each \( S \subset 2^\omega \), the function space \( C_\pi(\Sigma_S) \) is a first category subset of \( R^{\Sigma_S} \).

PROOF. We define an array \( A(m, n) \) as follows using the tree \( T = \bigcup_1^{\infty} T_n \);

(i) \( A(1, n) = T_n \) for \( n \geq 1 \);
(ii) \( A(2, 1) = T_1 \cup T_2, A(2, 2) = T_3 \cup T_4, A(2, 3) = T_5 \cup T_6, \ldots \);
(iii) in general, \( A(m, n) = T_{(n-1)m+1} \cup \cdots \cup T_{nm} \).

Obviously each \( A(m, n) \) is finite and because the sets \( T_1, T_2, \ldots \) are pairwise disjoint, each row \( A(m, 1), A(m, 2), \ldots \) of the array is pairwise disjoint. Suppose \( k(1), k(2), \ldots \) is a sequence of positive integers and suppose \( U = T - (B_{x_1} \cup \cdots \cup B_{x_r} \cup F) \), where \( x_i \in S \) and \( F \) is a finite subset of \( T \). If \( \emptyset = U \cap (\bigcup \{ A(m, k(m)) \mid m \geq 1 \}) \), then \( \bigcup \{ A(m, k(m)) \mid m \geq 1 \} \subset B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_r} \cup F \). Observe that for a fixed level \( T_j \) of the tree \( T \), \( \text{card}(B_{x_1} \cap T_n) = 1 \) so that \( \text{card}(T_j \cap (B_{x_1} \cup \cdots \cup B_{x_r} \cup F)) \leq n + \text{card}(F) \). Choose \( m > n + \text{card}(F) \). Then the set \( A(m, k(m)) \) contains a level \( T_j \) of \( T \) where \( \text{card}(T_j) \geq 2^m \) so that \( T_j \cap (B_{x_1} \cup \cdots \cup B_{x_r} \cup F) \) must have cardinality greater than \( n + \text{card}(F) \), contrary to our observation above. \( \square \)

In closing let us give one more example of a countable regular space \( X \) with a unique isolated point \( \infty \) which has a “bad” function space. Unlike the examples so far, \( C_\pi(X) \) is a second category subset of \( R^X \).

5.5 EXAMPLE. Let \( p \) be a free ultrafilter on \( \omega \) and topologize the set \( X = \omega \cup \{ \infty \} \) by isolating all points of \( \omega \) and by using all sets of the form \( \{ \infty \} \cup U \), where \( U \in p \), as neighborhoods of \( \infty \). Then \( C_\pi(X) \) is a second category subset of \( R^X \) and \( C_\pi(X) \notin \mathcal{L}_1(R^X) \cup \mathcal{L}_2(R^X) \).

PROOF. That \( C_\pi(X) \) is a second category subset of \( R^X \) follows from the equivalence of (a) and (c) in 5.1 (cf. [LM, 5.1 and 6.3] for details). Suppose
$C_\pi(X) \in \mathcal{L}_n(\mathbb{R}^X)$, where $n \in \{1, 2\}$. Define $j: 2^\omega \to \mathbb{R}^X$ by the rule that if $f \in 2^\omega$ then $j(f) = \hat{f} \in \mathbb{R}^X$ where $\hat{f}$ is given by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \omega, \\ 1 & \text{if } x = \infty. \end{cases}$$

Then $j$ is continuous so that by [K, §38, III, Proposition 2], $j^{-1}[C_\pi(X)] \in \mathcal{L}_n(2^\omega)$. Hence $j^{-1}[C_\pi(X)]$ is a measurable subset of $2^\omega$ (with respect to product measure $\mu$) because all analytic and co-analytic subsets of $2^\omega$ are measurable [L, p. 243, Proposition 3.24]. But $j^{-1}[C_\pi(X)] = \{x \in 2^\omega\}$ for some $U \in p, x(n) = 1$ for each $n \in U$ so that $j^{-1}[C_\pi(X)]$ is seen to be a tailset in $2^\omega$. Hence Kolmogorov's "0-1 law" guarantees that $\mu[j^{-1}[C_\pi(X)]] = 0$ or $\mu[j^{-1}[C_\pi(X)]] = 1$ [Ox2, p. 84]. However, consider the function $J: 2^\omega \to 2^\omega$ given by $J(f) = f \oplus 1$, where $1 \in 2^\omega$ is constantly equal to 1 and $\oplus$ denotes coordinatewise addition modulo 2, i.e., the usual group operation of $2^\omega$. Since $\mu$ is translation invariant, $J$ is a measure preserving transformation on $2^\omega$. Because $p$ is an ultrafilter, $J[j^{-1}[C_\pi(X)]] = 2^\omega - j^{-1}[C_\pi(X)]$ so that both $\mu[j^{-1}[C_\pi(X)]] = 0$ and $\mu[j^{-1}[C_\pi(X)]] = 1$ are impossible. Therefore $C_\pi(X) \notin \mathcal{L}_0(\mathbb{R}^X) \cup \mathcal{L}_1(\mathbb{R}^X) \cup \mathcal{L}_2(\mathbb{R}^X)$, as claimed. \[ \square \]