

PROJECTIVE MODULES IN THE CATEGORY \mathcal{O}_S : LOEWY SERIES

BY

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ABSTRACT. Let \mathfrak{g} be a complex, semisimple Lie algebra with a parabolic subalgebra \mathfrak{p}_S . The Loewy lengths and Loewy series of generalized Verma modules and of their projective covers in \mathcal{O}_S are studied with primary emphasis on the case in which \mathfrak{p}_S is a Borel subalgebra and \mathcal{O}_S is the category \mathcal{O} . An examination of the change in Loewy length of modules under translation leads to the calculation of Loewy length for Verma modules and for self-dual projectives in \mathcal{O} , assuming the Kazhdan-Lusztig conjecture (in an equivalent formulation due to Vogan). In turn, it is shown that the Loewy length results imply Vogan's statement, and lead to the determination of Loewy length for the self-dual projectives and certain generalized Verma modules in \mathcal{O}_S . Under the stronger assumption of Jantzen's conjecture, the radical and socle series are computed for self-dual projectives in \mathcal{O} . An analogous result is formulated for self-dual projectives in \mathcal{O}_S and proved in certain cases.

1. Introduction. In this paper, the study of generalized Verma modules and their projective covers in the category \mathcal{O}_S which was begun in [12] is continued. We examine the Loewy lengths of the generalized Verma modules and the Loewy series of the projectives which are self-dual. Let \mathfrak{g} be a fixed complex, semisimple Lie algebra with Cartan subalgebra \mathfrak{h} , root system R , and choice of simple roots B . Associated to a subset S of B is a parabolic subalgebra \mathfrak{p}_S and a category \mathcal{O}_S of finite length modules which is a natural setting for the study of generalized Verma modules—the \mathfrak{g} -modules induced from finite-dimensional \mathfrak{p}_S -modules. In case $S = \emptyset$, we are reduced to the case of a Borel subalgebra \mathfrak{b} , the category \mathcal{O} , and Verma modules. (For further explanation of any notation used, we refer to [14 or 12].)

Let λ be a dominant regular weight in \mathfrak{h}^* and let \mathcal{O}^λ be the corresponding block of \mathcal{O} . The Kazhdan-Lusztig conjecture describes the multiplicities of simple modules in Verma modules [15], and has been proved (for the case of λ integral, see [3 or 7]). D. Vogan has proved that the validity of the conjecture in \mathcal{O}^λ is equivalent to the statement that for $\alpha \in B_\lambda$ and $w \in W_\lambda$, the module $\theta_\alpha L(w \cdot \lambda)$ has Loewy length at most 3, where θ_α is the functor of translation through the α -wall [19]. The latter statement is called Vogan's conjecture, and we show in §3 that it can be extended for any module M in \mathcal{O}^λ to the statement that $\theta_\alpha M$ has Loewy length at most $l(M) + 2$, where we write $l(M)$ for the Loewy length of M . From this we prove, for w_λ the longest element of W_λ and $w \in W_\lambda$, that $lM(w \cdot \lambda) = l_\lambda(w_\lambda w) + 1$ and $lP(w_\lambda \cdot \lambda) = 2l_\lambda(w_\lambda) + 1$. Conversely, we will see in 5.2 that if $lP(w_\lambda \cdot \lambda) = 2l_\lambda(w_\lambda) + 1$, then

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\mathcal{O}^λ satisfies Vogan's conjecture. Thus, the determination of Loewy lengths is of the same order of difficulty as the solution of the Kazhdan-Lusztig conjecture. Section 3 contains additional results, which are used to extend the Loewy length results to nonregular weights and are needed later.

The module $P(w_\lambda \cdot \lambda)$ is the unique projective, indecomposable module in \mathcal{O}^λ which is self-dual [12, §3], and the results of §3 suggest a natural candidate for its two Loewy series. If for each $w \in W_\lambda$ the radical and socle series of $M(w \cdot \lambda)$ were known to coincide, one could easily verify that the candidate is correct. (In particular, this is the case if all multiplicities $(M(w \cdot \lambda): L(y \cdot \lambda))$ are 0 or 1, leading to a simple proof that for W_λ indecomposable of rank > 2 , some multiplicity is > 1 .) Instead, we must assume that \mathcal{O}^λ satisfies Jantzen's conjecture. With this hypothesis, the Jantzen and socle filtrations of each $M(w \cdot \lambda)$ coincide (see 5.3 or [2]), and the layers of the socle filtration can be described by Kazhdan-Lusztig polynomials [10, 4.9]. This makes it possible to verify that the two Loewy series of $P(w_\lambda \cdot \lambda)$ coincide with the natural candidate. These results are in §5. We should add that the Jantzen conjecture has been proved by J. Bernstein, although there is no published proof.

It is conjectured in [12] that the self-duality of $P(w_\lambda \cdot \lambda)$ generalizes to the statement for \mathcal{O}_S^λ that $P_S(w \cdot \lambda)$ is self-dual for all $w \in W_\lambda$ such that $L(w \cdot \lambda)$ is a summand of the socle of some generalized Verma module (*socular* w 's). We will refer to this as the *self-duality conjecture*; its verification in many cases can be found in [12]. In fact, as observed by D. Garfinkle, Theorem 4.4 of [12] can be used to prove the conjecture in general. For each w in the set ${}^S X_\lambda$ of *socular* elements of ${}^S W_\lambda$ there is a unique element \check{w} minimal under the Bruhat order in the set $\{y \in {}^S W_\lambda \mid (M_S(y \cdot \lambda): L(w \cdot \lambda)) \neq 0\}$, and $(M_S(\check{w} \cdot \lambda): L(w \cdot \lambda)) = 1$ with $L(w \cdot \lambda) = \text{soc } M_S(\check{w} \cdot \lambda)$ [12, 4.6]. Using the Loewy length results on $\theta_a M$ of §3, we show in §4 that the modules $M_S(\check{w} \cdot \lambda)$, for $w \in {}^S X_\lambda$, all have the same Loewy length. It is easy to see that this common Loewy length is at least $t + 1$ for a positive integer t naturally associated to ${}^S W_\lambda$. If ${}^S X_\lambda$ contains an element satisfying a specific hypothesis which holds in all the examples of [12], then $t + 1$ is exactly the common Loewy length. Moreover, for all other $y \in {}^S W_\lambda$ the Loewy length of $M_S(y \cdot \lambda)$ is $\leq t$, and for $w \in {}^S X_\lambda$ the Loewy length of $P_S(w \cdot \lambda)$ is $2t + 1$. In case $S = \emptyset$, the hypothesis is satisfied and the conclusions specialize to the results of §3.

For each $w \in {}^S X_\lambda$, in analogy with the case of $S = \emptyset$, there is again a natural candidate for the Loewy series of $P_S(w \cdot \lambda)$. In §6 we describe a procedure for verifying that the candidate is correct if one knows the radical series of the generalized Verma modules arising in a Verma flag for $P_S(w \cdot \lambda)$, and indicate how one might obtain this information. A list is provided of the cases for which this procedure has been carried out. For a general proof that the candidate is correct, one would presumably need general information on the Loewy series of generalized Verma modules, paralleling the information available in case $S = \emptyset$.

REMARKS. (1) The results of §§3 and 5 were originally in [11, §7], although some of the proofs here are less ponderous. I wish to repeat the acknowledgements made in that paper. I am indebted to Maurice Auslander for first bringing to my attention in

1978 the problem of describing the category \mathcal{O} from the point of view of finite-dimensional algebras. James Humphreys indirectly revived the problem for me in a colloquium talk at Brandeis in the spring of 1980, and I profited from both his expository writings on the subject and several conversations with him. I am also indebted to Christine Riedtmann, with whom I had many conversations on the material of [11], and from whom I learned a great deal about finite-dimensional algebras.

(2) Various parts of the work were supported by N.S.F. grants, including an N.S.F. post-doctoral fellowship at the University of Chicago, where the work was begun.

(3) I repeat from [11] the dedication of this paper to my father on the occasion of his 65th birthday.

2. Notation and terminology.

2.1. The notational conventions of [12] will generally be used. In particular, in a context in which a dominant weight λ is fixed, l is the length function on W_λ , but we will write w_λ for the longest element of W_λ instead of w_0 . In case there is any chance of confusion, we may use l_λ instead of l . The subgroup $\{w \in W_\lambda \mid w \cdot \lambda = \lambda\}$ is denoted W_λ^0 .

It will be convenient to introduce a *height* function on the orbit $W_\lambda \cdot \lambda$ of a dominant weight λ , related to the ordering on weights denoted \uparrow . Recall that for two weights $\mu < \eta$ in h^* , we write $\mu \uparrow \eta$ if there are roots $\alpha_1, \dots, \alpha_r$ such that

$$\mu < s_{\alpha_1} \cdot \mu < s_{\alpha_2} s_{\alpha_1} \cdot \mu < \dots < s_{\alpha_r} \dots s_{\alpha_1} \cdot \mu = \eta.$$

In case λ is regular and y, w are elements of W_λ , we have $y \cdot \lambda \uparrow w \cdot \lambda$ if and only if $y > w$, with respect to the Bruhat order. For this case, we define the height of $w \cdot \lambda$ to be $l(w_\lambda w)$. In general, $h(w \cdot \lambda)$ is the length of the interval between the antidominant weight $w_\lambda \cdot \lambda$ and $w \cdot \lambda$ under the \uparrow ordering. More precisely, it is $l(y)$ for the element y of minimal length satisfying $yw_\lambda \cdot \lambda = w \cdot \lambda$; in other words, $y = zw_\lambda$ for z the unique longest element in the coset wW_λ^0 of W_λ .

By the fundamental theorem of I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [4], the \uparrow relation describes which simples occur as composition factors in Verma modules: $(M(\eta): L(\mu)) \neq 0$ if and only if $\mu \uparrow \eta$ if and only if $M(\mu) \subseteq M(\eta)$. This fact is tacitly used throughout the paper.

Another fundamental theorem of Bernstein, Gelfand and Gelfand is the existence of a p -filtration for the projective modules of \mathcal{O} . Following [9], as in [12], we will refer to p -filtrations as Verma flags. Let $0 = M_0 \subset M_1 \subset \dots \subset M_k = P(\mu)$ be a Verma flag for the indecomposable, projective $P(\mu)$, with $M_{i+1}/M_i = M(\eta_i)$. If $\eta_i \not\prec \eta_j$ for any i and j with $0 \leq i < j \leq k - 1$, we will call the Verma flag an *ordered* flag. If in addition, for any such i and j , the heights satisfy $h(\eta_i) \geq h(\eta_j)$, we will call it a *well-ordered* flag. It follows from Lemma 1 of the proof of Proposition 2 in [5] that a well-ordered flag always exists. Similar results apply to the category \mathcal{O}_S corresponding to some choice of parabolic subalgebra (see [9 and 17]). We will use these facts often, without any further explicit reference.

2.2. Given a finite length module M over some algebra, the *socle* of M is the largest semisimple submodule, denoted $\text{soc } M$, and the *radical* of M is the intersection of the maximal submodules, denoted $\text{rad } M$. Equivalently, $\text{rad } M$ is the smallest submodule K such that M/K is semisimple, and we will call $M/\text{rad } M$ the *top* of M . These notions lead to two filtrations of M with semisimple quotients, the *socle filtration* (or series)

$$0 \subset \text{soc } M \subset \text{soc}^2 M \subset \dots \subset \text{soc}^r M$$

defined by $\text{soc}^{i+1} M / \text{soc}^i M = \text{soc}(M / \text{soc}^i M)$, and the *radical filtration* (or series)

$$0 = \text{rad}^s M \subset \text{rad}^{s-1} M \subset \dots \subset \text{rad } M \subset M$$

defined by $\text{rad}^{i+1} M = \text{rad}(\text{rad}^i M)$. These two series are sometimes called the lower and upper Loewy series respectively [1, 9.4], and we will use *Loewy series* to refer to the two of them. (Some authors call the radical series the Loewy series.)

It is easy to prove that $r = s$, this number being called the *Loewy length* of M , which we will denote $ll(M)$. It is the shortest possible length of any filtration of M with semisimple quotients, and given such a filtration, $0 \subset M_1 \subset \dots \subset M_r = M$, we have

$$\text{rad}^{r-i} M \subset M_i \subset \text{soc}^i M$$

for all i [1, 9.4]. Thus if the two Loewy series coincide, there is only one such filtration.

Let $\text{soc}_i M = \text{soc}^i M / \text{soc}^{i-1} M$ and $\text{rad}_i M = \text{rad}^i M / \text{rad}^{i+1} M$. We will call these sequences of semisimple modules the *layers* of the respective Loewy series. We also use a parallel terminology for the Jantzen filtration on a Verma module $M(\mu)$, a filtration $(0) = M(\mu)^n \subset \dots \subset M(\mu)^1 \subset M(\mu)$ which is defined in [13]. We write $M(\mu)_i$ for the i th layer $M(\mu)^i / M(\mu)^{i+1}$.

3. Loewy length in \mathcal{O} . In 3.1 we will obtain obvious lower bounds on the Loewy lengths of Verma modules and of the self-dual projective modules in \mathcal{O} . The rest of the section is devoted to the question of whether these bounds are the actual values. After some preliminary results in 3.2, we reduce the problem to regular weights in 3.3 by studying how Loewy length changes under translation. Under the assumption of Vogan’s conjecture, we obtain more refined information on translation in 3.4, leading in 3.5 to a proof that the Loewy lengths are given by the lower bounds.

3.1. PROPOSITION. *Let μ be a weight and λ a dominant weight.*

- (i) *The Loewy length of $M(\mu)$ is at least $h(\mu) + 1$.*
- (ii) *The Loewy length of $P(w_\lambda \cdot \lambda)$ is at least $2llM(\lambda) - 1$.*

PROOF. (i) The module $M(\mu)$ has simple top, so for any weight η with $\eta \uparrow \mu$, the submodule $M(\eta)$ lies in the radical of $M(\mu)$. The inequality $llM(\mu) \geq h(\mu) + 1$ follows by induction on height.

(ii) The projective $P(w_\lambda \cdot \lambda)$ is isomorphic to its dual module [12, §3], and $M(\lambda)$ is a submodule, so $DM(\lambda)$ is a homomorphic image. Let $r = llM(\lambda)$. It follows that $\text{rad}_{r-1} P(w_\lambda \cdot \lambda)$ involves $L(\lambda)$ as a composition factor. By BGG reciprocity, $(P(w_\lambda \cdot \lambda): L(\lambda)) = 1$, so that the appearance of $L(\lambda)$ in $\text{rad}_{r-1} P(w_\lambda \cdot \lambda)$ is its

only appearance in $P(w_\lambda \cdot \lambda)$. Hence it corresponds to the top of the submodule $M(\lambda)$, and the socle $L(w_\lambda \cdot \lambda)$ must lie in $\text{rad}^{2r-2} P(w_\lambda \cdot \lambda)$. In particular,

$$\text{ll}P(w_\lambda \cdot \lambda) \geq 2r - 1. \quad \square$$

Prompted by this result, we ask

Question. (i) Does a Verma module $M(\mu)$ have Loewy length $h(\mu) + 1$?

(ii) For λ dominant, does $P(w_\lambda \cdot \lambda)$ have Loewy length $2h(\lambda) + 1$?

Notice that a positive answer to (ii) implies $\text{ll}M(\lambda) = h(\lambda) + 1$, by the proposition, yielding a positive answer to (i).

3.2. Let $r(\mu)$ denote the Loewy length of $M(\mu)$.

PROPOSITION. Let λ be a dominant weight, and let μ be a weight in $W_\lambda \cdot \lambda$.

(i) $(P(w_\lambda \cdot \lambda)/\text{rad}^{r(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = (\text{rad}_{r(\mu)-1} P(w_\lambda \cdot \lambda): L(\mu)) = 1$.

(ii) $(\text{soc}^{r(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = (\text{soc}_{r(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = 1$.

PROOF. By self-duality of $P(w_\lambda \cdot \lambda)$, the two statements are equivalent. We will prove them by induction on $h(\lambda) - h(\mu)$, beginning with the 0th case, $\mu = \lambda$. The proof of Proposition 3.1 shows that the unique occurrence of $L(\lambda)$ as a composition factor occurs in layer $r(\lambda)$ of the socle series and, dually, layer $r(\lambda) - 1$ of the radical series.

For any weight μ , the fact that $M(\mu)$ is a submodule of $M(\lambda)$, which is in turn a submodule of $P(w_0 \cdot \lambda)$, shows that $(\text{soc}_{r(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) \geq 1$. It suffices to prove that $(P(w_\lambda \cdot \lambda)/\text{rad}^{r(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) \leq 1$, using self-duality again. By induction on $h(\lambda) - h(\mu)$ for any weight η with $\mu \uparrow \eta$ and $\mu < \eta$, we have $(P(w_\lambda \cdot \lambda)/\text{rad}^{r(\mu)} P(w_\lambda \cdot \lambda): L(\eta)) = 0$. Therefore, given an ordered Verma flag $0 \subset N_1 \subset \dots \subset N_k = P(w_\lambda \cdot \lambda)$, the smallest filtration submodule N_i having all $M(\eta)$'s with $\mu \uparrow \eta$ and $\mu < \eta$ as factors must lie in $\text{rad}^{r(\mu)} P(w_\lambda \cdot \lambda)$. But

$$(P(w_\lambda \cdot \lambda)/N_i: L(\mu)) = 1,$$

and the conclusion follows. \square

REMARK. As the proof indicates, the proposition may be interpreted as saying that for each μ , there is a unique lowest appearance of $L(\mu)$ in the socle series of $P(w_\lambda \cdot \lambda)$, corresponding to the top of the Verma submodule $M(\mu)$, and a unique highest appearance in the radical series, corresponding to the top of the Verma flag, factor $M(\mu)$. We note one consequence of this. Let $N(\mu)$ denote the unique homomorphic image of $P(\mu)$ in $\text{rad}^{r(\mu)-1} P(w_\lambda \cdot \lambda)$ with image not in $\text{rad}^{r(\mu)} P(w_\lambda \cdot \lambda)$. The existence of $N(\mu)$, and its uniqueness, follows from the proposition. For a pair of weights μ and η in $W_\lambda \cdot \lambda$, with $\mu \uparrow \eta$, it is clear (as in the proof of 3.1(i)) that the submodule $M(\mu)$ of $M(\eta)$ lies in $\text{rad}^{h(\eta)-h(\mu)} M(\eta)$. By self-duality of $P(w_\lambda \cdot \lambda)$, this translates into the following statement: $N(\eta)$ is a submodule of $N(\mu)$ lying inside $\text{rad}^{h(\eta)-h(\mu)} N(\mu)$. The point is that the tops of $N(\eta)$ and $N(\mu)$ are dual to the tops of the submodules $M(\eta)$ and $M(\mu)$.

3.3. In this subsection we reduce Question (ii) of 3.1 to the case of regular weights, via the following result, more easily expressed via antidominant weights.

PROPOSITION. *Let ν and μ be antidominant weights such that $\nu - \mu$ is integral. Assume that ν is regular and μ lies in the upper closure of the facette of ν . Let w_1 be the longest element in W_μ^0 .*

(i) *For any $w \in W_\nu$, the module $T_\mu^\nu L(w \cdot \mu)$ has Loewy length at least $2l(w_1) + 1$ and a simple top.*

(ii) *Given a module M of Loewy length s in \mathcal{O}^μ , the Loewy length of $T_\mu^\nu M$ is at least $2l(w_1) + s$.*

PROOF. By [13, 2.24], $T_\mu^\nu P(\mu) = P(\nu)$. A Verma flag for $P(\mu)$ will be carried by T_μ^ν to a filtration of $P(\nu)$ with quotients of the form $T_\mu^\nu M(y \cdot \mu)$, and by [13, 2.17], each such quotient has a Verma flag with constituents $\{M(yz \cdot \nu) | z \in W_\mu^0\}$. Let us consider the particular subquotient $T_\mu^\nu M(w \cdot \mu)$ of $P(\nu)$, and assume w is the element of minimal length in wW_μ^0 (so $l(w) = h(w \cdot \mu)$). The module $T_\mu^\nu L(w \cdot \mu)$ is a quotient of $T_\mu^\nu M(w \cdot \mu)$, which must contain all the composition factors of $T_\mu^\nu M(w \cdot \mu)$ isomorphic to $L(w \cdot \nu)$ [13, 2.11]. Since $(M(wz \cdot \nu) : L(w \cdot \nu)) = 1$ for all $z \in W_\mu^0$ [13, 2.16], there is exactly one such $L(w \cdot \nu)$ contributed by each Verma flag constituent $M(wz \cdot \nu)$. In particular, for each $z \in W_\mu^0$, the module $T_\mu^\nu L(w \cdot \nu)$ must involve $L(wz \cdot \nu)$ as a composition factor at least once, corresponding to its appearance as the top of a Verma flag constituent in $T_\mu^\nu M(w \cdot \mu)$. In the notation of Remark 3.2, we may conclude that $N(w \cdot \nu)$ maps onto $T_\mu^\nu L(w \cdot \nu)$ and the submodule $N(wz \cdot \nu)$ is not in the kernel.

In case $z = w_1$, we may conclude that $\text{rad}^{l(ww_1) - l(w)} T_\mu^\nu L(w \cdot \mu)$ contains $L(ww_1 \cdot \nu)$ as a composition factor, corresponding to the top of the Verma flag constituent, $M(ww_1 \cdot \nu)$, and $L(w \cdot \nu)$ occurs as a composition factor in

$$\text{rad}^{2(l(ww_1) - l(w))} T_\mu^\nu L(w \cdot \mu).$$

Since $l(ww_1) - l(w) = l(w_1)$, and $N(w \cdot \nu)$ has simple top, the desired result follows.

(ii) Let K be a nontrivial extension of $L(y \cdot \mu)$ by $L(w \cdot \mu)$ for y, w in W_μ , and suppose $y \cdot \mu \uparrow w \cdot \mu$. Then $T_\mu^\nu K$ is a homomorphic image of $T_\mu^\nu M(w \cdot \mu)$, which has a simple top by the argument of (i). Hence $T_\mu^\nu K$ and its dual $T_\mu^\nu DK$ are indecomposable. Part (ii) is proved by induction on lM , using this observation, the induction being started by (i). We may replace M by a submodule of the same Loewy length with simple top. Let $L(w \cdot \mu)$ be the top and $L(y \cdot \mu)$ a simple in the top of $\text{rad } M$, with K' the resulting length two homomorphic image of M . We may regard $T_\mu^\nu M$ as an extension of $T_\mu^\nu(\text{rad } M)$ by $T_\mu^\nu L(w \cdot \mu)$, and this extension cannot split or else $T_\mu^\nu K'$ does, contrary to our observation. Hence $lT_\mu^\nu M \geq lT_\mu^\nu(\text{rad } M) + 1$, and the proof is complete by induction. \square

COROLLARY. *Let ν and μ be antidominant weights with $\nu - \mu$ integral. Assume that ν is regular and μ lies in the upper closure of ν . If $P(\nu)$ has Loewy length $2l(w_\nu) + 1$, then $P(\mu)$ has Loewy length $2h(w_\nu \cdot \mu) + 1$.*

PROOF. By Proposition 3.1, the Loewy length t of $P(\mu)$ is at least $2h(w_\nu \cdot \mu) + 1$, and by the Proposition above, $lP(\nu) \geq 2l(w_1) + t$, since $P(\nu) = T_\mu^\nu P(\mu)$. Thus we have

$$2l(w_1) + 2h(w_\nu \cdot \mu) + 1 \leq 2l(w_1) + t \leq 2l(w_\nu) + 1.$$

But by the definition of height, $l(w_\nu) = h(w_\nu \cdot \mu) + l(w_1)$, so the inequalities are all equalities. \square

3.4. By 3.3, we may restrict to regular weights in our further consideration of Question 3.1. The results of 3.3 indicate that there is a relationship between the question and how Loewy length grows under translation. In this light, a connection between the solution to Question 3.1 and Vogan’s conjecture is not surprising. Let us recall the conjecture.

Given a dominant, regular weight λ and $\alpha \in B_\lambda$, the translation functor θ_α on \mathcal{O}^λ is defined as the composition $T_{w_\lambda \cdot \lambda + \nu_\alpha}^{w_\lambda \cdot \lambda} \circ T_{w_\lambda \cdot \lambda + \nu_\alpha}^{w_\lambda \cdot \lambda + \nu_\alpha}$, with ν_α chosen so that $W_{w_\lambda \cdot \lambda + \nu_\alpha}^0 = \{e, s_\alpha\}$. Since for $w \in W_\lambda$ the module $T_{w_\lambda \cdot \lambda + \nu_\alpha}^{w_\lambda \cdot \lambda} L(w \cdot \lambda)$ is either (0) or simple, Proposition 3.3(i) implies that $\theta_\alpha L(w \cdot \lambda)$ is either (0) or of Loewy length ≥ 3 . Vogan’s conjecture for \mathcal{O}^λ states that for all $\alpha \in B_\lambda$ and $w \in W_\lambda$, if $\theta_\alpha L(w \cdot \lambda)$ is nonzero, it has Loewy length exactly 3. This is equivalent to the Kazhdan-Lusztig conjecture for \mathcal{O}^λ .

Our goal in this subsection is to extend the conjecture to the statement that $\theta_\alpha M$ has Loewy length at most $ll(M) + 2$ for M in \mathcal{O}^λ . The proof is accomplished in three steps, after two preliminary results. Let λ be a fixed dominant, regular weight with α a fixed root in B_λ , and assume the action of θ_α on \mathcal{O}^λ satisfies Vogan’s conjecture.

LEMMA. *A module M of finite length embeds in a finite direct sum of homomorphic images of M , each having a simple socle. Dually, M is a homomorphic image of a finite direct sum of submodules, each having simple top.* \square

A certain type of module lies at the heart of our analysis in this subsection, and we will give it a name. Let y, w be elements of W_λ such that $ws_\alpha < w, ys_\alpha < y$ and $y > w$. Then a highest weight module N in \mathcal{O}^λ is called (y, w) -basic if N has simple top $L(w \cdot \lambda)$, simple socle $L(y \cdot \lambda)$, and all composition factors of $\text{rad } N / \text{soc } N$ are annihilated by θ_α . A module is called *basic* if it is (y, w) -basic for suitable y and w .

PROPOSITION. *Let N be a basic module in \mathcal{O}^λ . The $\theta_\alpha N$ has N as a submodule and as a homomorphic image.*

PROOF. (i) Assume that N is (y, w) -basic. Then $\theta_\alpha N$ has simple top $L(w \cdot \lambda)$, since $\theta_\alpha N$ is a homomorphic image of $\theta_\alpha M(w \cdot \lambda)$, which has simple top. We will prove that $\text{soc } \theta_\alpha N = L(y \cdot \lambda)$. By the definition of basic, $\theta_\alpha N$ is an extension of $\theta_\alpha L(y \cdot \lambda)$ by $\theta_\alpha L(w \cdot \lambda)$. In general, for z in W_λ with $zs_\alpha < z$, the top and socle of $\theta_\alpha L(z \cdot \lambda)$ are isomorphic to $L(z \cdot \lambda)$, and the other composition factors are annihilated by θ_α [13, Kapitel 2]. Hence, the socle of $\theta_\alpha N$ is either $L(y \cdot \lambda)$ or $L(y \cdot \lambda) \oplus L(w \cdot \lambda)$. In the latter case, because θ_α is a selfadjoint functor, we obtain

$$\text{Hom}_{\mathcal{O}}(\theta_\alpha L(w \cdot \lambda), N) \neq (0).$$

But N has $L(y \cdot \lambda)$ as socle and $(\theta_\alpha L(w \cdot \lambda): L(y \cdot \lambda)) = 0$, a contradiction.

(ii) Suppose $\theta_\alpha N$ maps onto N . By selfadjointness of θ_α , we obtain a nonzero homomorphism $f: N \rightarrow \theta_\alpha N$, and since $\theta_\alpha N$ has simple socle $L(y \cdot \lambda)$ by (i), the image of f must contain it. The only occurrence of $L(y \cdot \lambda)$ in N is as the socle, forcing f to be an embedding. Thus, it suffices to prove that $\theta_\alpha N$ maps onto N .

(iii) N is a homomorphic image of $M(w \cdot \lambda)$, and therefore also of $\theta_\alpha M(w \cdot \lambda)$, which is an extension of $M(ws_\alpha \cdot \lambda)$ by $M(w \cdot \lambda)$ with simple top. Let K be a submodule of $M(w \cdot \lambda)$ maximal with respect to the property that $\theta_\alpha(M(w \cdot \lambda)/K)$ maps onto N . We claim that $M(w \cdot \lambda)/K$ is (y, w) -basic.

It is evident that $M(w \cdot \lambda)/K$ has simple socle $L(y \cdot \lambda)$, with no other appearances by $L(y \cdot \lambda)$ as a composition factor. If $M(w \cdot \lambda)/K$ is not basic, there is a proper submodule L with top $L(z \cdot \lambda)$ such that $z > w$, $zs_\alpha < z$ and either L or DL is basic. In either case, by (i), $\theta_\alpha L$ has simple top $L(z \cdot \lambda)$, which must lie in the kernel of the map of $\theta_\alpha(M(w \cdot \lambda)/K)$ onto N . But $\theta_\alpha L$ involves all the composition factors of $M(w \cdot \lambda)/K$ which are isomorphic to $L(y \cdot \lambda)$, yielding a contradiction. We may conclude that any (y, w) -basic module N is a homomorphic image of $\theta_\alpha N'$ for some (y, w) -basic module N' .

(iv) Given N and N' as in the conclusion of (iii), we find by the arguments of (i) and (ii) that N' embeds in $\theta_\alpha N$. The quotient $\theta_\alpha N/N'$ has top $L(w \cdot \lambda)$, and except for that and one occurrence of $L(y \cdot \lambda)$, all composition factors are annihilated by θ_α . Thus there is a (y, w) -basic homomorphic image N'' , yielding the converse to the conclusion of (iii).

(v) The proof is easily completed. By [10, 3.6], $\theta_\alpha^2 N' \cong \theta_\alpha N' \oplus \theta_\alpha N'$, and by (iii) and (iv) we obtain a sequence of surjections $\theta_\alpha^2 N' \rightarrow \theta_\alpha N \rightarrow N''$. Hence $\theta_\alpha N' \oplus \theta_\alpha N'$ maps onto both N and N'' . From this it follows that $N \cong N''$, yielding the desired surjection of $\theta_\alpha N$ onto N . \square

COROLLARY. *Given a module M in \mathcal{O}^λ of Loewy length two, the module $\theta_\alpha M$ has Loewy length at most 4.*

PROOF. By the Lemma and the exactness of θ_α , we may assume M has simple socle. Another use of the Lemma permits us to assume M has simple top. Thus, M is a nontrivial extension of $L(y \cdot \lambda)$ by $L(w \cdot \lambda)$ for some elements y, w in W_λ . If θ_α annihilates either composition factor, the Corollary reduces to Vogan's conjecture, so we may assume $ys_\alpha < y$ and $ws_\alpha < w$. Since $D\theta_\alpha M = \theta_\alpha DM$, we may assume $y > w$ and M is (y, w) -basic. It will suffice to prove that the socle series of $\theta_\alpha M$ has successive layers as in the diagram below:

$$\begin{array}{rcc}
 L(w \cdot \lambda) & & \\
 U_\alpha L(w \cdot \lambda) & \oplus & L(y \cdot \lambda) \\
 L(w \cdot \lambda) & \oplus & U_\alpha L(y \cdot \lambda) \\
 & & L(y \cdot \lambda)
 \end{array}$$

Vogan's conjecture essentially states that the socle series of $\theta_\alpha L(w \cdot \lambda)$ and $\theta_\alpha L(y \cdot \lambda)$ are depicted by the two columns of the diagram, and according to the Proposition, M is a submodule of $\theta_\alpha M$. This implies that $\text{soc}_1 \theta_\alpha M$ and $\text{soc}_2 \theta_\alpha M$ are correctly depicted. If $\text{soc}_3 \theta_\alpha M$ is not as in the diagram, then some summand of $U_\alpha L(w \cdot \lambda)$ lies in $\text{soc}_4 \theta_\alpha M$, pushing $L(w \cdot \lambda)$ up to $\text{soc}_5 \theta_\alpha M$. Hence $L(y \cdot \lambda)$ must lie in $\text{rad}_2 \theta_\alpha M$ and $\theta_\alpha M$ cannot have M as homomorphic image, contradicting the Proposition. \square

COROLLARY. *Given a module M in \mathcal{O}^λ of Loewy length 3, the module $\theta_\alpha M$ has Loewy length at most 5.*

PROOF. As in the preceding proof, we may assume M has top $L(w \cdot \lambda)$ and socle $L(y \cdot \lambda)$, with $w > ws_\alpha$, $y > ys_\alpha$ and $y < w$. If θ_α kills the middle layer, then M is basic, and $\theta_\alpha M$ is an extension of $\theta_\alpha L(y \cdot \lambda)$ by $\theta_\alpha L(w \cdot \lambda)$. In this case, if $\theta_\alpha M$ fails to have Loewy length ≤ 5 , it has Loewy length 6 with $\text{soc}_3 \theta_\alpha M = L(y \cdot \lambda)$ and $\text{soc}_4 \theta_\alpha M = L(w \cdot \lambda)$. But then $\theta_\alpha M$ cannot map onto M , contradicting the Proposition.

We may assume M is not basic, so $\text{soc}_2 M = (\bigoplus_{i=1}^r L(z_i \cdot \lambda)) \oplus K$ for some elements $z_i \in W_\lambda$ with $z_i s_\alpha < z_i$, and a module K such that $\theta_\alpha K = (0)$. No z_i can equal y or w , or $L(z_i \cdot \lambda)$ would extend itself nontrivially. We claim that $\theta_\alpha M$ has a socle series with layers as in the diagram below:

$$\begin{array}{c}
 L(w \cdot \lambda) \\
 U_\alpha L(w \cdot \lambda) \oplus \bigoplus_{i=1}^r L(z_i \cdot \lambda) \\
 L(w \cdot \lambda) \oplus \bigoplus_{i=1}^r U_\alpha L(z_i \cdot \lambda) \oplus L(y \cdot \lambda) \\
 \qquad \qquad \qquad \bigoplus_{i=1}^r L(z_i \cdot \lambda) \quad \oplus U_\alpha L(y \cdot \lambda) \\
 \qquad \qquad \qquad \qquad \qquad \qquad L(y \cdot \lambda)
 \end{array}$$

By the previous Corollary, $\theta_\alpha(\text{rad } M)$ has Loewy length 4 with socle series as in the two right-hand columns of the diagram, and $\theta_\alpha(M/\text{soc } M)$ has Loewy length 4 with the two left-hand columns as its socle series. Thus, the claim can fail only if the lower $L(w \cdot \lambda)$ nontrivially extends the higher $L(y \cdot \lambda)$; that is, $\text{soc}_4 \theta_\alpha M$ has $L(w \cdot \lambda)$ as a summand, the other copy of $L(w \cdot \lambda)$ occurring in $\text{soc}_6 \theta_\alpha M$. In particular, $\theta_\alpha M/\text{soc}^2 \theta_\alpha M$ has a submodule N which is a nontrivial extension of $L(y \cdot \lambda)$ by $L(w \cdot \lambda)$, and there is no other subquotient of $\theta_\alpha M$ consisting of an extension of these two simples. Since $\theta_\alpha^2 M \cong \theta_\alpha M \oplus \theta_\alpha M$, the only subquotients of $\theta_\alpha^2 M$ isomorphic to N occur as submodules of $\text{soc}^4 \theta_\alpha^2 M/\text{soc}^2 \theta_\alpha^2 M$.

On the other hand, by the exactness of θ_α , there is a subquotient of $\theta_\alpha^2 M$ isomorphic to $\theta_\alpha N$, and by the Proposition, $\theta_\alpha N$ has simple top and socle, with Loewy length > 2 and N as submodule and as quotient. According to the previous observation, $\theta_\alpha N$ must be a subquotient of $\text{soc}^4 \theta_\alpha^2 M/\text{soc}^2 \theta_\alpha^2 M$, a contradiction. \square

THEOREM. *Given a module M in \mathcal{O}^λ of Loewy length s , the module $\theta_\alpha M$ has Loewy length at most $s + 2$.*

PROOF. Vogan’s conjecture and the two corollaries yield the desired result for $s \leq 3$, permitting us to assume that $s \geq 4$ and that the Theorem is true inductively for modules of Loewy length $< s$. It will suffice to prove that $\theta_\alpha M/\text{soc}^{s-1} \theta_\alpha M$ has Loewy length 3. By induction, $\theta_\alpha(\text{soc}^{s-3} M)$ lies in $\text{soc}^{s-1} \theta_\alpha M$, so $\theta_\alpha M/\text{soc}^{s-1} \theta_\alpha M$ is a homomorphic image of $\theta_\alpha(M/\text{soc}^{s-3} M)$.

Given a semisimple module N in \mathcal{O}^λ , let N^+ denote the largest submodule for which no nonzero summand is annihilated by θ_α . With this notation, we may deduce from the previous corollaries that the diagram below depicts a lower bound for the

PROOF. (i) The image of f must lie in $\text{rad } M_S(ws_\alpha \cdot \lambda)$ and have Loewy length $\text{ll}M_S(w \cdot \lambda)$. The lemma yields the desired result.

(ii) Let K be the kernel of f and C the cokernel. Since θ_α is exact and $\theta_\alpha M_S(w \cdot \lambda) = \theta_\alpha M_S(ws_\alpha \cdot \lambda)$, the modules K and C are annihilated by θ_α . Hence $\theta_\alpha M_S(w \cdot \lambda) = \theta_\alpha (M_S(w \cdot \lambda) / \text{rad}^k M_S(w \cdot \lambda))$ and the proof of the lemma yields $\text{ll}M_S(ws_\alpha \cdot \lambda) \leq \text{ll}M_S(w \cdot \lambda)$.

Suppose strict inequality holds. Since $\theta_\alpha C = (0)$, we have $\theta_\alpha M_S(ws_\alpha \cdot \lambda) = \theta_\alpha \text{rad } M_S(ws_\alpha \cdot \lambda)$, and by Theorem 3.5, the Loewy length of this module is at most $\text{ll}M_S(w \cdot \lambda)$. But $\theta_\alpha M_S(w \cdot \lambda)$ has $M_S(w \cdot \lambda)$ as homomorphic image, implying that the Loewy lengths are the same, so that some summand $L(y \cdot \lambda)$ of $\text{rad}^k M(w \cdot \lambda)$ lies in $\text{soc } \theta_\alpha M(w \cdot \lambda)$. By assumption, $L(y \cdot \lambda)$ lies in K and is annihilated by θ_α . But then $L(y \cdot \lambda)$ cannot occur in the socle of $\theta_\alpha N$ for any module N , yielding a contradiction. \square

4.2. We will first obtain lower bounds for certain Loewy lengths, analogous to the lower bounds of 3.1. Let ${}^S Z_\lambda$ denote the set $\{\check{w} | w \in {}^S X_\lambda\}$, where \check{w} is defined in the introduction. To each $w \in {}^S W_\lambda$ we can associate the element \bar{w} , following the notation of [12, 4.5]. This is the longest element of the Weyl group $W_{\tau_\lambda(w)}$ generated by the reflections about simple roots in $\tau_\lambda(w)$, where $\tau_\lambda(w) = \{\alpha \in B_\lambda | ws_\alpha < w\}$. Let $t = \max\{l_\lambda(\bar{w}) | w \in {}^S W_\lambda\}$. In case $S = \emptyset$, we have $\tau_\lambda(w_\lambda) = B_\lambda$ and $\bar{w}_\lambda = w_\lambda$, so $t = l(w_\lambda)$.

LEMMA. *Let w be an element of ${}^S W_\lambda$. The module $M_S(w\bar{w} \cdot \lambda)$ has Loewy length at least $l(\bar{w}) + 1$.*

PROOF. For any $\beta \in S$, iterated use of [13, 2.16] yields $(M(w\bar{w} \cdot \lambda): L(w \cdot \lambda)) = (M(w \cdot \lambda): L(w \cdot \lambda))$ and $(M(s_\beta w\bar{w} \cdot \lambda): L(w \cdot \lambda)) = (M(s_\beta w \cdot \lambda): L(w \cdot \lambda))$. Since $s_\beta w > w$, the latter multiplicity is 0, while the former multiplicity is 1. It follows that $(M_S(w\bar{w} \cdot \lambda): L(w \cdot \lambda)) = 1$, and that the one occurrence of $L(w \cdot \lambda)$ in $M_S(w\bar{w} \cdot \lambda)$ occurs in the image of the standard homomorphism from $M_S(w \cdot \lambda)$ to $M_S(w\bar{w} \cdot \lambda)$. The argument of 3.1(i) implies $(\text{rad}^{l(\bar{w})} M_S(w\bar{w} \cdot \lambda): L(w \cdot \lambda)) = 1$, proving the Lemma. \square

PROPOSITION. (i) *For any $y, z \in {}^S Z_\lambda$, the Loewy lengths of $M_S(y \cdot \lambda)$ and $M_S(z \cdot \lambda)$ are equal, and this common value is at least $t + 1$.*

(ii) *For any $w \in {}^S X_\lambda$, the Loewy length of $P_S(w \cdot \lambda)$ is at least $2\text{ll}M_S(\check{w} \cdot \lambda) - 1$. Hence, it is at least $2t + 1$.*

PROOF. (i) Given $\check{w} \in {}^S Z_\lambda$ and $\alpha \in B_\lambda$ satisfying $\check{w}s_\alpha < \check{w}$, the element $\check{w}s_\alpha$ also lies in ${}^S Z_\lambda$ [12, 4.6]. By the defining property of \check{w} , we have $\text{soc } M_S(\check{w} \cdot \lambda) = L(w \cdot \lambda)$ and $(M_S(\check{w}s_\alpha \cdot \lambda): L(w \cdot \lambda)) = 0$. Therefore, $\text{soc } M_S(\check{w} \cdot \lambda)$ lies in the kernel of the standard homomorphism of $M_S(\check{w} \cdot \lambda)$ to $M_S(\check{w}s_\alpha \cdot \lambda)$, and by Proposition 4.1, the two generalized Verma modules have the same Loewy length. Proceeding by induction on $l(\check{w})$, we obtain $\text{ll}M_S(\check{w} \cdot \lambda) = \text{ll}M_S(\lambda)$. Proposition 4.1 implies that $\text{ll}M_S(\lambda) \geq \text{ll}M_S(x \cdot \lambda)$ for any $x \in {}^S W_\lambda$. By the Lemma, $\text{ll}M_S(x \cdot \lambda) \geq t + 1$ for some x , and this yields the result.

(ii) By assumption, $P_S(w \cdot \lambda)$ is self-dual, and the properties of \check{w} are exactly what is required to repeat the proof of Proposition 3.1(ii), mutatis mutandis. \square

4.3. Under an additional hypothesis, the lower bounds of Proposition 4.2 must be the actual values.

THEOREM. *Assume that ${}^S X_\lambda$ contains an element x such that $x\bar{x} = \check{x}$.*

- (i) *The Loewy length of $M_S(w \cdot \lambda)$ is $t + 1$ for all $w \in {}^S Z_\lambda$.*
- (ii) *The Loewy length of $M_S(w \cdot \lambda)$ is less than $t + 1$ for all $w \notin {}^S Z_\lambda$.*
- (iii) *The Loewy length of $P_S(w \cdot \lambda)$ is $2t + 1$ for all $w \in {}^S X_\lambda$.*

PROOF. (i) Let \bar{x} have a minimal factorization $s_{\alpha_1} \cdots s_{\alpha_r}$ with $\alpha_i \in B_\lambda$ and $r = l(\bar{x})$, and let $\theta = \theta_{\alpha_1} \circ \cdots \circ \theta_{\alpha_r}$. The only elements y satisfying $\check{x} \leq y \leq x$ are those of the form xz for some $z \leq \bar{x}$, and for such y we have $(M_S(y \cdot \lambda) : L(x \cdot \lambda)) = 1$ by [13, 2.16]. Thus, by BGG reciprocity, a Verma flag for $P_S(x \cdot \lambda)$ has as its constituents $\{M_S(xz \cdot \lambda) : z \leq \bar{x}\}$, each occurring once.

By [12, 2.3], $P_S(x \cdot \lambda)$ occurs as a summand of $\theta P_S(\check{x} \cdot \lambda)$. But $P_S(\check{x} \cdot \lambda)$ has a Verma flag with top constituent $M_S(\check{x} \cdot \lambda)$ and all other constituents of the form $M_S(w \cdot \lambda)$ for elements $w < \check{x}$. Given such a w and any $\alpha \in B_\lambda$, the element ws_α cannot be $\geq \check{x}$, and so neither $M_S(w \cdot \lambda)$ nor $M_S(ws_\alpha \cdot \lambda)$ can occur in a Verma flag for $P_S(x \cdot \lambda)$. Let P be the penultimate submodule in a Verma flag for $P_S(\check{x} \cdot \lambda)$, with $P_S(\check{x} \cdot \lambda)/P \cong M_S(\check{x} \cdot \lambda)$. It follows that the surjection of $\theta P_S(\check{x} \cdot \lambda)$ onto $P_S(x \cdot \lambda)$ factors through $\theta P_S(\check{x} \cdot \lambda)/\theta P$, which equals $\theta M_S(\check{x} \cdot \lambda)$.

For each i with $0 \leq i \leq r$, let $x_i = \check{x}s_{\alpha_r} \cdots s_{\alpha_i}$ and let Q_i be the homomorphic image of $P_S(x_i \cdot \lambda)$ obtained as follows: Choose a Verma flag for $P_S(x_i \cdot \lambda)$ such that all constituents of the form $M_S(w \cdot \lambda)$ with $\check{x} \leq w \leq x_i$ occur above all others, and let Q_i be the image obtained modulo the other constituents. Let $m = llM_S(\check{x} \cdot \lambda)$. We claim that $llQ_i \leq m + i$. The case $i = 0$ is trivial, since $Q_0 = M_S(\check{x} \cdot \lambda)$. By the observation of the preceding paragraph, and the fact that $P_S(x_{i+1} \cdot \lambda)$ is a summand of $\theta_{\alpha_i} P_S(x_i \cdot \lambda)$ [12, 2.3], we find that Q_{i+1} is a homomorphic image of $\theta_{\alpha_i} Q_i$. But Q_i has simple top $L(x_i \cdot \lambda)$, which is annihilated by θ_{α_i} , and so $llQ_{i+1} \leq ll\theta_{\alpha_i}(\text{rad } Q_i)$. The claim follows by induction and Theorem 3.5.

The case $i = r$ of the claim yields $llQ_r \leq m + r$. But $Q_r = P_S(x \cdot \lambda)$, and $llP_S(x \cdot \lambda) \geq 2m - 1$ by Proposition 4.2(ii). Therefore $m - 1 \leq r$. On the other hand, the definition of t yields $r \leq t$, and $t \leq m - 1$ by Proposition 4.2(i). We may conclude that $r = t = m - 1$. In other words, the function $l(\bar{w})$ achieves its maximum on x , and $t + 1$ is the Loewy length of $M_S(\check{x} \cdot \lambda)$. We have proved (i), as well as (iii) in the special case of $w = x$.

(iii) Let w be an element of ${}^S X_\lambda$ and α a root of B_λ satisfying $ws_\alpha > w$. By self-duality, $P_S(w \cdot \lambda)$ has $L(w \cdot \lambda)$ as top and socle, and θ_α annihilates $L(w \cdot \lambda)$. Therefore Theorem 3.5 implies

$$ll\theta_\alpha P_S(w \cdot \lambda) = ll\theta_\alpha(\text{rad } P_S(w \cdot \lambda) / \text{soc } P_S(w \cdot \lambda)) \leq llP_S(w \cdot \lambda).$$

In particular, if $P_S(w \cdot \lambda)$ has Loewy length $2t + 1$, this is a bound for the Loewy length of each summand of $\theta_\alpha P_S(w \cdot \lambda)$. But the summands are themselves self-dual

projective modules, so by Proposition 4.2 they have Loewy length exactly $2t + 1$. The proof of [12, 9.2] shows for $y \in {}^S X_\lambda$ that $P_S(y \cdot \lambda)$ is a summand of $\theta_{\beta_1} \cdots \theta_{\beta_k} P_S(x \cdot \lambda)$ for a suitable sequence of roots β_1, \dots, β_k in B_λ , proving (iii).

(ii) Let y be an element in ${}^S W_\lambda - {}^S Z_\lambda$ for which $\text{ll} M_S(y \cdot \lambda) = t + 1$ and let L be a simple summand of $\text{rad}^t M_S(y \cdot \lambda)$ isomorphic to $L(z \cdot \lambda)$. Let K be a submodule of $M_S(y \cdot \lambda)$ maximal with respect to the condition $K \cap L = (0)$, and let $M = M_S(y \cdot \lambda)/K$. Then M embeds in the injective envelope $P_S(z \cdot \lambda)$ of L , and $\text{ll} M = t + 1$.

Let $0 \subset M_1 \subset \cdots \subset M_t = P_S(z \cdot \lambda)$ be a Verma flag, chosen so that for some k , all vectors of weight $> y \cdot \lambda$ lie in M_k , but for no $i \leq k$ is $M_i/M_{i-1} \cong M_S(y \cdot \lambda)$. By duality, $P_S(z \cdot \lambda)$ maps onto DM , and by the choice of the flag, this map factors through $P_S(z \cdot \lambda)/M_k$. Hence, $\text{rad}^t(P_S(z \cdot \lambda)/M_k)$ contains a vector v of weight $y \cdot \lambda$, which must be a highest weight vector. The submodule generated by v is isomorphic to $M_S(y \cdot \lambda)$, forcing $P_S(z \cdot \lambda)/M_k$ to have Loewy length at least $2t + 1$. On the other hand, $P_S(z \cdot \lambda)$ has Loewy length $2t + 1$ and simple socle, the socle lying in the submodule $M_S(\check{z} \cdot \lambda)$. But $M_S(\check{z} \cdot \lambda)$ lies in M_k , so that $P_S(z \cdot \lambda)/M_k$ has Loewy length at most $2t$, a contradiction. \square

REMARKS. (1) The self-duality conjecture is verified explicitly for a number of choices of S and R in [12], and in *all* of these cases, there is an element x of ${}^S X_\lambda$ satisfying $x\bar{x} = \check{x}$, so that the conclusions of the Theorem apply. Notice also that if $S = \emptyset$, then ${}^S X_\lambda = \{w_\lambda\}$ and $\check{w}_\lambda = e$, with $\bar{w}_\lambda = w_\lambda$. Thus the Theorem applies, and the conclusions coincide with those of 3.5.

(2) The result quoted from [12, 9.2] in the proof of (iii) above depends on Jantzen's conjecture, but this can often be avoided. Given w as in the proof, it is elementary that $P_S(w s_\alpha \cdot \lambda)$ is a summand of $\theta_\alpha P_S(w \cdot \lambda)$ [12, 2.3]. Thus, if we actually know that $\check{x} = x\bar{x}$ for *all* x minimal in ${}^S X_\lambda$ under the \ll ordering, then (iii) follows as in the proof, without using [12, 9.2]. This is a common situation, and in fact is satisfied by every example in [12] but one. The exception occurs with R of type C_3 and S consisting of the long simple root, and for this case one can verify (iii) directly.

5. Loewy series of $P(w_\lambda \cdot \lambda)$.

5.1. Let λ be a dominant weight. In this subsection, we refine the results of 3.2 under the assumption that $\text{ll} P(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$. By Corollary 3.3 and Theorem 2.5, this is a consequence of Vogan's conjecture. We need the results not only to study Loewy series, but also in order to obtain a converse to Theorem 3.5. For this reason, we do not assume Vogan's conjecture outright, but only its consequences on Loewy length.

PROPOSITION 1. *Let λ be a dominant weight and let μ be a weight in $W_\lambda \cdot \lambda$. Assume that $\text{ll} P(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$.*

(i) $(\text{soc}_{2h(\lambda)+1-h(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = (P(w_\lambda \cdot \lambda)/\text{soc}^{2h(\lambda)-h(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = 1$.

(ii) $(\text{rad}_{2h(\lambda)-h(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = (\text{rad}^{2h(\lambda)-h(\mu)} P(w_\lambda \cdot \lambda): L(\mu)) = 1$.

REMARK. Since $P(w_\lambda \cdot \lambda)$ is self-dual, the two statements are equivalent to each other. They complement Proposition 3.2. Under the assumption that $llP(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$, as noted in 3.1, we obtain $llM(\mu) = h(\mu) + 1$. Thus Proposition 3.2 states that the unique lowest appearance of $L(\mu)$ in the socle series of $P(w_\lambda \cdot \lambda)$ is in layer $h(\mu) + 1$, and the unique highest appearance in the radical series is in layer $h(\mu)$. The present proposition goes further in indicating that there is a unique lowest appearance of $L(\mu)$ in the radical series at the same level as in the socle series, and a unique highest appearance in the socle series at the same level as in the radical series.

PROOF. By assumption, $M(\lambda)$ has Loewy length $h(\lambda) + 1$ with a copy of $L(\mu)$ in $\text{soc}_{h(\mu)+1} M(\lambda)$ corresponding to the top of the submodule $M(\mu)$, and by the argument of 3.1, this $L(\mu)$ also lies in $\text{rad}_{h(\lambda)-h(\mu)} M(\lambda)$. Using the inclusion of $M(\lambda)$ in $P(w_\lambda \cdot \lambda)$ and self-duality, we obtain a copy of $L(\mu)$ in $\text{soc}_{2h(\lambda)+1-h(\mu)} P(w_\lambda \cdot \lambda)$, and there can be no other $L(\mu)$ in layer $2h(\lambda) + 1 - h(\mu)$ or above by Proposition 3.2 (and the Loewy length assumption). \square

The above proposition and Proposition 3.2 pin down, for the two Loewy series of $P(w_\lambda \cdot \lambda)$, the location of the simple modules corresponding to the tops of Verma modules in a Verma flag (see Remark 3.2). We next locate the simples corresponding to the socles of Verma flag constituents.

PROPOSITION 2. *Let λ be a dominant weight with $llP(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$, and for $0 \leq s \leq h(\lambda)$, let $n_s = \#\{\mu \in W_\lambda \cdot \lambda \mid h(\mu) = s\}$.*

(i) *($\text{rad}_{2s} P(w_\lambda \cdot \lambda): L(w_\lambda \cdot \lambda) = n_s = (\text{soc}_{2h(\lambda)+1-2s} P(w_\lambda \cdot \lambda): L(w_\lambda \cdot \lambda)$), and this accounts for all appearances of $L(w_\lambda \cdot \lambda)$ in $P(w_\lambda \cdot \lambda)$.*

(ii) *Let $0 = M_0 \subset M_1 \subset \dots \subset M_k = P(w_\lambda \cdot \lambda)$ be a well-ordered Verma flag, with $M_{i+1}/M_i \cong M(\eta_i)$, and let $r_i = 2h(\lambda) - 2h(\eta_i) + 1$. Then $(\text{soc}_{r_i} M_{i+1}: L(w_\lambda \cdot \lambda)) = (\text{soc}_{r_i} M_i: L(w_\lambda \cdot \lambda)) + 1$.*

PROOF. Let $\mu \in W_\lambda \cdot \lambda$. By Proposition 3.2, the lowest numbered layer of the radical series of $P(w_\lambda \cdot \lambda)$ containing a copy of $L(\mu)$ is layer $h(\mu)$, and the copy of $L(\mu)$ corresponds to the top of the Verma flag constituent $M(\mu)$. Therefore, since $llM(\mu) = h(\mu) + 1$, a copy of $L(w_\lambda \cdot \lambda)$ corresponding to the socle of $M(\mu)$ occurs in $\text{rad}^{2h(\mu)} P(w_\lambda \cdot \lambda)$. By self-duality, a copy of $L(w_\lambda \cdot \lambda)$ occurs in a layer numbered $\geq 2h(\mu) + 1$ of the socle series, and in the radical series, this copy of $L(w_\lambda \cdot \lambda)$ lies in a layer numbered $\leq 2h(\lambda) - 2h(\mu)$.

For each s between 0 and $h(\lambda)$, this produces at least n_s copies of $L(w_\lambda \cdot \lambda)$ in $\text{rad}^{2s} P(w_\lambda \cdot \lambda)$ as well as in $P(w_\lambda \cdot \lambda)/\text{rad}^{2h(\lambda)-2s+1} P(w_\lambda \cdot \lambda)$. By the symmetry of the Bruhat order under an upside down flip [18, p. 181], $n_s = n_{h(\lambda)-s}$, so there are at least n_s copies of $L(w_\lambda \cdot \lambda)$ in $P(w_\lambda \cdot \lambda)/\text{rad}^{2s+1} P(w_\lambda \cdot \lambda)$. This yields the inequalities

$$\sum_{i \geq s} n_i \leq (\text{rad}^{2s} P(w_\lambda \cdot \lambda): L(w_\lambda \cdot \lambda))$$

and

$$\sum_{i < s} n_i \leq (P(w_\lambda \cdot \lambda)/\text{rad}^{2s} P(w_\lambda \cdot \lambda): L(w_\lambda \cdot \lambda)).$$

But $\sum_{i=0}^{h(\lambda)} n_i = \#W_\lambda \cdot \lambda = (P(w_\lambda \cdot \lambda) : L(w_\lambda \cdot \lambda))$, forcing the inequalities above to be equalities. Part (i) follows for the radical series by induction on $h(\lambda) - s$; for the socle series, the conclusion follows by self-duality and the formula $n_s = n_{h(\lambda)-s}$.

Part (ii) is a consequence of (i) via induction on i : In extending M_i by $M(\eta_i)$ to form M_{i+1} , the additional copy of $L(w_\lambda \cdot \lambda)$ to be added cannot lie in a lower layer of the socle series than r_i , because by (i) and the hypothesis on the flag, for $m \leq r_i$ we have by induction $(\text{soc}_m M_i : L(w_\lambda \cdot \lambda)) = (\text{soc}_m P(w_\lambda \cdot \lambda) : L(w_\lambda \cdot \lambda))$. It cannot lie in a higher layer, for if it does, the top $L(\eta_i)$ is forced into a layer numbered $\geq 2h(\lambda) - h(\eta_i) + 2$, contradicting (i) of Proposition 1. \square

COROLLARY. *Let λ be a dominant, regular weight such that $llP(w_\lambda \cdot \lambda) = 2l(w_\lambda) + 1$. Then \mathcal{O}^λ satisfies Vogan's conjecture.*

PROOF. Let $w \in W_\lambda$ and $\alpha \in B_\lambda$ satisfy $ws_\alpha < w$. We can choose a well-ordered Verma flag $0 \subset M_1 \subset \cdots \subset M_k = P(w_\lambda \cdot \lambda)$ so that for some i , the quotients M_i/M_{i-1} and M_{i+1}/M_i are $M(ws_\alpha \cdot \lambda)$ and $M(w \cdot \lambda)$ respectively. By Propositions 1 and 2, and Remark 3.2, M_{i+1}/M_{i-1} is a subquotient of

$$\text{soc}^{2l(w_\lambda)-l(w)+1} P(w_\lambda \cdot \lambda) / \text{soc}^{2l(w_\lambda)-2l(w)-2} P(w_\lambda \cdot \lambda),$$

yielding $llM_{i+1}/M_{i-1} \leq l(w) + 3$. The module M_{i+1}/M_{i-1} is an extension of $M(ws_\alpha \cdot \lambda)$ by $M(w \cdot \lambda)$ with simple top. There is a unique homomorphic image of $P(w \cdot \lambda)$ of this form, that obtained modulo the submodule generated by all weight vectors of weight not $\leq ws_\alpha \lambda$. But $\theta_\alpha M(w \cdot \lambda)$ is another module of this form, so $\theta_\alpha M(w \cdot \lambda) \cong M_{i+1}/M_{i-1}$.

The assumption on Loewy length yields $llM(w \cdot \lambda) = l(w) + 1$ and $llM(ws_\alpha \cdot \lambda) = l(w) + 2$. Hence, $ll\theta_\alpha M(w \cdot \lambda)$ is exactly $l(w) + 3$, and the submodule $M(ws_\alpha \cdot \lambda)$ produces a copy of $L(w \cdot \lambda)$ in $\text{soc}_{l(w)+1} \theta_\alpha M(w \cdot \lambda)$. But $(\theta_\alpha M(w \cdot \lambda) : L(w \cdot \lambda)) = 2$, so both copies of $L(w \cdot \lambda)$ lie in $\theta_\alpha M(w \cdot \lambda) / \text{soc}^{l(w)} \theta_\alpha M(w \cdot \lambda)$, which has Loewy length 3. The module $\theta_\alpha L(w \cdot \lambda)$ is a homomorphic image of $\theta_\alpha M(w \cdot \lambda)$, with simple socle $L(w \cdot \lambda)$, implying that $\theta_\alpha L(w \cdot \lambda)$ is a homomorphic image of

$$\theta_\alpha M(w \cdot \lambda) / \text{soc}^{l(w)} \theta_\alpha M(w \cdot \lambda).$$

We may conclude that $\theta_\alpha L(w \cdot \lambda)$ has Loewy length 3 and Vogan's conjecture is satisfied. \square

5.2. Given a dominant weight λ such that $llP(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$, the propositions of 5.1 imply that the socle series of $P(w_\lambda \cdot \lambda)$ can be built in stages, with respect to a well-ordered flag: For each $\mu \in W_\lambda \cdot \lambda$, the module $M(\mu)$ is laid down with its socle in layer $2h(\lambda) - 2h(\mu) + 1$ and its top in layer $2h(\lambda) - h(\mu) + 1$. Analogously, in the radical series of $P(w_\lambda \cdot \lambda)$, the top of $M(\mu)$ is placed in layer $h(\mu)$ and the socle in layer $2h(\mu)$. The other layers of $M(\mu)$ are placed in the intermediate layers in a way which we cannot yet specify. But since the number of layers available is exactly $llM(\mu)$, the arrangement must correspond to some series of $M(\mu)$ between the socle and radical series (2.2). In particular, if the two Loewy series coincide, there is no choice, yielding the following result.

PROPOSITION. *Let λ be a dominant weight such that $lP(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$ and, for every $\mu \in W_\lambda \cdot \lambda$, the socle and radical series of $M(\mu)$ coincide. Then the two Loewy series of $P(w_\lambda \cdot \lambda)$ coincide, and their layers are as follows:*

$$\text{soc}_r P(w_\lambda \cdot \lambda) = \bigoplus_{\mu \in W_\lambda \cdot \lambda} \text{soc}_{r-2h(\lambda)+2h(\mu)} M(\mu). \quad \square$$

It is unknown whether the two Loewy series of a Verma module must coincide. We will be able to extend the conclusion of the Proposition to regular λ in 5.4, under the assumption of Jantzen’s conjecture. For now, let us note one case to which the Proposition applies.

LEMMA. *Let λ be a dominant weight such that $(M(\lambda): L(\mu)) = 1$ for all $\mu \in W_\lambda \cdot \lambda$. Then the two Loewy series of each $M(\mu)$ coincide, and $lP(w_\lambda \cdot \lambda) = 2h(\lambda) + 1$.*

PROOF. Since the only composition factors of $M(\mu)$ are those corresponding to tops of Verma submodules, the structure of $M(\mu)$ is transparent, and the Loewy series clearly coincide. It follows for μ, η in $W_\lambda \cdot \lambda$ that $\text{Ext}^1(L(\eta), L(\mu)) \neq 0$ if and only if $\mu \uparrow \eta$ or $\eta \uparrow \mu$ and $|h(\mu) - h(\eta)| = 1$, in which case there is exactly one nontrivial extension module up to isomorphism. Using this observation one can argue inductively, ascending through the steps of a well-ordered Verma flag, that the Loewy length of each step is not too large, obtaining $lP(w_\lambda \cdot \lambda) = 2h(w_\lambda) + 1$. Details can be found in [11, 8.1], or easily filled in. \square

REMARK. For λ as in the Lemma, the Proposition and the self-duality of $P(w_\lambda \cdot \lambda)$ places a restrictive condition on $W_\lambda \cdot \lambda$. For each $\mu \in W_\lambda \cdot \lambda$ and i with $0 \leq i \leq h(\lambda) - h(\mu)$, the following equality holds:

$$\begin{aligned} \# \{ \eta \in W_\lambda \cdot \lambda \mid \mu \uparrow \eta \text{ and } h(\eta) = h(\mu) + i \} \\ = \# \{ \eta \in W_\lambda \cdot \lambda \mid \mu \uparrow \eta \text{ and } h(\eta) = h(\lambda) - i \}. \end{aligned}$$

For W_λ indecomposable, it is easily checked that these equalities hold only if W_λ has rank 2 or $W_\lambda \cdot \lambda$ is totally ordered. (For such λ , it is well known that $(M(\lambda): L(\mu)) = 1$; see [13, 3.17 and 5.20].) In particular, for λ regular with W_λ of rank > 2 , there must be some $w \in W_\lambda$ with $(M(\lambda): L(w \cdot \lambda)) > 1$, an old result of Jantzen and Deodhar-Lepowsky [13, 4.4; 8]. We see that this fact may be interpreted as a natural consequence of the self-duality of $P(w_\lambda \cdot \lambda)$ (see [11, §8] for more details).

5.3. The key to obtaining the conclusion of Proposition 5.2 for regular λ is the following result.

THEOREM. *Let λ be a dominant, regular weight such that the Verma modules in \mathcal{O}^λ satisfy Jantzen’s conjecture. For any $w \in W_\lambda$, the Jantzen and socle filtrations on $M(w \cdot \lambda)$ coincide.*

PROOF. (i) It will be more convenient to index weights via the antidominant weight $w_\lambda \cdot \lambda$, which we will denote by ν . Jantzen’s conjecture states that, for each $w \in W_\lambda$ and $\alpha \in B_\lambda$ with $ws_\alpha > w$, the Jantzen filtrations on $M(w \cdot \nu)$ and $M(ws_\alpha \cdot \nu)$ are compatible in the sense that $M(ws_\alpha \cdot \nu)^{j+1} \cap M(w \cdot \nu) = M(w \cdot \nu)^j$ for all j . One consequence is that the layers $M(w \cdot \nu)_j$ are all semisimple [10, 4.8], and an old result of Jantzen is that $M(w \cdot \nu)^{l(w)} = \text{soc } M(w \cdot \nu)$ [13, 5.3]. Since $lM(w \cdot \nu) \geq l(w) + 1$ by Proposition 3.1, we immediately obtain equality (without resorting to

Vogan's conjecture, which anyway is a consequence of Jantzen's conjecture [10, 4.8]). Thus, $M(w \cdot \nu)^j \subset \text{soc}^{l(w)+1-j} M(w \cdot \nu)$, and we must prove the two modules are equal for all $w \in W_\lambda$ and $j \leq l(w)$. We will proceed by induction on $l(w)$, the case of length 0 being trivial. Fix $w \in W_\lambda$ so that the Theorem is true for $M(w \cdot \nu)$ and let α be a root of B_λ satisfying $ws_\alpha > w$. We will prove that $M(ws_\alpha \cdot \nu)^j = \text{soc}^{l(ws_\alpha)+1-j} M(ws_\alpha \cdot \nu)$ by induction on $l(ws_\alpha) + 1 - j$, the case of $j = l(ws_\alpha)$ following by Jantzen's aforementioned result.

(ii) Following the notation in [10, §4], let $X = M(ws_\alpha \cdot \nu)$ and $Z = M(w \cdot \nu)$, with Y equal to their extension $\theta_\alpha X$ and $\pi: Y \rightarrow Z$ the canonical surjection. Our goal is to prove that $X^j = \text{soc}^{l(ws_\alpha)+1-j} X$, and since there is an inclusion of the two modules, it will suffice to prove for each $y \in W_\lambda$ that $(X^j: L(y \cdot \nu)) = (\text{soc}^{l(ws_\alpha)+1-j} X: L(y \cdot \nu))$. Let us assume first that y satisfies $ys_\alpha > y$, so that $(X^j: L(y \cdot \nu)) = (Z^{j-1}: L(y \cdot \nu))$ by [10, 4.3(v)]. Jantzen's conjecture states that $X^j \cap Z = Z^{j-1}$, and by the inductive hypothesis on w , we have $Z^{j-1} = \text{soc}^{l(w)-j+2} Z$. But $Z \cap \text{soc}^{l(w)-j+2} X = \text{soc}^{l(w)-j+2} Z$, and $(X: L(y \cdot \nu)) = (Z: L(y \cdot \nu))$, so we may conclude for all j that

$$(X^j: L(y \cdot \nu)) = (Z \cap \text{soc}^{l(w)-j+2} X: L(y \cdot \nu)) = (\text{soc}^{l(ws_\alpha)-j+1} X: L(y \cdot \nu)).$$

(iii) We may assume for the remainder of the proof that y is an element of W_λ satisfying $ys_\alpha < y$. Let us recall some results of [10, 4.3–4.6]. The module Y inherits a filtration satisfying $Y^j = \theta_\alpha(Z^j) = \theta_\alpha(X^{j+1})$, with

$$X^{j+1} \subset Y^j \cap X \subset X^j \quad \text{and} \quad Z^{j+1} \subset \pi(Y^j) \subset Z^j.$$

Moreover, the composition factors of $Y^j \cap X$ which the first inclusion does not specify are the summands of X^j/X^{j+1} annihilated by θ_α , and the unspecified factors of $\pi(Y^j)$ are the summands of Z^j/Z^{j+1} not annihilated by θ_α . Hence, $(Y^j \cap X: L(y \cdot \nu)) = (X^j: L(y \cdot \nu))$, so that it suffices to prove that $(Y^j \cap X: L(y \cdot \nu)) = (\text{soc}^{l(ws_\alpha)+1-j} X: L(y \cdot \nu))$. One inequality follows from the inclusion $Y^j \cap X \subset X^j \subset \text{soc}^{l(ws_\alpha)+1-j} X$.

By induction on $l(ws_\alpha) + 1 - j$, we have $\text{soc}^{l(ws_\alpha)-j} X = X^{j+1}$, so $\text{soc}^{l(ws_\alpha)-j} X$ lies in $Y^j \cap X$. Let N be the minimal submodule of $\text{soc}^{l(ws_\alpha)+1-j} X$ containing $\text{soc}^{l(ws_\alpha)-j} X$ such that $(\text{soc}^{l(ws_\alpha)+1-j} X/N: L(y \cdot \nu)) = 0$. To prove the other inequality, it will suffice to show that $N \subset Y^j$. Suppose not, and choose k with $N \subset Y^{k-1}$ but $N \not\subset Y^k$. We have $k \leq j$, so Y^k contains $\text{soc}^{l(ws_\alpha)-j} X$. Therefore, under the natural map of Y^{k-1} onto Y^{k-1}/Y^k , the image of N is a nonzero semisimple module, consisting of a direct sum of copies of $L(y \cdot \nu)$. But $Y^{k-1} = \theta_\alpha(X^k)$ and $Y^k = \theta_\alpha(X^{k+1})$, so $Y^{k-1}/Y^k = \theta_\alpha(X^k/X^{k+1})$. This implies that Y^{k-1}/Y^k has a filtration with quotients of the form $\theta_\alpha L(z \cdot \nu)$ for elements $z \in W_\lambda$ with $zs_\alpha > z$. In particular, the socle of Y^{k-1}/Y^k is a direct sum of $L(z \cdot \nu)$'s with $zs_\alpha > z$. This contradicts the existence of $L(y \cdot \nu)$ in the socle, and completes the proof. \square

As noted in 5.2, it is plausible that the radical and socle series of a Verma module coincide, in which case the preceding theorem is trivial. As a consequence of the theorem, a partial result can be obtained on the coincidence of the two series.

COROLLARY. *Let λ be a dominant, regular weight such that the Verma modules in \mathcal{O}^λ satisfy Jantzen's conjecture. Then $\text{rad}^2 M(w \cdot \lambda) = \text{soc}^{l(w_\lambda w)^{-1}} M(w \cdot \lambda)$ for any $w \in W_\lambda$.*

PROOF. Let $\nu = w_\lambda \cdot \lambda$. We must prove $\text{rad}^2 M(w \cdot \nu) = \text{soc}^{l(w)^{-1}} M(w \cdot \nu)$. Since $M(w \cdot \nu)$ has Loewy length $l(w) + 1$ and simple top,

$$\text{rad } M(w \cdot \nu) = \text{soc}^{l(w)} M(w \cdot \nu),$$

and it suffices to show for any y in W_λ with $y < w$ that

$$(\text{rad}_1 M(w \cdot \nu) : L(y \cdot \nu)) = (\text{soc}_{l(w)} M(w \cdot \nu) : L(y \cdot \nu)).$$

Let r and s denote the respective integers; the inequality $r \geq s$ holds automatically.

The Theorem yields $s = (M(w \cdot \nu)_1 : L(y \cdot \nu))$, and by [10, 4.9], this is the coefficient of $q^{(l(w)-l(y)-1)/2}$ in the Kazhdan-Lusztig polynomial $P_{w_\lambda w, w_\lambda y}(q)$. This coefficient is denoted $\mu(w_\lambda w, w_\lambda y)$ (see [15] for information on Kazhdan-Lusztig polynomials). On the other hand, $r = \dim \text{Ext}^1(L(w \cdot \nu), L(y \cdot \nu))$, since any non-trivial extension of $L(y \cdot \nu)$ by $L(w \cdot \nu)$ is a highest weight module, making it a homomorphic image of $M(w \cdot \nu)$.

An upper bound is easily obtained for r . By duality,

$$r = \dim \text{Ext}^1(L(y \cdot \nu), L(w \cdot \nu)),$$

and applying $\text{Hom}(\ , L(w \cdot \nu))$ to the short exact sequence $0 \rightarrow \text{rad } M(y \cdot \nu) \rightarrow M(y \cdot \nu) \rightarrow L(y \cdot \nu) \rightarrow 0$ yields $r \leq \dim \text{Ext}^1(M(y \cdot \nu), L(w \cdot \nu))$ via the long exact sequence of Ext 's. By [15, 1.6], the latter dimension is $\mu(y, w)$, yielding

$$s = \mu(w_\lambda w, w_\lambda y) \leq r \leq \mu(y, w).$$

The symmetric argument yields $\mu(y, w) \leq \mu(w_\lambda w, w_\lambda y)$ (or, see [15, 3.2]), so all the integers are equal. \square

REMARK. The Corollary is essentially in [10, 3.16]. Both the Theorem and Corollary are also proved in [2]. They appeared in [11], but for the Theorem only the idea of proof was indicated. The above proof makes this idea precise.

5.4. We can now prove the desired result on Loewy series for $P(w_\lambda \cdot \lambda)$.

THEOREM. *Let λ be a dominant, regular weight such that the Verma modules in \mathcal{O}^λ satisfy Jantzen's conjecture. Then the radical and socle series of $P(w_\lambda \cdot \lambda)$ coincide, and their layers are as follows:*

$$\text{soc}_r P(w_\lambda \cdot \lambda) = \bigoplus_{w \in W_\lambda} \text{soc}_{r-2l(w)} M(w \cdot \lambda).$$

PROOF. The formula corresponds to the statement that the layers of the socle series for $P(w_\lambda \cdot \lambda)$ are obtained by laying down, for each $w \in W_\lambda$, the $l(w_\lambda w) + 1$ layers of the socle series of $M(w \cdot \lambda)$, with an upward shift of $2l(w)$. By 5.1, this is a lower bound, in the sense that if it is incorrect, the correct socle series is obtained by shifting some simple modules upward.

Let us switch notation, working with the antidominant weight $\nu = w_\lambda \cdot \lambda$. The formula can be restated as follows:

$$\text{soc}_r P(\nu) = \bigoplus_{w \in W_\lambda} \text{soc}_{r-2l(w_\lambda w)} M(w \cdot \nu).$$

For each $y \in W_\lambda$, the location of composition factors isomorphic to $L(y \cdot \nu)$ in $P(\nu)$ will be coded by the following polynomial in $\mathbf{Z}[q^{1/2}, q^{-1/2}]$:

$$T_y(q) = \sum_j (\text{soc}_j P(\nu) : L(y \cdot \nu)) q^{1/2(j-1-l(y))}.$$

By Proposition 1 of 5.1 and Proposition 3.2, the extreme occurrences of $L(y \cdot \nu)$ in the socle series of $P(\nu)$ are in layers $l(y) + 1$ and $2l(w_\lambda) - l(y) + 1$, with multiplicity 1. Thus $T_y(q)$ is in $\mathbf{Z}[q^{1/2}]$ with constant term 1 and degree $l(w_\lambda) - l(y)$. Let $U_y(q)$ be the analogous polynomial for the radical series:

$$U_y(q) = \sum_j (\text{rad}_j P(\nu) : L(y \cdot \nu)) q^{1/2(2l(w_\lambda)-l(y)-j)}.$$

It too lies in $\mathbf{Z}[q^{1/2}]$ with degree $l(w_\lambda) - l(y)$, by 3.2 and 5.1.

Let $E_y(q)$ be the polynomial which counts the locations of $L(y \cdot \nu)$ in $P(\nu)$ as predicted by the Theorem. In other words, the coefficient of $q^{m/2}$ in $E_y(q)$ is the number of copies of $L(y \cdot \nu)$ claimed to be in layer $m - 1 - l(y)$ of the socle series of $P(\nu)$. All three polynomials have degree $l(w_\lambda) - l(y)$ with 1 as constant term and as highest degree coefficient, and the Theorem states that they are all equal.

Consider the set of polynomials $R(q)$ in $\mathbf{Z}[q^{1/2}]$ of degree $l(w_\lambda) - l(y)$, with $R(1) = T_y(1)$ and all coefficients nonnegative. If $R(q) = \sum a_i q^i$, let $\tilde{R}(q)$ denote the dual polynomial $\sum a_{l(w_\lambda)-l(y)-i} q^i$. Given another such polynomial $S(q)$, we introduce an ordering $R(q) \geq S(q)$ if for every $j \leq 2(l(w_\lambda) - l(y))$, the sum of the coefficients of the terms in R of degree $\leq j/2$ is no more than the corresponding sum for S . In other words, R is obtained from S by shifting integral pieces of the coefficients of S upwards to higher degree terms. Notice that if $R(q) \geq S(q)$, then $\tilde{R}(q) \leq \tilde{S}(q)$.

The statement at the beginning of the proof that the claimed socle series is a lower bound for the actual socle series can be summarized by the inequality $E_y(q) \leq T_y(q)$, and $T_y(q) \leq U_y(q)$ by 2.2. By the self-duality of $P(\nu)$, we have $\tilde{T}_y(q) = U_y(q)$. Therefore, if $E_y(q) = \tilde{E}_y(q)$, it would follow that $\tilde{T}_y(q) \leq E_y(q)$, and all the polynomials would be equal, proving the Theorem.

In fact, $E_y(q)$ is easily computed; a theorem of Gabber and Joseph [10, 4.9], when combined with Theorem 5.3, yields the following interpretation of the Kazhdan-Lusztig polynomials $P_{w_\lambda w, w_\lambda y}(q)$, with $w \in W_\lambda$ and $y \leq w$:

$$P_{w_\lambda w, w_\lambda y}(q) = \sum_j (\text{soc}_j M(w \cdot \nu) : L(y \cdot \nu)) q^{1/2(j-1-l(y))}.$$

The predicted socle series for $P(\nu)$ is obtained by shifting the socle series of each $M(w \cdot \nu)$ upwards $2l(w_\lambda w)$ layers. Thus the contribution which $M(w \cdot \nu)$ makes to $E_y(q)$ is exactly $q^{l(w_\lambda w)} P_{w_\lambda w, w_\lambda y}(q)$, and

$$E_y(q) = \sum_{\substack{w \in W_\lambda \\ y \leq w}} q^{l(w_\lambda w)} P_{w_\lambda w, w_\lambda y}(q).$$

From this description, the equality $E_y(q) = \tilde{E}_y(q)$ follows from [16, 3.2 and 4.9]. (Alternatively, one can argue directly by defining an augmentation ψ on the Hecke

algebra \mathcal{H} associated to W_λ (see [15]). Let $\psi(T_y) = (-1)^{\langle y \rangle}$. Then ψ is a homomorphism of \mathcal{H} to $\mathbf{Z}[q^{1/2}, q^{-1/2}]$ commuting with the involution on \mathcal{H} . Applying ψ to $C_{w_\lambda, y}$ and using $C_{w_\lambda, y} = \overline{C_{w_\lambda, y}}$, one obtains the equality, because $\psi(\overline{C_{w_\lambda, y}}) = E_y(q)$. \square

6. Loewy series of $P_S(w \cdot \lambda)$.

6.1. Let S be a subset of B , let λ be a dominant, regular weight in P_S^+ , and recall that $P_S(w \cdot \lambda)$ is self-dual for w in ${}^S X_\lambda$. A natural candidate for the radical series of $P_S(w \cdot \lambda)$ can be described, which specializes in case $S = \emptyset$ to the series considered in 5.2 and 5.4. In order to describe the candidate, we will use the notation kN for k a positive integer and N a module to denote the direct sum of k copies of N . Let t be the invariant associated to ${}^S W_\lambda$ in §4, and assume further that $llM_S(\lambda) = t + 1$. Then $P_S(w \cdot \lambda)$ has Loewy length $2t + 1$, and our candidate is as follows:

(*)

$$\text{rad}_r P_S(w \cdot \lambda) = \bigoplus_{z \in {}^S W_\lambda} \bigoplus_{i < llM_S(z \cdot \lambda)} (\text{rad}_i M_S(z \cdot \lambda) : L(w \cdot \lambda)) \text{rad}_{r-i} M_S(z \cdot \lambda).$$

This candidate arises as follows. For each occurrence of $L(w \cdot \lambda)$ in $\text{rad}_i M_S(z \cdot \lambda)$, there is a homomorphic image N of $M_S(z \cdot \lambda)$ of Loewy length $i + 1$ with $L(w \cdot \lambda)$ as socle. The projective module $P_S(w \cdot \lambda)$ must map onto DN , yielding a copy of $L(z \cdot \lambda)$ in $\text{rad}_i P_S(w \cdot \lambda)$. This $L(z \cdot \lambda)$ must correspond to the top of a constituent $M_S(z \cdot \lambda)$ in a Verma flag for $P_S(w \cdot \lambda)$ (since it survives in the homomorphic image of $P_S(w \cdot \lambda)$ modulo all Verma flag constituents of higher weight). Thus, an ‘‘upper bound’’ for the radical series of $P_S(w \cdot \lambda)$ is obtained by placing the layers of the radical series of each such $M_S(z \cdot \lambda)$ in the radical series of $P_S(w \cdot \lambda)$ with a shift i layers downward. The formula (*) encapsulates this description, and the actual radical series can differ from the one described only by shifting some simples downward. In light of this, the first of the following questions is natural:

Questions. (1) For $w \in {}^S X_\lambda$, are the layers of the radical series of $P_S(w \cdot \lambda)$ described by (*)?

(2) For $w \in {}^S X_\lambda$, do the radical and socle series of $P_S(w \cdot \lambda)$ coincide?

(3) For $w \in {}^S W_\lambda$, do the radical and socle series of $M_S(w \cdot \lambda)$ coincide?

6.2. In specific examples, Questions (1) and (2) can often be answered positively if one knows the structure of the generalized Verma modules. Suppose, for a given λ , that the radical series of $M_S(y \cdot \lambda)$ is known for each $y \in {}^S W_\lambda$. Suppose also that $llM_S(\lambda) = t + 1$. Then (*) and self-duality yield candidates for the radical and the socle series of $P_S(w \cdot \lambda)$ for each $w \in {}^S X_\lambda$, and these candidates have the length required by Theorem 4.3. If in fact the two candidates coincide, then they are correct, answering (1) and (2) simultaneously. The point is that (*) is an upper bound for the radical series, and its dual is a lower bound for the socle series, so their coincidence forces the two Loewy series, which are squeezed in-between, to coincide. (This was the point of the proof of Theorem 5.4.) This strategy works in many cases, leading to the result below.

THEOREM. *Let λ be a dominant, integral weight in P_S^+ . For the choices of (R, S) listed below, and for each $w \in {}^S X$, the module $P_S(w \cdot \lambda)$ has the same radical and socle series, with layers described by (*). Moreover, in all cases, each generalized Verma module has the same radical and socle series.*

$$\begin{aligned} &(A_n, A_{n-1}), (A_n, A_{n-2} \times A_1), (A_n, A_{n-2}), \\ &(C_3, \text{any } S), (C_4, A_3), \\ &(D_n, D_{n-1}), (D_5, A_4), (D_4, A_1 \times A_1 \times A_1), \\ &(E_6, D_5). \quad \square \end{aligned}$$

The proof is a computation; the results of [12] yield the self-duality conjecture and desired Loewy length information in all cases, leaving only the structure of the generalized Verma modules to determine. Once this is done, the “self-duality” of (*) is easily verified. In fact, in diagrams 3, 6, 8, 10, 12, and 14 of [12], the composition factors of generalized Verma modules have been listed so that the rows correctly describe the radical and socle series.

Let us indicate how the structure of generalized Verma modules $M_S(w \cdot \lambda)$ can be computed inductively. The composition factors can be obtained simultaneously with the factors of various $\theta_\alpha L(y \cdot \lambda)$, as described for instance in [6]. Suppose that the Loewy series of $M_S(y \cdot \lambda)$ is already computed for all y with $l(y) > l(w)$. Typically, Theorem 4.3 yields $llM_S(w \cdot \lambda)$, and the standard composition factors of $M_S(w \cdot \lambda)$ are easily found. If $w \in {}^S Z$, then $\text{soc } M_S(w \cdot \lambda)$ is simple by [12, 4.6]. At this point, only a few composition factors will remain with their location undetermined. For any y, z with $y > z > w$, we already know whether there are any nontrivial extensions of $L(z \cdot \lambda)$ and $L(y \cdot \lambda)$, since this occurs if and only if $L(y \cdot \lambda)$ is a summand of $\text{rad}_1 M_S(z \cdot \lambda)$. This will eliminate certain possibilities for the location of composition factors. (For instance, once $\text{soc } M_S(w \cdot \lambda)$ is known, we can eliminate certain simples from $\text{soc}_2 M_S(w \cdot \lambda)$.) Finally, an examination of various subquotients $\theta_\alpha L(y \cdot \lambda)$ of $\theta_\alpha M_S(ws_\alpha \cdot \lambda)$ will pin down the location of some simples. For instance, suppose $ys_\alpha < y$ and $ws_\alpha > w$, and

$$(M_S(w \cdot \lambda) : L(y \cdot \lambda)) = (M_S(ws_\alpha \cdot \lambda) : L(y \cdot \lambda)) = 1.$$

Suppose also that we know the location of $L(y \cdot \lambda)$ in the socle series of $M_S(w \cdot \lambda)$. Then the summands of $U_\alpha L(y \cdot \lambda)$ lie one layer above in the extension $\theta_\alpha M_S(w \cdot \lambda)$ of $M_S(w \cdot \lambda)$ by $M_S(ws_\alpha \cdot \lambda)$. A summand with multiplicity 0 in $M_S(ws_\alpha \cdot \lambda)$ will therefore be one layer above $L(y \cdot \lambda)$ in the socle series of $M_S(w \cdot \lambda)$. In the cases of the Theorem, these approaches combine to produce the structure of the generalized Verma modules.

REMARKS. (1) The example $(A_n, A_{n-2} \times A_1)$ is not explicitly treated in [12], and the composition factors of generalized Verma modules were not determined for this case or the (A_n, A_{n-2}) case in which the roots of $B \setminus S$ are not orthogonal, but in both these cases this is easily done. The structure of generalized Verma modules follows easily, since there is at most one nonstandard composition factor to place properly in each generalized Verma module.

(2) As remarked at the close of [12], in all the examples treated, any multiplicity $(M_S(\eta) : L(\xi))$ is 0 or 1. This may produce a deceptive picture. However, the description of self-dual projectives in (*) makes sense regardless of multiplicity, and can be transferred to categories \mathcal{O}_S^λ with λ nonregular. In particular, as observed in

[12, 9.6], there is a D_4 example with μ dominant in P_S^+ but not regular, such that there is only one socular weight ν , and $\text{soc } M_S(\eta) = L(\nu) \oplus L(\nu)$ for a specific η . In this example, the apparent structure of the generalized Verma modules produces a candidate, via the analogue of (*), for the radical series of $P(\nu)$ which is self-dual, suggesting that the questions raised in 6.1 (suitably rephrased) may have positive answers for all blocks of \mathcal{O}_S . I am not aware of any example to the contrary.

NOTE ADDED IN PROOF. It is easy to see that the hypothesis of the Theorem in §4.3 is satisfied by ${}^S X_\lambda$ if and only if there is a dominant weight μ such that $\lambda - \mu$ is integral and \mathcal{O}_S^μ contains a simple projective module. This yields the validity of the Theorem for any parabolic subalgebra of an A_n Lie algebra, as well as many other cases, and failure of the Theorem for certain parabolic subalgebras of the D_4 Lie algebra. Details will appear in a joint paper with Brad Shelton, along with proposed statements on Loewy length in \mathcal{O}_S^λ which should be valid for any dominant, regular λ .

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