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PROPAGATION ESTIMATES FOR SCHRÖDINGER-TYPE OPERATORS

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ABSTRACT. Propagation estimates for a Schrödinger-type operator are obtained using multiple commutator techniques. A new method is given for obtaining estimates for powers of the resolvent. As an application, micro-local propagation estimates are obtained for two-body Schrödinger operators with smooth long-range potentials.

1. Introduction. In this paper we obtain results on propagation properties for Schrödinger operators. We first give results in an abstract setting, and then use these results to derive micro-local propagation estimates for two-body Schrödinger operators with smooth long-range potentials. The abstract results generalize those in [**11**]. A new technique is introduced to obtain these generalizations.[1]

A propagation estimate for a Schrödinger operator H is an estimate of the form

$$\|Xe^{-itH}g(H)Y\| \leq C_\Phi \Phi(t). \tag{1.1}$$

Here g is a smooth function with compact support contained in the absolutely continuous spectrum of H. The operators X, Y localize in appropriate regions of phase space. The choice of X and Y depends on the physics of the problem at hand. X and Y can depend on t (see [**4**]), but we do not consider such results here. The estimate (1.1) may be one-sided or two-sided, depending on the choice of X and Y. In most cases we have $\Phi(t) = (1 + |t|)^{-s}$, $s > 0$.

Propagation estimates have played a crucial role in recent results on N-body Schrödinger operators (see e.g. [**2**, **5**, **15**, **16**]).

Here we obtain estimates (1.1) from resolvent estimates ($R(z) = (H - z)^{-1}$)

$$\|XR(z)^l Y\| \leq c_l \tag{1.2}$$

using Fourier analysis (cf. [**19**]). c_l is uniform in z for $\operatorname{Re} z$ in bounded intervals of the absolutely continuous spectrum of H and $\operatorname{Im} z \neq 0$ with restrictions on the sign for one-sided estimates.

In the abstract setting, estimates (1.2) are obtained using the commutator method developed by Mourre [**12–16**]. To state the results, let H be a selfadjoint operator in a Hilbert space $\mathcal{H}$, and let A be another selfadjoint operator which is conjugate to H at $E \in \mathbf{R}$ [**13**]. The essential condition for A to be conjugate to H is the Mourre estimate

$$E_H(J)i[H, A]E_H(J) \geq \alpha E_H(J) + K. \tag{1.3}$$

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Here $\alpha > 0$, $J = (E-\delta, E+\delta)$, K is compact on $\mathcal{H}$, and E_H is the spectral measure of H. We also need some smoothness of H with respect to A (see Definition 3.1). As operators X, Y we use powers of $\rho = (1+A^2)^{-1/2}$ and the spectral projections of A for $(-\infty, 0)$ and $(0, \infty)$, denoted P_A^- and P_A^+, respectively. The main results can be stated as follows (see Theorem 3.10): For $g \in C_0^\infty(J \backslash \sigma_p(H))$, $s \geq 0$, $0 \leq \mu' < \mu$, we have for $\pm t > 0$

$$\|\rho^{-s} P_A^{\mp} e^{-itH} g(H) \rho^{s+\mu}\| \leq c(1+|t|)^{-\mu'}. \tag{1.4}$$

For $s_1, s_2 \in \mathbf{R}$, $N > 0$, we have for $\pm t > 0$

$$\|\rho^{s_1} P_A^{\mp} e^{-itH} g(H) P_A^{\pm} \rho^{s_2}\| \leq c(1+|t|)^{-N}. \tag{1.5}$$

We derive (1.4) and (1.5) from the following resolvent estimates. $I \subset J \setminus \sigma_p(H)$ denotes a relatively compact interval.

(a_l) For $s > l - 1/2$, $\operatorname{Re} z \in I$ and $\operatorname{Im} z \neq 0$

$$\|\rho^s R(z)^l \rho^s\| \leq c_l.$$

($\mathrm{b}_l^{\pm}$) For $s > l - 1/2$, $\operatorname{Re} z \in I$ and $\mp \operatorname{Im} z > 0$

$$\|\rho^{l-s} P_A^{\pm} R(z)^l \rho^s\| \leq c_l.$$

($\mathrm{c}_l^{\pm}$) For $s_1, s_2 \in \mathbf{R}$, $\operatorname{Re} z \in I$ and $\pm \operatorname{Im} z > 0$

$$\|\rho^{s_1} P_A^{\mp} R(z)^l P_A^{\pm} \rho^{s_2}\| \leq c_l.$$

Estimates of this type were previously obtained in [**11**]. There ($\mathrm{b}_l^{\pm}$) were proved for $s > l$ without the left factor ρ^{l-s}, and ($\mathrm{c}_l^{\pm}$) were proved for $s_1 = s_2 = 0$.

The estimates (a_l), ($\mathrm{b}_l^{\pm}$) and ($\mathrm{c}_l^{\pm}$) are *natural resolvent estimates* in the sense that they follow from the same estimates with $l = 1$. This observation is the key to our new approach to the resolvent estimates (see Lemma 2.1). Thus the problem is reduced to the $l = 1$ case. (a_1) was proved in [**13**, **20**], and special cases of ($\mathrm{b}_1^{\pm}$), ($\mathrm{c}_1^{\pm}$) in [**16**] by Mourre. Our proofs of the general ($\mathrm{b}_1^{\pm}$), ($\mathrm{c}_1^{\pm}$) use results and techniques from [**11**, **16**, **17**].

In §4 we obtain micro-local estimates for a two-body Schrödinger operator with a smooth long-range potential,

$$H = -\Delta + V \quad \text{in } \mathcal{H} = L^2(\mathbf{R}^n).$$

With $A = (2i)^{-1}(x \cdot \nabla + \nabla \cdot x)$ the estimates (1.4), (1.5) hold. We then show how to replace ρ by $\langle x \rangle^{-1} = (1+x^2)^{-1/2}$ and $P_A^{\pm}$ by phase space localizations given by pseudo-differential operators (see §4 for precise statements).

Micro-local estimates for $R(z)$ were obtained in [**9**] by Isozaki and Kitada, using the calculus of pseudo-differential operators. They gave results on e^{-itH} in [**10**] using Fourier integral operators. Recently, Isozaki [**8**] gave micro-local estimates for $R(z)^l$ using the calculus of pseudo-differential operators and an approach similar to our Lemma 2.1.

The idea of relating $P_A^{\pm}$ to phase space localizations is due to Mourre [**14**]. He gave further results and applications in [**15**, **16**], and in particular constructed a sufficiently large family of conjugate operators. We use this result in the proof of our Theorem 4.5.

The abstract results can be applied to N-body Schrödinger operators with smooth long-range two-body potentials. Applications to these operators will be given elsewhere.

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2. Estimates for powers of an operator. This section contains results on estimates for powers of an operator obtainable from estimates for the operator to the first power. An important aspect of the results is the explicit control over the constants in the estimates.

Let R be a bounded operator on a Hilbert space $\mathcal{H}$. For $l \geq 1$, R^l is the lth power as usual. Let ρ be a bounded selfadjoint operator. We assume $\rho^{-1} \geq 1$ for simplicity and define ρ^s for any $s \in \mathbf{R}$ using the functional calculus. For $s < 0$, ρ^s may be unbounded. For any bounded operator X, the statement $X\rho^s$ (or $\rho^s X$) bounded means that this densely defined operator extends to a bounded operator on $\mathcal{H}$. For $s \in \mathbf{R}$ we use the notation $(s)_+ = \max\{0, s\}$.

LEMMA 2.1. *Let R, ρ, P_+ and P_- be bounded operators on $\mathcal{H}$. Assume ρ selfadjoint with $\rho^{-1} \geq 1$, $P_+ + P_- = 1$ and that the following four estimates hold for some $c_0 > 0$, $0 < \sigma \leq 1$ and integer $n > \sigma$:*

(a_1) $\|\rho^\sigma R \rho^\sigma\| \leq c_0$.

(b_1^-) $\|\rho^{1-s} P_- R \rho^s\| \leq c_0$ *for* $\sigma \leq s < n$.

(b_1^+) $\|\rho^s R P_+ \rho^{1-s}\| \leq c_0$ *for* $\sigma \leq s < n$.

(c_1) $\|\rho^{-s_1} P_- R P_+ \rho^{-s_2}\| \leq c_0$ *for* $s_1, s_2 \in \mathbf{R}$ *with* $(s_1)_+ + (s_2)_+ < n - 1$.

Then for any integer $l \geq 1$, $l < n + 1 - \sigma$, the following estimates hold:

(a_l) $\|\rho^{l-1+\sigma} R^l \rho^{l-1+\sigma}\| \leq 2^l c_0^l$.

(b_l^-) $\|\rho^{l-s} P_- R^l \rho^s\| \leq 2^l c_0^l$ *for* $l - 1 + \sigma \leq s < n$.

(b_l^+) $\|\rho^s R^l P_+ \rho^{l-s}\| \leq 2^l c_0^l$ *for* $l - 1 + \sigma \leq s < n$.

(c_l) $\|\rho^{-s_1} P_- R^l P_+ \rho^{-s_2}\| \leq 2^l c_0^l$ *for* $s_1, s_2 \in \mathbf{R}$ *with* $(s_1)_+ + (s_2)_+ < n - l$.

PROOF. The idea of the proof is to insert $1 = P_+ + P_-$ between the factors in R^l and write $P_- = \rho^t \rho^{-t} P_-$, $P_+ = P_+ \rho^{-\tau} \rho^\tau$. It is then a matter of bookkeeping to get the conditions on l and s, s_1, s_2. The most convenient way to carry out the proof is by induction, showing that (a_{l-1}), ($\mathrm{b}_{l-1}^\pm$), (c_{l-1}) imply (a_l), ($\mathrm{b}_l^\pm$), (c_l). Let us give the start of this argument. Assume the $l - 1$ estimates hold and $l < n + 1 - \sigma$. To prove (a_l) we write

$$\begin{aligned}
\rho^{l-1+\sigma} R^l \rho^{l-1+\sigma} &= \rho^{l-1+\sigma} R^{l-1} (P_+ + P_-) R \rho^{l-1+\sigma} \\
&= (\rho^{l-1+\sigma} R^{l-1} P_+ \rho^{-\sigma})(\rho^\sigma R \rho^{l-1+\sigma}) \\
&\quad + (\rho^{l-1+\sigma} R^{l-1} \rho^{l-2+\sigma})(\rho^{1-(l-1+\sigma)} P_- R \rho^{l-1+\sigma}) \\
&= A_1 A_2 + B_1 B_2.
\end{aligned}$$

Since $-\sigma = (l-1) - (l - 1 + \sigma)$, ($\mathrm{b}_{l-1}^+$) with $s = l - 1 + \sigma$ implies A_1 bounded with $\|A_1\| \leq 2^{l-1} c_0^{l-1}$. The assumptions on ρ imply $\|\rho^s\| \leq \|\rho^\sigma\|$ for any $s \geq \dot{\sigma}$, so (a_1) implies $\|A_2\| \leq c_0$. Similarly (a_{l-1}) implies $\|B_1\| \leq 2^{l-1} c_0^{l-1}$. The result ($\mathrm{b}_1^-$) with $s = l - 1 + \sigma \geq \sigma$ ($s < n$ by the assumption on l) implies $\|B_2\| \leq c_0$. Combining these results we obtain (a_l).

The remainder of the proof is similar, and is omitted. $\square$

The following lemma is used to obtain continuity of boundary values. It gives a general result on estimates for the difference of powers of two commuting operators.

LEMMA 2.2. *Let R_1, R_2, ρ, P_+ and P_- be bounded operators on $\mathcal{H}$. Assume $R_1R_2 = R_2R_1$, ρ selfadjoint with $\rho^{-1} \geq 1$, and $P_+ + P_- = 1$. Assume that R_1, R_2 satisfy* (a_1), ($\mathrm{b}_1^{\pm}$), (c_1) *in Lemma* 2.1 *with the same constants σ, c_0, n, and that $R_1 - R_2$ satisfies* (a_1), ($\mathrm{b}_1^{\pm}$), (c_1) *with σ, n, and a constant c_0 denoted d here. Then the following estimates hold for $l \geq 1$, $l < n + 1 - \sigma$:*

(A_l) $\|\rho^{l-1+\sigma}(R_1^l - R_2^l)\rho^{l-1+\sigma}\| \leq d \cdot l \cdot 2^{l+1} \cdot c_0^{l-1}$.

(B_l^-) $\|\rho^{l-s}P_-(R_1^l - R_2^l)\rho^s\| \leq d \cdot l \cdot 2^{l+1}c_0^{l-1}$ *for* $l - 1 + \sigma \leq s < n$.

(B_l^+) $\|\rho^s(R_1^l - R_2^l)P_+\rho^{l-s}\| \leq d \cdot l \cdot 2^{l+1}c_0^{l-1}$ *for* $l - 1 + \sigma \leq s < n$.

(C_l) $\|\rho^{-s_1}P_-(R_1^l - R_2^l)P_+\rho^{-s_2}\| \leq d \cdot l \cdot 2^{l+1}c_0^{l-1}$ *for* $s_1, s_2 \in \mathbf{R}$ *with* $(s_1)_+ + (s_2)_+ < n - l$.

PROOF. The idea of the proof is to use the identity

$$R_1^l - R_2^l = (R_1 - R_2) \cdot \sum_{j=0}^{l-1} R_1^j R_2^{l-1-j}$$

and insert $1 = P_- + P_+$ between factors R_1, R_2 in the above identity. It is then straightforward to obtain (A_l), ($\mathrm{B}_l^{\pm}$) and (C_l) using Lemma 2.1. Details of the computation are omitted. □

3. Estimates for powers of the resolvent of a selfadjoint operator. In this section we obtain estimates of the type (a_1), ($\mathrm{b}_1^{\pm}$) and (c_1) in Lemma 2.1 for the resolvent of a selfadjoint operator, and thus estimates for powers of the resolvent. These results generalize those in [**11**]. We use some results and techniques from [**11**, **13**, **16**, **17**], and shall briefly recall some definitions and results from these papers.

Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$. E_H denotes the spectral measure of H. $\mathcal{H}_n$ denotes the completion of vectors satisfying

$$\|\psi\|_n^2 = \int_{-\infty}^{\infty} (1 + \lambda^2)^{n/2}\, d\|E_H(\lambda)\psi\|^2 < \infty.$$

The space $\mathcal{H}_{+2}$ is the domain $\mathcal{D}(H)$ with the graph norm, and $\mathcal{H}_{-2}$ can be identified with the dual of $\mathcal{H}_{+2}$ via the inner product on $\mathcal{H}$.

DEFINITION 3.1. A selfadjoint operator A is said to be conjugate to H at $E \in \mathbf{R}$, and H is said to be n-smooth with respect to A, if the following conditions are satisfied:

(a) $\mathcal{D}(A) \cap \mathcal{D}(H)$ is dense in $\mathcal{H}_{+2}$.

(b) $e^{i\theta A}$ maps $\mathcal{D}(H)$ into $\mathcal{D}(H)$, and $\sup_{|\theta|\leq 1} \|He^{i\theta A}\psi\| < \infty$ for each $\psi \in \mathcal{D}(H)$.

(c_n) The form $i[H, A]$, defined on $\mathcal{D}(A) \cap \mathcal{D}(H)$, is bounded from below and closable. The selfadjoint operator associated with its closure is denoted iB_1. Assume $\mathcal{D}(B_1) \supset \mathcal{D}(H)$. If $n > 1$, assume for $j = 2, 3, \ldots, n$ that the form $i[iB_{j-1}, A]$, defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$, is bounded from below and closable. The associated selfadjoint operator is denoted iB_j, and $\mathcal{D}(B_j) \supset \mathcal{D}(H)$ is assumed.

(d_n) The form $i[B_n, A]$, defined on $\mathcal{D}(A) \cap \mathcal{D}(H)$, extends to a bounded operator from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$.

(e) There exist $\alpha > 0$, $\delta > 0$, and a compact operator K on $\mathcal{H}$ such that

$$E_H(J)iB_1E_H(J) \geq \alpha E_H(J) + E_H(J)KE_H(J),$$

where $J = (E - \delta, E + \delta)$.

The interval J is called the interval of conjugacy. If (c_n) holds for all integers $n \geq 1$, H is said to be ∞-smooth with respect to A.

This definition for $n = 1$ was introduced by Mourre [**13**]. For $n > 1$ is was given in [**11**]. The condition (c_n) essentially means that the map $\theta \mapsto e^{i\theta A}He^{-i\theta A}$ from $\mathbf{R}$ to $\mathcal{B}(\mathcal{H}_{+2}, \mathcal{H})$ (the bounded operators from $\mathcal{H}_{+2}$ to $\mathcal{H}$) is C^n in norm.

Existence of a conjugate operator A to H at $E \in \mathbf{R}$ implies that the point spectrum $\sigma_p(H)$ of H is discrete in J [**13**]. If $I \subset J \setminus \sigma_p(H)$ is relatively compact, we have with $\rho = (1 + A^2)^{-1/2}$, $s > 1/2$, the a priori estimate $(R(z) = (H - z)^{-1})$

$$\|\rho^s R(z)\rho^s\| \leq c \tag{3.1}$$

for z with $\operatorname{Re} z \in I$, $\operatorname{Im} z \neq 0$ [**13**, **20**]. In particular, H has no singular continuous spectrum in J.

Further results on H and A were given in [**16**]. We recall some needed below. Assume A is conjugate to H at $E \in \mathbf{R}$, and H is 1-smooth w.r.t. A. Let P_A^+ (P_A^-) denote the spectral projections for A for $(0, \infty)$ $((-\infty, 0))$. Let $\chi_n = \chi_{[n,n+1)}(A)$ denote the spectral projection for $[n, n+1)$, n an integer. The following three results are proved in [**16**]. Let $I \subset J \setminus \sigma_p(H)$ be relatively compact.

$$\|\chi_n R(z)\chi_m\| \leq c \tag{3.2}$$

for $\operatorname{Re} z \in I$, $\operatorname{Im} z \neq 0$, $n, m \in \mathbf{Z}$ with c independent of n, m. For $s > 1$

$$\|P_A^{\mp} R(z)\rho^s\| \leq c \tag{3.3}$$

for $\operatorname{Re} z \in I$, $\pm\operatorname{Im} z > 0$. Assume furthermore that H is 2-smooth w.r.t. A. Then

$$\|P_A^{\mp} R(z)P_A^{\pm}\| \leq c \tag{3.4}$$

for $\operatorname{Re} z \in I$, $\pm\operatorname{Im} z > 0$.

In this section we first prove extensions of (3.3) and (3.4) which are natural in view of Lemma 2.1.

THEOREM 3.2. *Assume A conjugate to H at $E \in \mathbf{R}$, and H n-smooth with respect to A for some $n \geq 2$. Let J be the interval of conjugacy, and let $I \subset J \setminus \sigma_p(H)$ be a relatively compact interval. Then for $s_1, s_2 \in \mathbf{R}$ with $(s_1)_+ + (s_2)_+ < n - 1$ the estimate*

$$\|\rho^{-s_1} P_A^{\mp} R(z)P_A^{\pm}\rho^{-s_2}\| \leq c \tag{3.5}$$

holds for all z with $\operatorname{Re} z \in I$, $\pm\operatorname{Im} z > 0$. Boundary values for $z = \lambda \pm i\varepsilon \to \lambda \pm i0$ as $\varepsilon \downarrow 0$ exist and are Hölder continuous.

REMARK 3.3. Precisely stated, (3.5) means that the quadratic form

$$q(f, g) = \langle \rho^{-s_1} f, P_A^- R(z)P_A^+ \rho^{-s_2} g\rangle$$

defined on $\mathcal{D}(\rho^{-s_1}) \times \mathcal{D}(\rho^{-s_2})$, satisfies the estimate $|q(f, g)| \leq c\|f\| \cdot \|g\|$, and thus extends to a bounded quadratic form on $\mathcal{H} \times \mathcal{H}$. The associated operator is denoted $\rho^{-s_1} P_A^- R(z)P_A^+ \rho^{-s_2}$.

PROOF. For $s_1, s_2 \leq 0$ the result is a consequence of (3.4). Assume $s_1, s_2 > 0$. The proof is an extension of the proof of Theorem 2.4 in [**11**]. Using the operators B_j from Definition 3.5(c_n), we define

$$C_n(\varepsilon) = \sum_{j=1}^{n} \frac{\varepsilon^j}{j!} B_j.$$

Assume $\operatorname{Re} z \in I$, $\operatorname{Im} z > 0$. By [**11**, Lemma 3.1], the operator

$$G_z(\varepsilon) = (H - z + C_n(\varepsilon))^{-1}$$

is bounded for $0 < \varepsilon < \varepsilon_0$. Furthermore, on $\mathcal{D}(A)$ (norm-derivative)

$$\frac{d}{d\varepsilon} G_z(\varepsilon) = [G_z(\varepsilon), A] + \frac{\varepsilon^n}{n!} G_z(\varepsilon)[B_n, A] G_z(\varepsilon). \tag{3.6}$$

Consider for $0 < \varepsilon < \varepsilon_0$ the family

$$F_z(\varepsilon) = \rho^{-s_1} e^{\varepsilon A} P_A^- G_z(\varepsilon) P_A^+ e^{-\varepsilon A} \rho^{-s_2}$$

of bounded operators. Compute the weak derivative using (3.6):

$$\begin{aligned}
\frac{d}{d\varepsilon} F_z(\varepsilon) &= \rho^{-s_1} e^{\varepsilon A} P_A^- [A, G_z(\varepsilon)] P_A^+ e^{-\varepsilon A} \rho^{-s_2} \\
&\quad + \rho^{-s_1} e^{\varepsilon A} P_A^- \left([G_z(\varepsilon), A] + \frac{\varepsilon^n}{n!} G_z(\varepsilon)[B_n, A] G_z(\varepsilon) \right) P_A^+ e^{-\varepsilon A} \rho^{-s_2} \\
&= \frac{\varepsilon^n}{n!} \rho^{-s_1} e^{\varepsilon A} P_A^- G_z(\varepsilon)[B_n, A] G_z(\varepsilon) P_A^+ e^{-\varepsilon A} \rho^{-s_2}.
\end{aligned}$$

We now use the estimate $\|(H+i)G_z(\varepsilon)\| \leq c/\varepsilon$ from [**11**] and the obvious estimates $\|\rho^{-s_1} e^{\varepsilon A} P_A^-\| \leq c\varepsilon^{-s_1}$ and $\|P_A^+ e^{-\varepsilon A} \rho^{-s_2}\| \leq c\varepsilon^{-s_2}$ together with (d_n) in Definition 3.1 to get

$$\left\| \frac{d}{d\varepsilon} F_z(\varepsilon) \right\| \leq c\varepsilon^{n - s_1 - s_2 - 2}.$$

The condition $s_1 + s_2 < n - 1$ implies $n - s_1 - s_2 - 2 > -1$, and hence integrability. The result (3.5) now follows as in [**11**]. Existence and Hölder continuity of boundary values also follows as in [**11**]. □

The following lemma gives a straightforward extension of the well-known Hardy inequality. $l^2 = l^2(\mathbf{N})$ is the usual sequence space.

LEMMA 3.4. *Let $s > 1/2$. The map $T\colon l^2 \to l^2$ defined by $T(a_n) = (b_n)$, $b_n = n^{-s} \sum_{m=1}^{n} m^{s-1} a_m$, is a bounded operator.*

PROOF. For $s = 1$ the result follows from Hardy's inequality [**6**]. The result for $s > 1$ follows from the $s = 1$ result, since

$$\left| n^{-s} \sum_{m=1}^{n} m^{s-1} a_m \right| \leq n^{-1} \sum_{m=1}^{n} |a_m|.$$

The proof for $1/2 < s < 1$ can be obtained by modifying the proof in [**6**]. Details are omitted. □

The idea of using Hardy's inequality to prove the following result in the $s = 1$ case is due to Mourre [**17**].

THEOREM 3.5. *Let A be conjugate to H at $E \in \mathbf{R}$, and H n-smooth with respect to A for some $n \geq 2$. Let J be the interval of conjugacy, and let $I \subset J \backslash \sigma_p(H)$ be a relatively compact interval. Then for $1/2 < s < n$ we have*

$$\|\rho^s R(z) P_A^{\pm} \rho^{1-s}\| \leq c \tag{3.7}$$

for $\operatorname{Re} z \in I$, $\pm \operatorname{Im} z > 0$. For any $s' < s$ boundary values of $\rho^s R(\lambda \pm i\varepsilon) P_A^{\pm} \rho^{1-s'}$ as $\varepsilon \downarrow 0$ exist and are Hölder continuous.

PROOF. Consider first $\operatorname{Im} z > 0$. Let $g \in \mathcal{D}(\rho^{1-s})$.

$$\rho^s R(z) P_A^+ \rho^{1-s} g = \rho^s P_A^+ R(z) P_A^+ \rho^{1-s} g + \rho^s P_A^- R(z) P_A^+ \rho^{1-s} g.$$

Since $s < n$, $(s-1)_+ < n-1$, and Theorem 3.2 implies

$$\|\rho^s P_A^- R(z) P_A^+ \rho^{1-s} g\| \leq c\|g\|.$$

Thus we have to consider the first term. Let $f \in \mathcal{H}$ and $\chi_n = \chi_{[n,n+1)}(A)$.

$$\langle f, \rho^s P_A^+ R(z) P_A^+ \rho^{1-s} g\rangle = \sum_{n=0}^{\infty} \langle \rho^s \chi_n f, \chi_n R(z) P_A^+ \rho^{1-s} g\rangle. \tag{3.8}$$

We write

$$\begin{aligned} \chi_n R(z) P_A^+ \rho^{1-s} g = {} & \chi_n R(z) \chi_{[0,n+1)}(A) \rho^{1-s} g \\ & + \chi_n R(z) \chi_{[n+1,\infty)}(A) \rho^{1-s} g. \end{aligned} \tag{3.9}$$

The first term in (3.9) is written as

$$\chi_n R(z) \chi_{[0,n+1)}(A) \rho^{1-s} g = \sum_{m=0}^{n} \chi_n R(z) \chi_m \rho^{1-s} g.$$

By Mourre's result (3.2) and $\|\chi_m \rho^{1-s} g\| \leq c(m+1)^{s-1} \|\chi_m g\|$, we have

$$\|\chi_n R(z) \chi_{[0,n+1)}(A) \rho^{1-s} g\| \leq c \sum_{m=0}^{n} (m+1)^{s-1} \|\chi_m g\|. \tag{3.10}$$

To estimate the second term in (3.9) we first consider $1/2 < s \leq 1$. In this case ρ^{1-s} is bounded, and we can estimate $\chi_n R(z) \chi_{[n+1,\infty)}(A)$ using Mourre's observation [**16**] that $A - n - 1$ is conjugate to H at $E \in \mathbf{R}$ with the same α in Definition 3.1(e) as for A. Thus (3.4) implies

$$\|\chi_n R(z) \chi_{[n+1,\infty)}(A)\| \leq c \tag{3.11}$$

with c independent of n. Thus in the case $1/2 < s \leq 1$, the second term in (3.9) is estimated by $c \cdot \|g\|$ with c independent of n. Using (3.10) and Lemma 3.4, we can estimate the expression in (3.8):

$$\begin{aligned} & |\langle f, \rho^s P_A^+ R(z) P_A^+ \rho^{1-s} g\rangle| \\ & \quad \leq c \cdot \sum_{n=0}^{\infty} \|\chi_n f\| (n+1)^{-s} \sum_{m=0}^{n} (m+1)^{s-1} \|\chi_m g\| + c\|g\| \sum_{n=0}^{\infty} (n+1)^{-s} \|\chi_n f\| \\ & \quad \leq c \left(\sum_{n=0}^{\infty} \|\chi_n f\|^2 \right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} \|\chi_n g\|^2 \right)^{1/2} \\ & \qquad + c\|g\| \left(\sum_{n=0}^{\infty} (n+1)^{-2s} \right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} \|\chi_n f\|^2 \right)^{1/2} \\ & \quad \leq c\|f\| \cdot \|g\|. \end{aligned}$$

This estimate proves (3.7) for $1/2 < s \leq 1$.

Assume now $1 < s < n$. In this case (3.10) is still valid, but (3.11) must be replaced by a different estimate. Since $0 < s-1 < n-1$, we have from Theorem 3.2 with $A-n-1$ as the conjugate operator,

$$\chi_n R(z)\chi_{[n+1,\infty)}(A)\rho^{1-s}g = \{\chi_n P^-_{A-n-1}R(z)P^+_{A-n-1}(1+(A-n-1)^2)^{(s-1)/2}\} \cdot \{(1+(A-n-1)^2)^{-(s-1)/2}(1+A^2)^{(s-1)/2}g\}.$$

The first factor in $\{\cdots\}$ is bounded with norm bounded by an n-independent constant. The second factor has a norm bounded by $c(n+1)^{s-1}$. Thus we get

$$\|\chi_n R(z)\chi_{[n+1,\infty)}(A)\rho^{1-s}g\| \le c(n+1)^{s-1}\|g\|. \tag{3.12}$$

Using (3.10) and (3.12) we get

$$\begin{aligned}|\langle f, \rho^s P^+_A R(z) P^+_A \rho^{1-s} g\rangle| &\le c\sum_{n=0}^{\infty}\|\chi_n f\|(n+1)^{-s}\sum_{m=0}^{n}(m+1)^{s-1}\|\chi_m g\| \\ &\quad + c\|g\|\sum_{n=0}^{\infty}\|\chi_n f\|(n+1)^{-s}(n+1)^{s-1} \\ &\le c\|f\|\cdot\|g\|.\end{aligned}$$

This estimate completes the proof of (3.7) in the case $1 < s < n$.

The last statement of the theorem follows by interpolation, if one notes that $\rho^s R(z)P^+_A\rho^s$ is Hölder continuous in z for $s > 1/2$ (see [**11**]). The $\operatorname{Im} z < 0$ result is proved similarly. □

Theorems 3.2 and 3.5, (3.1), and Lemmas 2.1 and 2.2 lead to extensions of the results of [**11**]. An important new feature is the explicit control of the constants in the power estimates obtained from Lemmas 2.1 and 2.2. Since these results are obvious from Lemmas 2.1 and 2.2, they will not be repeated below.

THEOREM 3.6. *Let A be conjugate to H at $E \in \mathbf{R}$, and H n-smooth with respect to A for some $n \ge 2$. Let J be the interval of conjugacy, and let $I \subset J\backslash\sigma_p(H)$ be a relatively compact interval. Then the following estimates hold for the resolvent $R(z) = (H-z)^{-1}$ for $1 \le l < n$:*

(i) *For $l-(1/2) < s_j$, $j = 1,2$, there exists $c = c(s_1, s_2, l)$ such that for all z with* $\operatorname{Re} z \in I$, $\operatorname{Im} \neq 0$

$$\|\rho^{s_1}R(z)^l\rho^{s_2}\| \le c.$$

(ii) *For $l-(1/2) < s < n$ there exists $c = c(s,l)$ such that for all z with* $\operatorname{Re} z \in I$, $\pm\operatorname{Im} z > 0$

$$\|\rho^s R(z)^l P^\pm_A \rho^{l-s}\| \le c.$$

(iii) *For $s_1, s_2 \in \mathbf{R}$ with $(s_1)_+ + (s_2)_+ < n-l$, there exists $c = c(s_1, s_2, l)$ such that for all z with* $\operatorname{Re} z \in I$, $\pm\operatorname{Im} z > 0$

$$\|\rho^{-s_1}P^\mp_A R(z)^l P^\pm_A \rho^{-s_2}\| \le c.$$

(iv) *In* (i) *and* (iii) *boundary values exist and are Hölder continuous. In* (ii) *boundary values exist and are Hölder continuous if ρ^{l-s} is replaced by $\rho^{l-s'}$ with $s' < s$.*

PROOF. The results (i), (ii) and (iii) follow from (3.1), (3.5), (3.7) and Lemma 2.1 (with $\sigma > 1/2$). To prove (iv) we use the same estimates and Lemma 2.2 (in case (ii) with a straightforward modification to take into account $\rho^{l-s'}$ with $s' < s$). □

REMARK 3.7. Using the notation $\rho^{s_1}R(\lambda \pm i0)\rho^{s_2}$ etc. to denote the boundary values for $l = 1$, we have the following results:

$$\lim_{\varepsilon \downarrow 0} \rho^{s_1} R(\lambda \pm i\varepsilon)^l \rho^{s_2} = \left(\frac{d}{d\lambda}\right)^{l-1} \rho^{s_1} R(\lambda \pm i0)\rho^{s_2},$$

$$\lim_{\varepsilon \downarrow 0} \rho^{s} R(\lambda \pm i\varepsilon)^l P_A^{\pm} \rho^{l-s'} = \left(\frac{d}{d\lambda}\right)^{l-1} \rho^{s} R(\lambda \pm i0) P_A^{\pm} \rho^{l-s'},$$

$$\lim_{\varepsilon \downarrow 0} \rho^{-s_1} P_A^{\mp} R(\lambda \pm i\varepsilon)^l P_A^{\pm} \rho^{-s_2} = \left(\frac{d}{d\lambda}\right)^{l-1} \rho^{-s_1} P_A^{\mp} R(\lambda \pm i0) P_A^{\pm} \rho^{-s_2}$$

with the appropriate restrictions on λ, s_1, s_2, s and s' from Theorem 3.6. The convergence is in operator norm on $\mathcal{B}(\mathcal{H})$.

The resolvent estimates in Theorem 3.6 lead to propagation estimates of the type discussed in the Introduction. To derive them, we need the following lemma. The first result is well known [**13**, **19**]. Results similar to the second part have been given in [**15**, **16**]. The simple proof given here is new.

LEMMA 3.8. *Let $n \geq 1$ be an integer. Let A and H satisfy* (a), (b) *and* (c_n) *in Definition 3.1. Let $g \in C_0^\infty(\mathbf{R})$.*

(i) *For $|s| \leq n$, the operator $\rho^{-s} g(H) \rho^s$ is bounded.*

(ii) *For $s_1, s_2 \in \mathbf{R}$ with $(s_1)_+ + (s_2)_+ < n$, the operators $\rho^{-s_1} P_A^- g(H) P_A^+ \rho^{-s_2}$ and $\rho^{-s_1} P_A^+ g(H) P_A^- \rho^{-s_2}$ are bounded.*

PROOF. For the proof of (i), see [**13**, **19**]. Let $\mathrm{ad}_A(g(H)) = [A, g(H)]$, and let $\mathrm{ad}_A^n(g(H))$ denote the n-fold commutator. Part (i) implies that $\mathrm{ad}_A^n(g(H))$ extends to a bounded operator on $\mathcal{H}$. It suffices to prove (ii) for $s_1, s_2 > 0$. Assume further that $s_1 + s_2 < n$. For $\varepsilon > 0$, consider the family of bounded operators

$$F(\varepsilon) = \rho^{-s_1} e^{\varepsilon A} P_A^- g(H) P_A^+ e^{-\varepsilon A} \rho^{-s_2}.$$

This family is weakly C^∞ w.r.t. ε on a dense set in $\mathcal{H}$ (e.g. take the C^∞-vectors for A). Take the nth weak derivative on this set to get

$$\left(\frac{d}{d\varepsilon}\right)^n F(\varepsilon) = \rho^{-s_1} e^{\varepsilon A} P_A^- \mathrm{ad}_A^n(g(H)) P_A^+ e^{-\varepsilon A} \rho^{-s_2}.$$

Estimate as in the proof of Theorem 3.2 to get

$$\left\| \left(\frac{d}{d\varepsilon}\right)^n F(\varepsilon) \right\| \leq c\varepsilon^{-s_1 - s_2}.$$

Since $s_1 + s_2 < n$, the result follows by integration. □

REMARK 3.9. The lemma holds under the assumption that $\mathrm{ad}_A^j(g(H))$ is bounded for $j = 1, 2, \ldots, n$, and the operator norms can be estimated by the sum of the norms of $\mathrm{ad}_A^j(g(H))$, $j = 1, 2, \ldots, n$.

The resolvent estimates in Theorem 3.6 lead to propagation estimates for e^{-itH}. We state these results below in case H is ∞-smooth with respect to A. Results for finite n-smoothness are readily obtainable from Theorem 3.6 (see [**11**] for some results of this type). Part (i) of the following theorem was obtained in [**11**] and is included here for the sake of completeness. Part (ii) generalizes results in [**11**]. Part (iii) was not given in [**11**].

THEOREM 3.10. *Let A be conjugate to H at $E \in \mathbf{R}$, and let H be ∞-smooth with respect to A. Let J denote the interval of conjugacy. The following estimates hold for any $g \in C_0^\infty(J \setminus \sigma_p(H))$:*

(i) *For $0 \le s' < s$*

$$\|\rho^s e^{-itH} g(H) \rho^s\| \le c(1+|t|)^{-s'}, \qquad t \in \mathbf{R}.$$

(ii) *For $s > 0$, $0 \le \mu' < \mu$*

$$\|\rho^{-s} P_A^{\mp} e^{-itH} g(H) \rho^{s+\mu}\| \le c(1+|t|)^{-\mu'}, \qquad \pm t > 0.$$

(iii) *For $s_1, s_2 \in \mathbf{R}$, $N > 0$:*

$$\|\rho^{-s_1} P_A^{\mp} e^{-itH} g(H) P_A^{\pm} \rho^{-s_2}\| \le c(1+|t|)^{-N}, \qquad \pm t > 0.$$

PROOF. For the proof of (i), see [**11**]. To prove (ii), we first note [**19**] for $\pm t > 0$

$$e^{-itH} g(H) = \lim_{\varepsilon \downarrow 0} \frac{(l-1)!}{2\pi i} \frac{-1}{(it)^{l-1}} \int_{-\infty}^{\infty} d\lambda e^{-it\lambda} R(\lambda \pm i\varepsilon)^l g(H). \tag{3.13}$$

Let $g \in C_0^\infty(J \setminus \sigma_p(H))$, $[a,b] \supseteq \operatorname{supp}(g)$, and $a_1 < a < b < b_1$, $[a_1, b_1] \subset J \setminus \sigma_p(H)$. We need the following estimates to prove (ii) for $t > 0$ ($s \ge 0$, $l \ge 1$):

$$\|\rho^{-s} P_A^- R(\lambda + i\varepsilon)^l g(H) \rho^{s+l}\| \le c, \qquad \lambda \in [a_1, b_1], \tag{3.14}$$

$$\|\rho^{-s} P_A^- R(\lambda + i\varepsilon)^l g(H) \rho^{s+l}\| \le c(1+|\lambda|)^{-l}, \qquad \lambda \in \mathbf{R} \setminus [a_1, b_1]. \tag{3.15}$$

The estimate (3.14) follows for $s, l \ge 0$, from Theorem 3.6(ii) and Lemma 3.8(i). The estimate (3.15) follows using Remark 3.9. The results (3.13), (3.14) and (3.15) imply for $t > 0$

$$\|\rho^{-s} P_A^- e^{-itH} g(H) \rho^{s+l}\| \le c(1+|t|)^{-l+1}$$

for $s \ge 0$, $l \ge 1$. The result (ii) now follows by interpolation. The $t < 0$ case is proved similarly.

To prove (iii) we first assume $t > 0$ and note that it suffices to consider $s_1, s_2 \ge 0$. We need to prove

$$\|\rho^{-s_1} P_A^- R(\lambda + i\varepsilon)^l g(H) P_A^+ \rho^{-s_2}\| \le c, \qquad \lambda \in [a_1, b_1], \tag{3.16}$$

$$\|\rho^{-s_1} P_A^- R(\lambda + i\varepsilon)^l g(H) P_A^+ \rho^{-s_2}\| \le c(1+|\lambda|)^{-l}, \qquad \lambda \in \mathbf{R} \setminus [a_1, b_1]. \tag{3.17}$$

To prove (3.16) we write

$$\begin{aligned} &\rho^{-s_1} P_A^- R(\lambda + i\varepsilon)^l g(H) P_A^+ \rho^{-s_2} \\ &\quad = (\rho^{-s_1} P_A^- R(\lambda + i\varepsilon)^l \rho^{s_1+l})(\rho^{-s_1-l} P_A^- g(H) P_A^+ \rho^{-s_2}) \\ &\quad + (\rho^{-s_1} P_A^- R(\lambda + i\varepsilon)^l P_A^+ \rho^{-s_2})(\rho^{s_2} g(H) \rho^{-s_2} P_A^+). \end{aligned}$$

The estimate now follows from Theorem 3.6 and Lemma 3.8. The estimate (3.17) follows from Remark 3.9 and straightforward computations. Result (iii) for $t > 0$ now follows from (3.13), (3.16) and (3.17) using interpolation. The $t < 0$ result is proved similarly. □

4. Micro-local estimates for two-body Schrödinger operators. In this section we use the results of §3 to obtain micro-local estimates for powers of the resolvent of a two-body Schrödinger operator. Before discussing two-body operators in detail, we note that the results in §3 can be applied to general N-body Schrödinger operators with smooth long-range potentials.

We consider Schrödinger operators $H = H_0 + V$ on $\mathcal{H} = L^2(\mathbf{R}^n)$. Here $H_0 = -\Delta$, V is a real-valued function, $V \in C^\infty(\mathbf{R}^n)$, and

$$|\partial_x^\alpha V(x)| \le c_\alpha (1 + |x|)^{-\varepsilon - |\alpha|} \tag{4.1}$$

for some $\varepsilon > 0$ and all multi-indices α. It is well known that H has no positive eigenvalues (see [**21**]). For an interval $J = (a, b)$, $0 < a < b < \infty$, a conjugate operator is the generator of dilations $A = (2i)^{-1}(x \cdot \nabla + \nabla \cdot x)$. Furthermore, H is ∞-smooth with respect to A due to (4.1). For details of the verification of the conditions in Definition 3.1, see e.g. [**11**]. Let $\rho = (1 + A^2)^{-1/2}$ and let P_A^+ (P_A^-) denote the spectral projections of A for $(0, \infty)$ ($(-\infty, 0)$). Then Theorem 3.6 holds for H and A.

The micro-local estimates are obtained by replacing the $P_A^\pm$ localizations by phase-space localizations given by pseudo-differential operators (ψdo's). The results are stated for powers of the resolvent $R(z) = (H - z)^{-1}$. Results for e^{-itH} follow as in the proof of Theorem 3.10, and will not be stated separately. A few facts on ψdo's are given below. Details and proofs of some lemmas can be found in the appendix.

Let $p \in C^\infty(\mathbf{R}^{2n})$ satisfy

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \le c_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}, \tag{4.2}$$

where $\langle x \rangle = (1 + x^2)^{1/2}$. Let $\mathcal{S}(\mathbf{R}^n)$ denote the Schwartz space of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbf{R}^n)$ the pseudo-differential operator P with symbol p is defined by

$$(Pf)(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi)\, d\xi. \tag{4.3}$$

Here $\hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x)\, dx$ denotes the Fourier transform. Condition (4.2) implies that P extends uniquely to a bounded operator on $\mathcal{H}$ (cf. Proposition A.3). Localizing operators are obtained by restricting the support of $p(x, \xi)$. We first give results for operators similar to $P_A^\pm$. Let $q_\pm \in C^\infty(\mathbf{R}^{2n})$ satisfy (4.2), and assume for some $\sigma, v > 0$,

$$\operatorname{supp} q_\pm(x, \xi) \subseteq \{(x, \xi) | \pm x \cdot \xi \ge \sigma |x|\, |\xi|, |\xi| \ge v\}. \tag{4.4}$$

The associated bounded operators are denoted $Q^\pm$.

The following two lemmas are needed to prove localization results. Lemma 4.1 is well known [**19**, **20**]. Lemma 4.2 will be proved in the appendix.

LEMMA 4.1. (i) *For $s \in \mathbf{R}$ and $l \ge 1$ an integer, the operator $\langle x \rangle^s (H + i)^{-l} \langle x \rangle^{-s}$ is bounded on $\mathcal{H}$.*

(ii) *For an integer $l \ge 1$ and $0 \le s \le 2l$, the operators $\langle x \rangle^{-s} (H + i)^{-l} \rho^{-s}$ and $\rho^{-s} (H + i)^{-l} \langle x \rangle^{-s}$ are bounded on $\mathcal{H}$.*

LEMMA 4.2. (i) *For $s \ge 0$, the operators $\langle x \rangle^s Q^\pm \rho^s$ and $\rho^s Q^\pm \langle x \rangle^s$ are bounded on $\mathcal{H}$.*

(ii) *For $s_1, s_2 \ge 0$, the operators $\langle x \rangle^{s_1} Q^\pm P_A^\mp \rho^{-s_2}$ and $\rho^{-s_2} P_A^\mp Q^\pm \langle x \rangle^{s_1}$ are bounded on $\mathcal{H}$.*

THEOREM 4.3. *Let H and $Q^\pm$ be the operators defined above. Let $J = (a, b)$ with $0 < a < b < \infty$, and let $l \ge 1$ be an integer. The following estimates hold for $R(z) = (H - z)^{-1}$ for all z with $\operatorname{Re} z \in J$, $\operatorname{Im} z > 0$:*

(i) $\|\langle x\rangle^{-s}R(z)^l\langle x\rangle^{-s}\| \le c$ *for* $s > l-(1/2)$.
(ii) $\|\langle x\rangle^{s-l}Q^-R(z)^l\langle x\rangle^{-s}\| \le c$ *and* $\|\langle x\rangle^{-s}R(z)^lQ^+\langle x\rangle^{s-l}\| \le c$ *for* $s > l-(1/2)$.
(iii) $\|\langle x\rangle^{s_1}Q^-R(z)^lQ^+\langle x\rangle^{s_2}\| \le c$ *for* $s_1, s_2 \in \mathbf{R}$.

The estimates (i), (ii) *and* (iii) *hold for* z *with* $\operatorname{Re} z \in J$, $\operatorname{Im} z < 0$, *if* Q^+ *and* Q^- *are interchanged.*

PROOF. The result (i) was derived for $l = 1$ in [**20**]. Using Lemma 4.1(i) the extension to $l > 1$ is straightforward. To prove (ii), consider first the Q^--estimate. We assume $\operatorname{Re} z \in J$ and $\operatorname{Im} z > 0$ in the following computations. The first resolvent equation gives

$$R(z) = (z+i)R(-i)R(z) + R(-i).$$

Iterate to get

$$R(z) = (z+i)^kR(-i)^kR(z) + \sum_{j=1}^{k} c_jR(-i)^j. \tag{4.5}$$

To prove (ii), consider first $l = 1$. Using (4.5) we get

$$\begin{aligned}\langle x\rangle^{s-1}Q^-R(z)\langle x\rangle^{-s} &= (z+i)^k\langle x\rangle^{s-1}Q^-R(z)(H+i)^{-k}\langle x\rangle^{-s}\\ &\quad+\sum_{j=1}^{k} c_j\langle x\rangle^{s-1}Q^-(H+i)^{-j}\langle x\rangle^{-s}.\end{aligned}$$

The sum above is a bounded operator by Lemma 4.1(i) if we note that $\langle x\rangle^sQ^-\langle x\rangle^{-s}$ is bounded (see the Appendix). For the first term, assume $1/2 < s \le 1$, and take $k = 2$:

$$\begin{aligned}&\langle x\rangle^{s-1}Q^-R(z)(H+i)^{-2}\langle x\rangle^{-s}\\ &\quad= (\langle x\rangle^{s-1}Q^-(H+i)^{-1}\rho^{s-1})(\rho^{1-s}P_A^-R(z)\rho^s)(\rho^{-s}(H+i)^{-1}\langle x\rangle^{-s})\\ &\qquad+ (\langle x\rangle^{s-1}Q^-P_A^+\rho^{-s})(\rho^sR(z)\rho^s)(\rho^{-s}(H+i)^{-2}\langle x\rangle^{-s}).\end{aligned}$$

The result (ii) for $1/2 < s \le 1$ now follows from Theorem 3.6 and Lemmas 4.1 and 4.2. If $s > 1$, choose an integer $k > s$. Then one has

$$\begin{aligned}&\langle x\rangle^{s-1}Q^-R(z)(H+i)^{-k}\langle x\rangle^{-s}\\ &\quad= (\langle x\rangle^{s-1}Q^-\rho^{s-1})(\rho^{1-s}P_A^-R(z)\rho^s)(\rho^{-s}(H+i)^{-k}\langle x\rangle^{-s})\\ &\qquad+ (\langle x\rangle^{s-1}Q^-P_A^+\rho^{-s})(\rho^sR(z)\rho^s)(\rho^{-s}(H+i)^{-k}\langle x\rangle^{-s}).\end{aligned}$$

The result now follows from Theorem 3.6 and Lemmas 4.1 and 4.2. To prove (ii) for $l > 1$, one takes the lth power of (4.5) and proceeds by induction. Details are omitted. The proof for Q^+ is similar and is omitted.

To prove (iii), it suffices for a given l to consider $s_1 = s_2 = s > l - 1/2$:

$$\begin{aligned}&\langle x\rangle^sQ^-R(z)^lQ^+\langle x\rangle^s\\ &\quad= \langle x\rangle^sQ^-(P_A^- + P_A^+)R(z)^l(P_A^- + P_A^+)Q^+\langle x\rangle^s\\ &\quad= (\langle x\rangle^sQ^-\rho^s)(\rho^{-s}P_A^-R(z)^l\rho^{s+l})(\rho^{-s-l}P_A^-Q^+\langle x\rangle^s)\\ &\qquad+ (\langle x\rangle^sQ^-P_A^+\rho^{-s})(\rho^sR(z)^l\rho^s)(\rho^{-s}P_A^-Q^+\langle x\rangle^s)\\ &\qquad+ (\langle x\rangle^sQ^-\rho^s)(\rho^{-s}P_A^-R(z)^lP_A^+\rho^{-s})(\rho^sQ^+\langle x\rangle^s)\\ &\qquad+ (\langle x\rangle^sQ^-P_A^+\rho^{-s-l})(\rho^{s+l}R(z)^lP_A^+\rho^{-s})(\rho^sQ^+\langle x\rangle^s).\end{aligned}$$

The result now follows from $s > l - 1/2$, Theorem 3.6 and Lemmas 4.1 and 4.2. □

It is possible to extend these results to $Q^{\pm}$ with a larger support. We shall briefly sketch an extension of Theorem 4.3(ii). We assume $p_{\pm} \in C^{\infty}(\mathbf{R}^{2n})$ satisfy (4.2) and for some $v > 0$, $-1 < \sigma < 1$:

$$\operatorname{supp}(p_{\pm}) \subseteq \{(x,\xi) \mid |\xi| \geq v, \pm\hat{x}\cdot\hat{\xi} \geq \sigma\}, \tag{4.6}$$

where $\hat{x} = x/|x|$. Operators $P^{\pm}$ are defined using (4.3).

The following lemma plays an essential role in [**9**]. A proof is given in the Appendix. Let $\mathcal{B}^{\infty}([0,\infty))$ denote the C^{∞}-functions on $[0,\infty)$ with bounded derivatives.

LEMMA 4.4. *Let $0 < a < b < \infty$, $I_0 = (a,b)$. Let $\phi \in \mathcal{B}^{\infty}([0,\infty))$ be a function which vanishes in a neighborhood of I_0. Let $l \geq 1$ and $s \in \mathbf{R}$. Then*

$$\langle x\rangle^{s}\phi(H_0)R(z)^{l}\langle x\rangle^{-s}$$

extends to a bounded operator on $\mathcal{H}$, with norm uniformly bounded in z for $\operatorname{Re} z \in I_0$, $\operatorname{Im} z \neq 0$.

This lemma is important because it allows us to localize in kinetic energy in proving the propagation estimates.

THEOREM 4.5. *Let $P^{\pm}$ be operators with symbols satisfying* (4.6). *Let $J = (a,b)$, $0 < a < b < \infty$, and $l \geq 1$ an integer. For $s > l - 1/2$ we have*

$$\|\langle x\rangle^{s-l}P^{-}R(z)^{l}\langle x\rangle^{-s}\| \leq c, \qquad \|\langle x\rangle^{-s}R(z)^{l}P^{+}\langle x\rangle^{s-l}\| \leq c$$

for all z with $\operatorname{Re} z \in J$, $\operatorname{Im} z > 0$.

PROOF. We shall only outline the proof. A result due to Mourre [**16**] plays an essential role. Let $\phi \in C_0^{\infty}(0,\infty)$ satisfy $\phi(\lambda) = 1$ for $\lambda \in J_{\delta} = (a-\delta, b+\delta)$ (δ sufficiently small). Then

$$\begin{aligned}\langle x\rangle^{s-l}P^{-}R(z)^{l}\langle x\rangle^{-s} &= \langle x\rangle^{s-l}P^{-}(1-\phi(H_0))R(z)^{l}\langle x\rangle^{-s}\\ &\quad + \langle x\rangle^{s-l}P^{-}\phi(H_0)R(z)^{l}\langle x\rangle^{-s}.\end{aligned}$$

The first term is bounded on $\mathcal{H}$ with norm uniformly bounded in z, $\operatorname{Re} z \in J$, $\operatorname{Im} z \neq 0$, by Lemma 4.4. The second term requires a decomposition of the symbol of $P^{-}\phi(H_0)$. Mimicking [**16**] we can get

$$p_{-}(x,\xi)\phi(\xi^2) = \sum_{j=1}^{N} q_j(x,\xi)$$

such that for each j and $E \in J$ there exists a vector $a_j \in \mathbf{R}^n$ with the property that $A_{a_j} = A + a_j\cdot x$ is conjugate to H at $E \in J$, and Lemma 4.2 holds for Q_j, A_{a_j}. Thus we can repeat the proof of Theorem 4.3 to get $\langle x\rangle^{s-l}Q_jR(z)^{l}\langle x\rangle^{-s}$ bounded with norm uniformly bounded in z, $\operatorname{Re} z \in (E-\delta, E+\delta)$, $\operatorname{Im} z > 0$. The proof is now completed by a covering argument. □

REMARK 4.6. The result (iii) in Theorem 4.3 can be extended to $P^{\pm}$ with symbols satisfying (4.6) and the additional condition

$$\inf_{x\in\mathbf{R}^n} \operatorname{dist}\left(\operatorname*{supp}_{\xi}(p_{-}(x,\xi)), \operatorname*{supp}_{\xi}(p_{+}(x,\xi))\right) > 0.$$

This result was first obtained for $l = 1$ in [**9**], using a different method.

REMARK 4.7. The results of Theorem 4.3 lead to propagation estimates similar to those in Theorem 3.10. Since the results are quite similar, they will not be stated in detail.

REMARK 4.8. (i) If one assumes $|\partial_x^\alpha V(x)| \leq c_\alpha(1+|x|)^{-\varepsilon-|\alpha|}$ for $|\alpha| \leq N$, then one obtains estimates for $R(z)^l$ for $l < N-1$ (N an integer) (see [**11**] for precise statements of related results).

(ii) One can also obtain resolvent estimates which are valid for $\operatorname{Re} z \in [a, \infty)$, $a > 0$ with a decay $(\operatorname{Re} z)^{-(l/2)}$ of the norm. This result requires construction of a different A (see [**3**] for results in this direction). Results for $\operatorname{Re} z \in [a, \infty)$ were also obtained in [**8**, **9**].

Appendix. In this Appendix we recall some well-known results on pseudo-differential operators and use these to prove Lemmas 4.2 and 4.4.

DEFINITION A.1. Let $\mu, m \in \mathbf{R}$. $S^m(\mu)$ denotes the functions $p \in C^\infty(\mathbf{R}^{2n})$ which satisfy

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq c_{\alpha\beta}\langle x\rangle^{\mu-|\alpha|}\langle \xi\rangle^{m-|\beta|}$$

for all multi-indices α, β, and all $(x,\xi) \in \mathbf{R}^{2n}$.

The functions $p \in S^m(\mu)$ are the symbols of a class of pseudo-differential operators on $\mathbf{R}^n$. This class was first used for Schrödinger operators by Agmon [**1**]. Much more general symbols have been considered by Hörmander [**7**], and all the general results on pseudo-differential operators used here can be obtained from [**7**, **22**].

The operator associated to $p \in S^m(\mu)$ is given for $f \in \mathcal{S}(\mathbf{R}^n)$ by

$$(Pf)(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat{f}(\xi)\,d\xi.$$

P is continuous on $\mathcal{S}(\mathbf{R}^n)$ and by duality on $\mathcal{S}'(\mathbf{R}^n)$ (strong topology). The class of operators with symbols in $S^m(\mu)$ is denoted $\mathrm{OP}S^m(\mu)$. For $P \in \mathrm{OP}S^m(\mu)$, its symbol is denoted $\sigma(P)$. The following properties of this class of ψdo's are needed below.

PROPOSITION A.2. *Let $P_j \in \mathrm{OP}S^{m_j}(\mu_j)$, $j = 1, 2$. Then*

$$P_1P_2 \in \mathrm{OP}S^{m_1+m_2}(\mu_1+\mu_2),$$

and the symbol of P_1P_2 has the asymptotic expansion

$$\sigma(P_1P_2) = \sum_{|\alpha|<N}\frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p_1(x,\xi)\partial_x^\alpha p_2(x,\xi) + r_N(x,\xi)$$

for any $N > 0$ with $r_N \in S^{m_1+m_2-N}(\mu_1+\mu_2-N)$.

PROPOSITION A.3. *Let $P \in \mathrm{OP}S^m(\mu)$ with $m, \mu \leq 0$. Then P extends to a bounded operator on $L^2(\mathbf{R}^n)$.*

The following subsets of $S^m(\mu)$ are used in the proofs of Lemmas 4.2 and 4.4.

DEFINITION A.4. $S_\pm^m(\mu)$ denotes $p \in S^m(\mu)$ such that there exist $\sigma, v > 0$ (depending on p) with

$$\operatorname{supp}(p) \subseteq \{(x,\xi) \mid |\xi| \geq v, \pm\hat{x}\cdot\hat{\xi} \geq \sigma\}.$$

As in §4, we let $A = (2i)^{-1}(x\cdot\nabla + \nabla\cdot x)$, $A \in \mathrm{OP}S^1(1)$, and $\sigma(A) = a(x,\xi) = x\cdot\xi - in/2$.

LEMMA A.5. *Let $Q^- \in \mathrm{OP}S^0_-(0)$, and let γ be a multi-index. There exist $P_j^- \in \mathrm{OP}S^0_-(0)$, $j = 0, 1, \ldots, |\gamma|$, such that*

$$x^\gamma Q^- = \sum_{j=0}^{|\gamma|} P_j^- A^j.$$

A similar result holds in the $+$-case.

PROOF. The proof is straightforward application of the symbol calculus. For the sake of completeness, the proof is outlined. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \le \phi(x) \le 1$, $\phi(x) = 1$ for $|x| \le 1$, $\phi(x) = 0$ for $|x| \ge 2$. $x^\gamma Q^- = x^\gamma \phi(x) Q^- + x^\gamma (1 - \phi(x)) Q^-$. The first term belongs to $\mathrm{OP}S^0_-(0)$. Thus we can assume that the symbol q of Q^- has support in

$$M = \{(x, \xi) | -\hat{x} \cdot \hat{\xi} \ge \sigma, |\xi| \ge v, |x| \ge 1\}.$$

The result is proved first for $|\gamma| = 1$. Let $u(x, \xi) = (1 - x \cdot \xi)^{-1} \cdot x_j$ for $(x, \xi) \in M$. Elementary computations show

$$|\partial_x^\alpha \partial_\xi^\beta u(x, \xi)| \le c_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-1-|\beta|}$$

for all α, β and all $(x, \xi) \in M$. Let $p(x, \xi) = q(x, \xi) u(x, \xi)$. Then $p \in S_-^{-1}(0)$. Let P be the associated operator. Proposition A.2 shows

$$\begin{aligned}\sigma(-PA) &= -p(x, \xi) a(x, \xi) + \sum_{j=1}^n i\xi_j \partial_{\xi_j} p(x, \xi) + r_2(x, \xi) \\ &= x_j q(x, \xi) + \left\{ ((in/2) - 1) p(x, \xi) + \sum_{j=1}^n i\xi_j \partial_{\xi_j} p(x, \xi) + r_2(x, \xi) \right\}.\end{aligned}$$

The terms in $\{\cdots\}$ belong to $S^0_-(0)$. From this equation the result follows for $|\gamma| = 1$. The general result follows by iteration. $\square$

PROOF OF LEMMA 4.2(i). For integer s the result follows from Lemma A.5. The general result follows using interpolation. $\square$

(ii) Consider first $\langle x \rangle^{s_1} Q^- P_A^+ \rho^{-s_2}$. It suffices to prove boundedness for $s_1 = 0$ by Lemma A.5. We use the Mellin transform to diagonalize A. A stationary phase argument then shows that $Q^- P_A^+ \rho^{-s_2}$ is bounded (see [**18**] for related results). A direct proof using the calculus of ψdo is difficult due to the presence of P_A^+. The remaining cases are treated similarly. $\square$

PROOF OF LEMMA 4.4. The proof is based on an idea in [**9**] and is included here for the sake of completeness. Let $\mathcal{A}$ denote the functions in $\mathcal{B}^\infty([0, \infty))$ which vanish on a neighborhood of $[a, b]$. Let $\mathcal{I} = \{z | \mathrm{Re}\, z \in (a, b), \mathrm{Im}\, z \ne 0\}$. It is obvious from Propositions A.2 and A.3 that the result holds for $R_0(z) = (H_0 - z)^{-1}$. Let $L_s^2 = \{f \in L^2_{\mathrm{loc}}(\mathbf{R}^n) | \langle x \rangle^s f \in L^2(\mathbf{R}^n)\}$ denote the weighted L^2-space. For $\phi \in \mathcal{A}$, Proposition A.3 implies

$$\phi(H_0) R_0(z) V (1 - \phi(H_0)) = X \psi(H_0)$$

for some $\psi \in \mathcal{A}$, where X maps L_s^2 into $L_{s+\varepsilon}^2$ for any $s \in \mathbf{R}$, with norm uniformly bounded in $z \in \mathcal{I}$. Let $s > 1/2$ and $u \in L_s^2$. Then $R(z)u \in L^2_{-s}$ with norm uniformly bounded in $z \in \mathcal{I}$ (see Theorem 4.3(i)). We have

$$\phi(H_0) R(z) u = \phi(H_0) R_0(z) u + X \psi(H_0) R(z) u + \phi(H_0) R_0(z) V \phi(H_0) R(z) u$$

and thus for any $\phi \in \mathcal{A}$, $\phi(H_0)R(z)u \in L^2_{\min\{-s+\varepsilon,s\}}$. The result now follows for $s > 1/2$ by a finite number of iterations. The same result holds for $R(z)\phi(H_0)$ for $s > 1/2$, and hence by duality the result holds for $s < -1/2$ for $\phi(H_0)R(z)$. The general case follows by interpolation. For $l > 1$ we proceed by induction. Details are omitted. □

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