ON THE DECOMPOSITION NUMBERS OF
THE FINITE GENERAL LINEAR GROUPS. II

BY

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Abstract. Let $q$ be a prime power, $G = \text{GL}_n(q)$ and let $r$ be a prime not dividing $q$. Using representations of Hecke algebras associated with symmetric groups over arbitrary fields, the $r$-modular irreducible $G$-modules are classified. The decomposition matrix $D$ of $G$ (with respect to $r$) is partly described in terms of decomposition matrices of Hecke algebras, and it is shown that $D$ is lower unitriangular, provided the irreducible characters and irreducible Brauer characters of $G$ are suitably ordered.

Introduction. It is a well-known theorem that the decomposition matrices of symmetric groups with respect to any prime $p$ are always lower unitriangular.

Our main theorem in this paper says that the same is true for the general linear groups for all primes different from the describing characteristic, i.e. for all primes $r$ not dividing $q$, where $G = \text{GL}_n(q)$ for some $n \in \mathbb{N}$ and some prime power $q$.

The Hecke algebra $R[W]$ of a symmetric group $W$ over an integral domain $R$ plays the key role for the proof of the theorem. If $r$ divides $q - 1$, then the occurring Hecke algebras $R[W]$ over a field $R$ of characteristic $r$ are isomorphic to the usual group algebra $R W$ over $R$. Thus the representation theory of symmetric groups has been used in [2] to give a close connection between the decomposition matrix of $W$ with respect to $r$ and the decomposition matrix $\text{GL}_n(q)$.

Meanwhile, representation theory of Hecke algebras $R[W]$ over arbitrary integral domains $R$ has been developed so far in [3] that we are able to generalize [2] to arbitrary primes $r$ not dividing $q$.

However the main work is done in [2 and 3]. So using [3] most proofs in [2] carry over with only slight modifications to the more general situation here. So we shall frequently refer to [2] and omit the proofs in this paper.

It should be remarked that G. James proved in [9], with entirely different methods, results on the decomposition modulo $r$ of unipotent characters of $G$, indicating a strong connection with representation theory of Hecke algebras as well. G. James and the author will combine both methods in a forthcoming paper, which gives a better insight into the connections mentioned above and more details on the decomposition matrices of general linear groups.

I wish to thank G. James for many helpful hints and suggestions.
1. Preliminaries. Let $G = \text{GL}_n(q)$, the general linear group of rank $n$ over the field $F$ with $q$ elements, $q$ a prime power. Throughout we will adopt the notation of [2]. In particular $r$ will be a prime not dividing $q$, and $(\bar{R}, R, K)$ is an $r$-modular split system for $G$.

If $s$ is a semisimple element of $G$, then $s$ is conjugate to $\prod m_\Lambda(s)(\Lambda)$, where the product has to be taken over the set $\mathcal{F}$ of monic irreducible polynomials over $F$, $(\Lambda)$ denotes the companion matrix of $\Lambda \in \mathcal{F}$, and $m_\Lambda(s)$ the multiplicity of $\Lambda$ as an elementary divisor of $s$. As in [2] we fix a linear order on $\mathcal{F}$ and take all occurring products, sums, etc. over $\mathcal{F}$ in this order, if not otherwise stated. Recall that the centralizer $C_G(s)$ of $s$ in $G$ is isomorphic to $\prod \text{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$. The Weyl group of $C_G(s)$ is denoted by $W_s$. As in [2] we consider all occurring Weyl groups as subgroups of $G$.

Define the Levi subgroup $L_s$ of $G$ setting $L_s = \prod \text{GL}_{\deg \Lambda}(q^{m_\Lambda(s)})$ and the parabolic subgroup $P_s$ with Levi decomposition $P_s = L_s U_s$ by $P_s = L_s U_0$, where $U_0$ is the subgroup of all upper unitriangular matrices in $G$.

To * we construct as in [2] a certain irreducible character $\psi$ of $L_s$ using Deligne-Lusztig operators. We choose the $\mathcal{R}$-$L_s$-lattice $S_0$ affording $\psi$ as in [2] and denote the pull back of $S_0$ to $P_s$ by $S$, respectively by $S_s$. Note that $\psi$ is a cuspidal character of $L_s$ by [2, 1.3], i.e. the trivial character $1_U$ does not occur as constituent in the restriction of $\psi$ to $U$ for the unipotent radical $U$ of any proper parabolic subgroup of $L_s$ (compare [11, Chapter 4]).

Let $b_s$ be the block of $\mathcal{R}P_s$, which contains $S$. In [2] the block idempotent of $b_s$ has been denoted by $e_s$. Since we need the letter $e$ in a different meaning in the following, we change notation and denote the block idempotent of $b_s$ by $e_s$. We shall frequently refer to the following:

1.1 Hypothesis. (i) $\bar{R}S$ is an irreducible $\bar{R}P_s$-module.
(ii) Let $V \in b_s$ be an irreducible constituent of $KSG$. Then $V = KS$.

By [7, 2] $\text{End}_{\bar{R}G}(\bar{R}S^G) = \bar{R}[W_s]$ for $\bar{R} = K$, and, if 1.1 holds, for every choice of $\bar{R} \in \{ \bar{R}, R, K \}$, where $\bar{R}[W_s]$ denotes the Hecke algebra of $W_s$ with respect to $\bar{R}$ as defined in [2, §3]. We recall that $K[W_s] = KW_s$, the usual group algebra of $W_s$ over $K$. Since $W_s = \prod \sigma_{m_\Lambda(s)}$, where $\sigma_{m_\Lambda(s)}$ denotes the symmetric group acting on $m_\Lambda(s)$ letters, the irreducible $K[W_s]$-modules are indexed by multipartitions $\lambda = \prod \lambda_\Lambda$, where $\lambda_\Lambda$ is a partition of $m_\Lambda(s)$ for $\Lambda \in \mathcal{F}$. The character of the corresponding irreducible constituent of $KSG$ is denoted by $\chi_{s,\lambda}$.

For a natural number $m$ we write shortly $\mu \vdash m$ if $\mu$ is a partition of $m$. Let $M_s = \{ \lambda = \prod \lambda_\Lambda | \lambda_\Lambda \vdash m_\Lambda(s) \}$. Then $\{ \chi_{s,\lambda} | \lambda \in M_s \}$ is called the geometric conjugacy class of $s$ and is denoted by $(s)^G$. If $t$ and $s$ are semisimple conjugate elements of $G$, then $(t)^G = (s)^G$. Moreover the set of characters $\chi_{s,\lambda}$, where $s$ runs through a set of representatives of $G$-conjugacy classes of semisimple elements and $\lambda$ through $M_s$, is a full set of nonisomorphic irreducible characters of $G$ by [5] (compare [4]).

In general $\bar{R}[W_s] \not\cong \bar{R}W_s (\bar{R} \in \{ \bar{R}, R \})$, but the representation theory of $\bar{R}[W_s]$ is a remarkable $q$-analogue to the representation theory of symmetric groups. It has
been developed in [6] for \( \tilde{R} = K \) and in [3] for arbitrary domains \( \tilde{R} \). We are going to give a brief survey on the main results proved in [3].

So let \( \tilde{R} \in \{ \tilde{R}, R, K \} \). For each \( \lambda \in M_s \) we may define a Specht module \( S^\lambda = S^\lambda_{\tilde{R}} \) generated by a certain element \( z_\lambda = z_{\lambda, \tilde{R}} \) of \( \tilde{R}[W_s] \). Note that for \( \tilde{R} = R \), the definitions of \( S^\lambda \) and \( z_\lambda \) in [2 and 3] coincide. \( S^\lambda \) has a standard basis (independent of \( \tilde{R} \) and \( q \)) [3, 5.6]. In particular, \( S^\lambda_{\tilde{R}} = z_{\lambda, \tilde{R}} \tilde{R}[W_s] \) is a reduction mod \( r \) of \( S^\lambda_K = z_{\lambda, K} K[W_s] \), where reduction mod \( r \) means here reduction via the \( R \)-order \( R[W_s] \) in \( K[W_s] \).

On \( S^\lambda_{\tilde{R}} \) a bilinear form is defined, which has certain invariance properties under the action of \( \tilde{R}[W_s] \). In particular, if \( \tilde{R} \in \{ \tilde{R}, K \} \), the radical \( S^\lambda_{\tilde{R}, \pm} \) of this bilinear form is a submodule of \( S^\lambda \), and \( D^\lambda = S^\lambda / S^\lambda_{\tilde{R}, \pm} \) is either zero or an irreducible \( \tilde{R}[W_s] \)-module.

For \( \lambda \in \mathcal{F} \) let \( e_\lambda \) be the minimal positive integer \( k \) such that \( k \) divides \( 1 + q^{\deg \lambda} + q^{2\deg \lambda} + \cdots + q^{(k-1)\deg \lambda} \) and define \( e \) to be the cartesian product \( \prod \lambda \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is a partition of \( k = m_\lambda(s) \), we say, \( \lambda \) is \( e_\lambda \)-regular if either \( m_\lambda(s) = 0 \) or \( r < e_\lambda \) for \( 1 \leq i \leq m_\lambda(s) \). Call \( \lambda = \prod \lambda \in M_s \) \( e \)-regular if \( \lambda \) is \( e_\lambda \)-regular for all \( \lambda \in \mathcal{F} \). We denote the set of \( e \)-regular \( \lambda \in M_s \) by \( M_{s, e} \), and \( M \subseteq M_{s, e} \). Note that for all \( \lambda \in \mathcal{F} \), \( 2 < e_\lambda < r \) and \( e_\lambda = r \) if and only if \( r \) divides \( q^{\deg \lambda} - 1 \).

On the set of partitions of a natural number define the dominance order \( \triangleright \) as in [8, 3.2] and extend this to \( M_s \) setting \( \lambda \triangleright \mu \) if \( \lambda \triangleright \mu \lambda \) for all \( \lambda \in \mathcal{F} \), where \( \lambda = \prod \lambda \), \( \mu = \prod \mu \lambda \). Finally fix a linear order \( \triangleright \) on \( M_s \) compatible with \( \triangleright \) (e.g. the lexicographical ordering).

1.2 Theorem [3, 7.7]. Let \( s \) be a semisimple element of \( G \).

(i) \( S^\lambda_{K, \pm} = (0) \) for all \( \lambda \in M_s \) and \( \{ S^\lambda_{K} | \lambda \in M_s \} \) is a full set of nonisomorphic irreducible \( K[W_s] \)-modules.

(ii) Let \( \tilde{R} = \tilde{R}, \lambda \in M_s \). Then \( D^\lambda \neq (0) \) if and only if \( \lambda \) is \( e \)-regular. \( \{ D^\lambda | \mu \in M_{s, e} \} \) is a full set of nonisomorphic irreducible \( \tilde{R}[W_s] \)-modules.

(iii) \( S^\lambda_{\tilde{R}} \) is an \( R[W_s] \)-lattice in \( S^\lambda_K \) and \( S^\lambda_{\tilde{R}} = \tilde{R} \otimes_R S^\lambda_K \). Ordering \( \{ S^\lambda_{\tilde{R}} | \lambda \in M_s \} \) and \( \{ D^\lambda | \mu \in M_{s, e} \} \) according to the linear order \( \triangleright \) on \( M_s \) (downwards), the decomposition matrix \( D \) of \( R[W_s] \) is lower unitriangular, i.e. \( D \) has the form

\[
D = \begin{pmatrix}
1 & 1 & \cdots & 0 \\
0 & \ddots & \cdots & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1
\end{pmatrix}.
\]

In the following \( S^\lambda \) always means \( S^\lambda_{\tilde{R}} \) and \( S^\lambda_K, S^\lambda_{\tilde{R}} \) is denoted by \( KS^\lambda, \overline{RS}^\lambda \) respectively.

Let \( s \in G \) be semisimple satisfying 1.1. Define the subalgebra \( A = A_s \) of \( RG \) as in [2], and recall that \( I \to I(S \otimes 1)\tilde{R}A \) defines an isomorphism between the lattice of all \( \tilde{R} \)-free right ideals \( I \) of \( \tilde{R}[W_s] \) and the lattice of all \( \tilde{R}S \)-homogeneous submodules of \( (S \otimes 1)\tilde{R}A = \tilde{R}[W_s](S \otimes 1) \otimes \tilde{R}S^G_{e_s} \), where \( \tilde{R} \in \{ \tilde{R}, R, K \} \), by [2, 3.8].

For \( \lambda \in M_s \) define the \( A \)-module \( S_\lambda \) setting

\[
S_\lambda = S^\lambda(S \otimes 1)A = z_\lambda R[W_s](S \otimes 1) \subseteq S^G_{e_s}.
\]
Then $S_\lambda RG = z_\lambda S^G$ is an $RG$-submodule of $S^G$. Moreover it is an $RG$-lattice in the simple $KG$-module $z_\lambda KS^G$, which affords the character $\chi_{z_\lambda}$. Set $D^\lambda = \overline{RS}^\lambda/J(\overline{RS}^\lambda)$, $D_\lambda = \overline{RS}_\lambda/J(\overline{RS}_\lambda)$ and $E_\lambda = \overline{R}(z_\lambda S^G)/J(\overline{R}(z_\lambda S^G))$, where, for an arbitrary module $M$, $J(M)$ denotes the Jacobson radical of $M$. Now 1.2 and [2, 4.6] imply immediately

1.3 Lemma. Suppose that $s \in G$ satisfies 1.1, and let $\lambda \in M_s$ be $e$-regular. Then $D_\lambda$ is a simple $\overline{RS}$-homogeneous $\overline{RA}$-module and $E_\lambda$ is a simple $\overline{RG}$-module. Moreover $D_\lambda$ occurs in $E_\lambda e_s$ as a composition factor.

Assume 1.1 and let $b_{\lambda \rho}$ be the multiplicity of $E_\rho$ as a composition factor of $\overline{R}(z_\lambda S^G)$, and $a_{\lambda \rho}$ the multiplicity of $D_\rho$ as a composition factor of $\overline{RS}^\lambda$ ($\lambda \in M_s$, $\rho \in M_{s,e}$). Note that the standard basis theorem [3, 5.6 and 2, 3.7] imply immediately that $\overline{R}(z_\lambda S^G e_s) = z_\lambda(\overline{RS}^G e_s)$ as $A$-modules, where $z_\lambda = z_\lambda R$. So we conclude as in [2, 5.5/5.6/5.7/5.8] using 1.2:

1.4 Lemma. Assume 1.1. Let $\lambda \in M_s$ and $\rho \in M_{s,e}$.

(i) If $\rho' \in M_{s,e}$, then $E_\rho \cong E_{\rho'}$ implies $\rho = \rho'$.

(ii) $b_{\lambda \rho} \leq a_{\lambda \rho}$. In particular $b_{\lambda \rho} \neq 0$ forces $\rho \supseteq \lambda$, and $b_{\rho \rho} = 1$.

(iii) Let $Q^\rho \leq \overline{R}[W]_\rho$ be a projective indecomposable $\overline{R}[W]_\rho$-module such that $Q^\rho/J(Q^\rho) = D^\rho$. Then $J(Q^\rho)\overline{RS}^\rho$ is the unique maximal submodule of the $\overline{RG}$-module $P_\rho = Q^\rho\overline{RS}^\rho$. In particular $E_\rho = P_\rho/J(P_\rho)$ is simple and $E_\rho e_s = D_\rho$ as $\overline{RA}$-modules.

For an arbitrary group $H$ with subgroups $H_1$, $H_2$ we write $H_1 \leq H_2$ if $H_1^h \leq H_2$ for some $h \in H$.

It is easy to generalize [2, 6.1] to the following

1.5 Lemma. Let $s$ and $t$ be semisimple elements of $G$ and assume that both satisfy 1.1. Let $\rho \in M_{s,e}$ and $\lambda \in M_t$.

(i) If $E_\rho$ occurs as a composition factor in $\overline{R}(z_\lambda S^G_t)$ (in $\overline{RS}^G_t$), then $L_t \leq G L_s$.

(ii) If $E_\rho e_s \neq (0)$, then $L_s \leq G L_t$.

Remark. Since $KS_s$ affords a cuspidal character of $L_s$ for a semisimple $s \in G$, 1.5 follows easily from [11, 4.5] as well.

2. Combinatorics. For $\Lambda \in \mathcal{F}$, let $\tilde{e}_\Lambda$ be the multiplicative order of $q^{\deg \Lambda}$ modulo $r$. Thus $\tilde{e}_\Lambda = e_\Lambda$ if $r$ does not divide $q^{\deg \Lambda} - 1$, and $\tilde{e}_\Lambda = 1$, $e_\Lambda = r$ otherwise. In particular $1 \leq \tilde{e}_\Lambda \leq r - 1$. Let $v_r$ be the exponential valuation of $\mathbb{Z}$ associated to $r$, normalized so that $v_r(r) = 1$. Let $a_\Lambda = v_r(q^{\deg \Lambda} - 1)$ if $r$ is odd. If $r = 2$, then $\tilde{e}_\Lambda = 1$. In this case we set $a_\Lambda = v_2(q^{\deg \Lambda} - 1)$ if $q^{\deg \Lambda} - 1 \equiv 0 \pmod{4}$ and $a_\Lambda = v_2(q^{\deg \Lambda} - 1) - 1$ otherwise. Note that $a_\Lambda \geq 2$ for $r = 2$.

2.1 Lemma. Let $i$ be a positive integer, $\Lambda \in \mathcal{F}$. Then $v_r(q^{i \deg \Lambda} - 1) > 0$ if and only if $\tilde{e}_\Lambda$ divides $i$. If so, then $v_r(q^{i \deg \Lambda} - 1) = a_\Lambda + v_r(i)$, or $r = 2$, $q = 3 \pmod{4}$ and $i$ is odd. In this case $v_2(q^{i \deg \Lambda} - 1) = 1$.

Proof. This is immediate from the definitions of $\tilde{e}_\Lambda$ and $a_\Lambda$. (For odd $r$ compare [4, 3A].) □

For $\Lambda \in \mathcal{F}$ let $\omega$ be an element of order $r^k$ ($1 \leq k \in \mathbb{N}$) in some extension field of $GF(q^{\deg \Lambda})$, and let $\Gamma$ be the minimum polynomial of $\omega$ over $GF(q^{\deg \Lambda})$. Define
i = i_\lambda(\omega) \in \mathbb{N} as follows: If r is odd or r = 2 and q = 1 \mod(4), let i = k - a_\lambda for \( k \geq a \) and i = 0 otherwise. If r = 2 and q = 3 \mod(4), let i = 0 for k = 1, i = 1 for 2 \leq k \leq a_\lambda and i = k - a_\lambda for a_\lambda < k.

### 2.2 Corollary

Let \( \Lambda \in \mathcal{F}, \omega, \Gamma, k \) and \( i = i_\lambda(\omega) \) as above. Then \( \deg \Gamma = \tilde{\varepsilon}_\lambda r' \).

For the rest of this section let \( s \) be a fixed semisimple element of \( G \), given in its rational canonical form. Moreover we suppose that \( s \) is an \( r' \)-element, i.e. its order is prime to \( r \). For an arbitrary finite group \( H \) we shall denote the set of \( r' \)-elements of \( H \) by \( H_{r'} \). Let \( y \in C_G(s) \), and \( t = sy \). Then \( t \) and \( y \) are semisimple and \( C_G(t) \leq C_G(s) = \prod GL_{m_\lambda(s)}(q^{\deg \Lambda}) \). Let \( \prod y_\lambda, \prod t_\lambda \) and \( \prod C_G(s)_\lambda \) be the corresponding decomposition of \( y, t \) and \( C_G(s) \) respectively, and let \( \Lambda \in \mathcal{F} \) be an elementary divisor of \( s \), i.e. \( m_\lambda(s) \neq 0 \). Let \( \tilde{\Gamma} \) be an elementary divisor of \( y \in GL_{m_\lambda(s)}(q^{\deg \Lambda}), \omega \) a fixed root of \( \tilde{\Gamma} \) and \( \sigma \) of \( \Lambda \). Then the order of \( \omega \) is a power of \( r \), say \( r^i \). Of course \( GF(q^{\deg \Lambda})[\omega] = GF(q^{\deg \Lambda - \deg \Gamma}) = GF(q)[\sigma \omega] \). Let \( \Gamma \) be a minimum polynomial of \( \sigma \omega \over F = GF(q) \), and let \( i = i_\lambda(\omega) \) be defined as above. Then by 2.2 \( \deg \Gamma = \deg \Lambda \cdot \tilde{\varepsilon}_\lambda \cdot r^i \). Now \( GL_{m_\lambda(s)}(q^{\deg \Lambda}) \leq GL_{m_\lambda(s)_{\deg \Lambda}}(q) \leq G \) canonically, and we may consider \( t_\lambda \) as an element of \( GL_{m_\lambda(s)_{\deg \Lambda}}(q) \). We conclude that \( m_\Gamma(t_\lambda) \) equals the multiplicity of \( \tilde{\Gamma} \) as an elementary divisor of \( y_\Lambda \in GL_{m_\lambda(s)}(q^{\deg \Lambda}) \), and we get each elementary divisor of \( t_\lambda \) in this way. In particular, for arbitrary \( \Lambda \neq \Gamma \in \mathcal{F}, m_\Gamma(t_\lambda) \neq 0 \) implies that \( \deg \Gamma = \deg \Lambda \cdot \tilde{\varepsilon}_\lambda \cdot r^i \) for some \( 0 \leq i \in \mathbb{Z} \).

Set \( N_i = \{ \Gamma \in \mathcal{F} | \deg \Gamma = \deg \Lambda \cdot \tilde{\varepsilon}_\lambda \cdot r^i \} \) and \( d_i = \sum_{\Gamma \in N_i} m_\Gamma(t_\lambda) \). Note that \( \Lambda \in N_0 \) if and only if \( \tilde{\varepsilon}_\lambda = 1 \). So define \( d_{-1} \) to be zero and \( N_{-1} = \emptyset \) if \( \tilde{\varepsilon}_\lambda = 1 \), and \( d_{-1} = m_\lambda(t_\lambda) \) and \( N_{-1} = \{ \Lambda \} \) otherwise. Then \( m_\Lambda(s) = d_{-1} + \tilde{\varepsilon}_\lambda \sum_{i=0}^{\infty} d_i r^i \).

We call \( t_\lambda \) reduction stable (with respect to \( r \)), if there exists at most one \( \Gamma \in N_i \) with \( m_\Gamma(t_\lambda) \neq 0 \) for all \( -1 \leq i \in \mathbb{Z} \). Call \( t = sy \in sC_G(s) \), reduction stable (with respect to \( r \)), if \( t_\lambda \) is reduction stable for all elementary divisors \( \Lambda \) of \( s \), and extend this definition to all semisimple elements \( t \) of \( G \) in the obvious way. We shall see later that \( t \) satisfies 1.1(i) always and 1.1(ii) if and only if \( t \) is reduction stable. Note in particular that semisimple \( r' \)-elements of \( G \) are reduction stable with respect to \( r \).

Let \( t = sy \) and \( \Lambda \) be as above. Define \( D_\Lambda \) to be the set of all sequences \( (d_i)_{-1 \leq i \in \mathbb{Z}} \) of nonnegative integers satisfying \( d_{-1} + \tilde{\varepsilon}_\lambda \sum_{i=0}^{\infty} d_i r^i = m_\lambda(s) \), where \( d_{-1} = 0 \), if \( \tilde{\varepsilon}_\lambda = 1 \). With \( d_\lambda = (d_i) \in D_\Lambda \) we may associate the partition
\[
\lambda_{d_\lambda} = \left( \deg \Lambda^d_{-1}, (\tilde{\varepsilon}_\lambda \deg \Lambda)^d_0, (\tilde{\varepsilon}_\lambda r \deg \Lambda)^d_1, \ldots, (\tilde{\varepsilon}_\lambda r^k \deg \Lambda)^d_k \right)
\]
of \( k = m_\lambda(s) \deg \Lambda \). We defined in [2, §6] a partial order \( < \) on the set of partitions of \( k \), which reflects the ordering \( < \) on the set of Levi subgroups of \( GL_k(q) \) given by \( L_1 < L_2 \) if \( L_1 \subseteq GL_{L_2}(q) \), \( L_1, L_2 \) are Levi subgroups of \( GL_k(q) \). So define as in [2]
\[
d_\lambda < d'_\lambda \iff (d_\lambda, d'_\lambda \in D_\Lambda) \text{ if } \lambda_{d_\lambda} < \lambda_{d'_\lambda}.
\]
Let \( D_s = \prod D_\Lambda \) (cartesian product) and define a partial order \( < \) on \( D_s \) setting \( d < d' \iff (d = \prod d_\lambda, d' = \prod d'_\lambda, d_\lambda, d'_\lambda \in D_\Lambda) \text{ if } d_\lambda < d'_\lambda \text{ for all } \Lambda \in \mathcal{F} \). Fix a linear order \( \preceq \) on \( D_s \) compatible with \( < \). However \( |D_s| = k < \infty \). So we may write the elements of \( D_s \) as the increasing chain \( d_1 \preceq d_2 \preceq \cdots \preceq d_k \).

So far we have constructed a map from \( sC_G(s) \), into \( D_s \). For \( t = sy, y \in C_G(s) \), the image of \( t \) under this map is denoted by \( d_t \). If \( d_t = d_j \) \( (1 \leq j \leq k) \), we call \( j = ht (y) \) the \( s \)-height of \( y \). Note that \( t' = sy' (y' \in C_G(s)) \) is conjugate in \( G \) to \( t \) if
and only if $y$ and $y'$ are conjugate in $C_G(s)$. In particular, if $t$ and $t'$ are conjugate in $G$, then $d_t = d_{t'}$. From the definition of $d_t$ and $L_t$, we get immediately

2.3 Lemma. Assume that $s$ has only one elementary divisor, and let $t = sy$, $t' = sy'$ ($y, y' \in C_G(s)$). Then $L_t \leq_G L_{t'}$ if and only if $d_t < d_{t'}$. In particular $L_t = L_{t'}$ if and only if $ht_s(y) = ht_s(y')$.

Let $s$ again be an arbitrary semisimple $r'$-element of $G$, and let $\Lambda \in \mathcal{F}$ be an elementary divisor of $s$, a root of $\Lambda$. For $0 \leq i \in \mathbb{Z}$ set $j_i = \tilde{e}_\Lambda \deg \Lambda \cdot r^i$. For $2 \leq i \in \mathbb{Z}$, fix an element $\omega_i$ of order $r^{a_\Lambda + i}$ in $GF(q^{j_i})$. If $r$ is odd or $r = 2$ and $q = 1 \mod(2)$, then let $\omega_1$ be an arbitrary $2$-element in $GF(q^{j_1})$ of order $r^{a_\Lambda + 1}$, and if $r = 2$ and $q = 3 \mod(4)$, then let $\omega_1$ be an arbitrary $2$-element in $GF(q^{j_1})$ of order $3$. If $\tilde{e}_\Lambda > 1$ fix an element $\omega_0$ of order $r^{a_\Lambda}$ in $GF(q^{j_0})$ and set $\omega_{-1} = \sigma$. If $\tilde{e}_\Lambda = 1$, then $\omega_1 = \sigma$ and $\omega_{-1} = 0$. Then $GF(q^{\deg \Lambda})[\omega_i] = GF(q^{j_i})$ for all $0 \leq i \in \mathbb{Z}$ with $\omega_i \neq \sigma$. For $0 \leq i \in \mathbb{Z}$ with $\omega_i \neq \sigma$ let $\Lambda_i$ be the minimum polynomial of $\sigma \omega_i$ over $F$ and $\Lambda'$ the minimum polynomial of $\omega_i$ over $GF(q^{\deg \Lambda})$. Set $\Lambda_{-1} = \Lambda$ and $\Lambda'_{-1} = X - 1$ if $\tilde{e}_\Lambda > 1$, and $\Lambda_{-1} = \Lambda'_{-1} = 0$, $\Lambda_0 = \Lambda$ and $\Lambda' = X - 1$ if $e = 1$, where we consider $0$ as an irreducible polynomial not occurring as an elementary divisor, i.e. $m_0(t) = 0$ for all semisimple elements $t$ of $G$.

For $d_\Lambda = (d_i) \in D_\Lambda$ define $y_\Lambda \in C_G(s)_\Lambda$ setting $y_\Lambda = \prod_{i=-1}^{d_\Lambda} d_i(y_i')$ (matrix direct sum), and for $d = \prod d_\Lambda \in D_s$ set $y_d = \prod y_\Lambda$. Note that $y_d \in C_G(s)_\Lambda$. All elementary divisors of $t_d = sy_d$ are of the form $\Lambda_j$ for some elementary divisor $\Lambda$ of $s$ and some $-1 \leq i \in \mathbb{Z}$. Moreover $m_{\Lambda'}(t_d) = d_j$, where $d = \prod d_\Lambda$ and $d_\Lambda = (d_i) \in D_\Lambda$. Now, for $1 \leq j \leq k$, let $y_j = y_{d_j}$ and $t_j = sy_j$. Note in particular that $t_1 = s$, since $d_1 = \prod d_\Lambda$ with $d_\Lambda = (d_i)$, where $d_{-1} = m_{\Lambda'}(s)$ and $d_i = 0$ ($0 \leq i \in \mathbb{Z}$) if $\tilde{e}_\Lambda > 1$, and $d_0 = m_{\Lambda'}(s)$ and $d_{-1} = 0 = d_i (1 \leq i \in \mathbb{Z})$ if $\tilde{e}_\Lambda = 1$.

Obviously [2, 6.3] now holds analogously replacing $s \delta(B_s)$ by $sC_G(s)_\Lambda$. The following lemma is immediate from the definitions as well:

2.4 Lemma. Let $s$ be as above and $1 \leq j \leq k$. Then $t_j$ is reduction stable.

For the rest of this paper we shall replace all occurring indices of the form $t_j$ ($1 \leq j \leq k$) by $j$ itself, e.g. $W_j = W_{t_j}$, $S_j = S_{t_j}$, etc. Note that in particular $M_1 = M_j$, $M_{1, e'} = M_{j, e'}$ and $W_1 = W_j$.

Let $\Lambda = \prod \Lambda \in M_1$, i.e. $\Lambda = \Lambda_1 \cdots \Lambda_\Lambda$. Let $\Lambda_\Lambda = (1, 2, \ldots, m_{\Lambda_\lambda})$, where $m = m_{\Lambda}(s)$. Then $\sum_{\nu=1}^m k_{\nu} v = m$. Let $k_{\nu} = \tilde{e}_\Lambda \Lambda_{\nu} + d_{\nu}$ for some $d_{\nu} \in \mathbb{Z}$ with $0 \leq d_{\nu} < \tilde{e}_\Lambda$ and let $k_{\nu} = \sum_{i=0}^{d_{\nu}} d_i^{(\nu)} r^i$ be the $r$-adic expansion of $k_{\nu}$. Set $d_i = \sum_{\nu=1}^m d_i^{(\nu)} v$. Then $d_{\nu} = \tilde{e}_\Lambda \sum_{i=0}^{d_i} d_i^{(\nu)} v = m = m_{\Lambda}(s)$. Thus $d_\Lambda = (d_i)_{1 \leq i \in \mathbb{Z}} \in D_\Lambda$ and $d = \prod d_\Lambda \in D_s$, i.e. $d = d_j$ for some $1 \leq j \leq k$. By [2, 6.3] $m_{\Lambda}(t_j) = d_j$, and obviously

$$d_i = \sum_{\nu=1}^m d_i^{(\nu)} v = \sum_{\nu=1}^{d_i} d_i^{(\nu)} v.$$ 

Therefore $\Lambda_{\Lambda_i} = (1^{(d_1)}, \ldots, v^{d_i^{(\nu)}}, \ldots, k^{d_i^{(d_i)}})$ is a partition of $d_i = k$. If $\tilde{e}_\Lambda = 1$, then $e_{\Lambda_i} = r$ for $0 \leq i \in \mathbb{Z}$, and $d_{-1} = 0$. By construction, $0 \leq d_i^{(\nu)} < r$ for all $1 \leq \nu \leq d_i$, hence $\Lambda_{\Lambda_i}$ is $e_{\Lambda_i}$-regular. If $\tilde{e}_\Lambda > 1$, then $e_{\Lambda_i} = r$ for $0 \leq i \in \mathbb{Z}$ by 2.1, and $\tilde{e}_\Lambda = e_{\Lambda_i} = e_{\Lambda_i} < r$. By construction $d_{\nu+1} < \tilde{e}_\Lambda = e_{\Lambda_i}$ and $0 < d_i^{(\nu)} < e_{\Lambda_i} = r$ for all $1 \leq \nu < d_i$. 

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hence in all cases $\lambda_{\Lambda_i}$ is $e_{\Lambda_i}$-regular for all $-1 \leq i \leq 1$. Taking the cartesian product over all $\Lambda \in \mathcal{F}$ and all $-1 \leq i \leq 1$ we have defined an $e$-regular element $\mu_{\Lambda_i} M_{j,e}$. This defines a map $\lambda \mapsto \mu_{\lambda}$ from $M_1$ into $\bigcup_{j=1}^k M_{j,e}$. Note that this union is disjoint by [2, 6.3].

As in [2] an inverse mapping may be constructed easily, so [2, 6.4] holds analogously, i.e. the map constructed above is a bijection. In the following we will identify $M_1$ with $\bigcup_{j=1}^k M_{j,e}$ by this bijection.

3. The theorem. Fix a semisimple $r$-element $s$ of $G$. Let $B_s$ be the union of all blocks of $RG$ with semisimple part $s$, where the semisimple part of an $r$-block of $G$ is defined as in [4, 5D].

As in [2] an inverse mapping may be constructed easily, so [2, 6.4] holds analogously, i.e. the map constructed above is a bijection. In the following we will identify $M_1$ with $\bigcup_{j=1}^k M_{j,e}$ by this bijection.

3. Theorem. Fix a semisimple $r$-element $s$ of $G$. Let $B_s$ be the union of all blocks of $RG$ with semisimple part $s$, where the semisimple part of an $r$-block of $G$ is defined as in [4, 5D].

3.1 Lemma. Let $y \in C_G(s)^r$, $t = sy$, and assume that $t$ is reduction stable. Let $y' \in C_G(s)^r \cap L$, and $t' = sy'$. Then $t$ and $t'$ are conjugate in $G$ if and only if they are conjugate in $L$.

Let $t = sy$, $y \in C_G(s)^r$. Let $V \in b_i$ be an irreducible constituent of $KS_{P_i}^G$ and $\psi$ the character afforded by $V$. Denote the restriction of $\psi$ to $L_i$ by $\chi$. Since the semisimple part of $b_i$ is $s$(where we identify blocks of $L_i$ with their liftings to $P_i$), $\chi \in (sy')^{L_i}$ for some $y' \in C_{L_i}(s)^r$. Thus the $KL^r$-module affording $\chi$ is a constituent of the $KL^r$-module $KS_{L_i}^{L_i}$, replacing $G$ by $L_i$.

Using [2, 1.2], the orthogonality relations for Deligne-Lusztig operators [11, 6.14] and Frobenius reciprocity we see that $sy'$ and $t$ has to be conjugate in $G$ (compare the proof of [2, 2.3] and [2, 6.9]). If $t$ is reduction stable, $t$ and $sy'$ has to be conjugate in $L_i$ by 3.1, hence $L_i = L_{sy'}$ by construction of $L_i$ and $L_{sy'}$ respectively. In particular $KS_{L_i}^{L_i} = V_{L_i}$ is irreducible and $V \neq KS_i$, showing that $t$ satisfies 1.1(ii).

Assume that $t$ is not reduction stable. So let $\Lambda \in \mathcal{F}$, $m_{\Lambda}(s) \neq 0$ and $-1 \leq i \leq 1$ such that there exist two different elementary divisors of $t$ in $N_i$, say $\Gamma$, $\Gamma'$. Then $t = B + (\Gamma') + (\Gamma') + C$ (matrix direct sum) for some matrices $B$ and $C$. Let $t' \in G$ be defined setting $t' = B + (\Gamma') + (\Gamma') + C$. Then $t$ and $t'$ are conjugate in $G$ but not in $L_i$. Moreover by [4, 8A] $S_i$ and $S_i$ are both contained in the block $b_i$ of $RP_i$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Using [2, 1.2, 11, 6.14] and Frobenius reciprocity once more, we see that the irreducible $KP_r$-module $KS_r$ contained in $b$, is different from $KS_r$ and occurs as a constituent of $KS^G_{p_r}$. We have shown

3.2 **Lemma.** Let $t \in G$ be semisimple. Then $t$ satisfies 1.1(ii) if and only if $t$ is reduction stable with respect to $r$.

3.3 **Lemma.** Let $t \in G$ be semisimple. Then $t$ satisfies 1.1(i). Moreover $\bar{\rho}_S \cong \bar{\rho}_{S_j}$, where $j = \text{ht}_s(y)$ for the $r'$-part $s$ and $r$-part $y$ of $t$.

**Proof.** We have to consider $S$, as an $RL_r$-module contained in the block $b$, of $L_r$.

So we may assume that $t$ has only one elementary divisor $\gamma$ of degree $n$. Consequently $s$ has only one elementary divisor as well, say $\Lambda$. If $\deg \Gamma = \deg \Lambda$, then $L_s = L_t = G$ and $B_s$ contains just the block $b_s$. By [4, 7A] $b_s$ has a cyclic defect group and contains only one irreducible Brauer character by [4, 8A]. From this the assertion follows immediately.

So assume that $\deg \Gamma > \deg \Lambda$. Then $n = \bar{\epsilon}_\Lambda \deg \Lambda \cdot r^h, s = \bar{\epsilon}_\Lambda r^h(\Lambda)$ and $t = (\Gamma)$ for some $0 \leq h \in \mathbb{Z}$. Let $t' = (\Lambda_h) \in G$, where $\Lambda_h \in \mathcal{F}$ is defined as in §2. Then $L_s = L_t' = G$, $d_s = d_t$, and $\text{ht}_s(y) = \text{ht}_s(y')$, where $y'$ denotes the $r'$-part of $t'$.

Moreover $M_s = M_{t', r}$ contains just one element, say $\lambda = \prod \lambda_{\Gamma'}$, with $\lambda_{\Gamma'} = (1)$ and $\lambda_{\gamma'} = (\gamma') = (\gamma') \in \mathcal{F}$.

As in §2 let $D_s = \{d_1, \ldots, d_k\}$. Then $\text{ht}_s(y) = k$. We may assume by induction on $n$ that the assertion holds for all $L_j, 1 \leq j \leq k - 1$, since $L_j = L_k$ implies $j = k$ by 2.3. By 2.4 and 3.2 $t_j (1 \leq j \leq k - 1)$ satisfies 1.1. So we have defined by 1.3 for each $\mu \in M_s = M_1 = \bigcup_{j=1}^{k-1} M_{j, r}$, with $\mu \neq \lambda$ an irreducible $\bar{\rho}_G$-module $E_\mu$ contained in $B_s$. Let $\lambda \neq \mu$, $\mu' \in M_1$ and assume that $E_\mu \cong E_{\mu'}$. Let $\mu \in M_{j, r}$, $\mu' \in M_{j', r}$, $1 \leq j, j' \leq k - 1$. Then by 1.5 $L_j \leq L_{j'}$ and $L_j \leq L_{j'}$, hence $j = j'$ by 2.3, and then $\mu = \mu'$ by 1.4. So far we have found all but one irreducible $\bar{\rho}_G$-module in $B_s$, by [4, 8A]. Let $E$ be the remaining irreducible $\bar{\rho}_G$-module in $B_s$. By 1.5 $E_\mu$ does not occur as a composition factor of $\bar{\rho}_S = \bar{\rho}_S^{G_{1}}$ for $\lambda \neq \mu \in M_1$. Thus $E$ is the only composition factor of $\bar{\rho}_S$, and as in [2, 6.9] we conclude from [2, 6.5/6.6] that the multiplicity of $E$ as composition factor of $\bar{\rho}_S$ is one, showing that $\bar{\rho}_S = E$ is irreducible. This applies to $t'$ as well, so $\bar{\rho}_S = \bar{\rho}_S^{G_{1}} = E$, and the lemma is proved.

As an immediate consequence we get the following theorems:

3.4 **Theorem.** Let $t \in G$ be semisimple. Then $t$ satisfies 1.1 if and only if $t$ is reduction stable. If the $r$-part of $t$ has $s$-height $j$ (the $r'$-part of $t$), then $\text{End}_{\bar{\rho}_G}(\bar{\rho}_S^{G_{1}}) = [W_j]$. Finally $\bar{\rho} \otimes \bar{\rho} \text{End}_{\bar{\rho}_G}(S_{p_r}^{G_{1}}) = \text{End}_{\bar{\rho}_G}(\bar{\rho}_S^{G_{1}})$ if and only if $t$ is reduction stable.

3.5 **Theorem.** Let $\chi$ be an irreducible cuspidal character of $G$. Then $\chi$ restricted to $G_r$ is an irreducible Brauer character for all primes $r$ not dividing $q (G = \text{GL}_n(q))$.

3.6 **Corollary.** Let $s$ be a semisimple $r'$-element of $G$, and let $t_j (1 \leq j \leq k)$ be defined as in §2. Then $t_j$ satisfies 1.1 for all $j \in \{1, \ldots, k\}$.

So far we have defined for each $\rho \in M_1$ an irreducible $\bar{\rho}_G$-module $E_\rho$ by 3.6, 1.3 and [2, 6.4]. We proceed now as in [2] to prove our main theorem: First we use 1.3,
2.3, 1.5 and 1.4(i) to show that \( \{ E_\rho | \rho \in M_1 \} \) is a full set of nonisomorphic irreducible \( \bar{R}G \)-modules in \( B_s \), provided \( s \) has only one elementary divisor \([2, 6.10]\). Then \( E_\rho \cong \bar{E}_\rho \) as well, where \( \bar{E}_\rho (\rho \in M_1) \) is defined as in 1.4(iii) \([2, 6.11]\).

Next we denote for an arbitrary semisimple \( r' \)-element \( s \) of \( G \) the multiplicity of \( E_\rho \) as a composition factor of \( \bar{R}(z_s F)^G \) by \( d_{\lambda, \rho} \), where \( \rho \in M_{j, e}, \lambda \in M_{j'}, 1 \leq j, j' \leq k \).

The \((|M_j| \times |M_{j, e}|)\)-matrix \((d_{\lambda, \rho})\) is denoted by \( D_{jj'} \) and the decomposition matrix of \( R[W_j] \) by \( D^j \). Replacing \( r' \) by \( e' \) the proofs of \([2, 6.12 \text{ and } 2, 6.13] \) carry over word by word, showing the following theorems.

3.7 Theorem. Let \( s \) be a semisimple element of \( G \). Then the following holds:

(i) \( \{ E_\rho | \rho \in M_1 \} \) is a full set of nonisomorphic irreducible \( \bar{R}G \)-modules in the union \( B_s \) of \( r \)-blocks of \( G \) with semisimple part \( s \).

(ii) Let \( 1 \leq j, j' \leq k, \rho \in M_{j, e}, \lambda \in M_j \). Then \( d_{\lambda, \rho} \neq 0 \) implies that \( d_{\lambda, j'} < d_{\lambda, j} \). In particular, if \( j < j' \), then \( D_{jj'} = 0 \).

(iii) \( D_{jj} = D^j \) for all \( j \in \{1, \ldots, k \} \).

3.8 Theorem. Let \( s \) be a semisimple \( r' \)-element of \( G \) and let \( B_s \) be the union of all \( r \)-blocks of \( G \) with semisimple part \( s \). Then there exist \( 1 = y_1, y_2, \ldots, y_k \in C_G(s)_r \) and a bijection between \( M_s \) and the disjoint union \( \bigcup_{j=1}^k M_{j, e} \) such that \( \{ E_\rho | \rho \in M_{j, e}, 1 \leq j \leq k \} \) is a full set of nonisomorphic irreducible \( \bar{R}G \)-modules in \( B_s \). Ordering \( 1, \ldots, k \) downwards, the decomposition matrix \( D \) of \( B_s \) has the following form:

\[
D = \begin{pmatrix}
D^k & 0 & \cdots & 0 \\
D^{k-1} & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix},
\]

where \( D^j (1 \leq j \leq k) \) denotes the decomposition matrix of the Hecke algebra \( R[W_j] \).

Furthermore, the irreducible characters and Brauer characters in \( B_s \) may be ordered so that \( D \) is lower unitriangular, i.e. has the form

\[
D = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]

3.9 Corollary. Let \( G \) be a finite general linear group defined over a field \( F \) of characteristic \( p \). Let \( r \neq p \) be a prime. Then the decomposition matrix of \( G \) with respect to \( r \) is lower unitriangular for a suitable ordering of the irreducible characters and Brauer characters of \( G \).

However, \([2, 6.13] \) generalizes as well:

3.10 Corollary. Let \( s = \prod m_{\Lambda}(s) \Lambda(\Lambda) \) be a semisimple \( r' \)-element of \( G = GL_{n}(q) \). Define the Levi subgroup \( \bar{L} \) of \( G \) setting \( \bar{L} = \prod_{\Lambda} GL_{n_{\Lambda}(q)} \) and let \( b \) be the union
of all blocks of $\tilde{L}$ with semisimple part $s$. Let $B_s$ be defined as in 3.8. Then the following hold:

(i) Let $V_0$ be an $\overline{RL}$-module in $b$ and $V \subseteq V_0$. Then $V \rightarrow V^G$ defines an isomorphism from the submodule lattice of $V_0$ onto the submodule lattice of $V^G_0$.

(ii) The decomposition matrices of $b$ and $B_s$ coincide.

Let $s$ be as in 3.10, $y \in C_G(s)$, and $t = sy$. Let $j = ht_s(y)$. By [4, 8A] $(t)^G \subseteq B_s$. So we may ask what we can say about the part of the decomposition matrix $D$ of $B_s$ which corresponds to the geometric conjugacy class $(t)^G$. We have seen in 3.3 that $\overline{RS}_s \cong \overline{RS}_t$. Let $\lambda \in M_t$. By [2, 4.5] $z_\lambda KS_t^G$ affords the irreducible character $\chi_{t, \lambda}$ of $G$. As in the first section let $b_t$ be the block of $L_t$ containing $KS_t$, and $\varepsilon_t$ be the block idempotent of $b_t$. However $L_t = L_j$ by construction of $d_j$, $L_t$ and $ht_t$, and $\varepsilon_t = \varepsilon_j$, since $\overline{RS}_t \cong \overline{RS}_j$.

Now $\overline{R}(z_\lambda S_t^G)$ is a reduction modulo $r$ of $z_\lambda KS_t^G$. Moreover we have seen in the first section that $\overline{R}(z_\lambda S_t^G, \varepsilon_t) = \overline{R}A_{tj}$-modules, where the subalgebra $A_t$ of $RG$ is defined as in [2, §3].

Since $\varepsilon_t = \varepsilon_j$, $\overline{R}(z_\lambda S_t^G, \varepsilon_t)$ is an $\overline{R}A_t$-submodule of $\overline{RS}_t^G$ as well. By 3.10 we may assume that $s$ has only one elementary divisor. So we conclude as in [2, 6.12] that the multiplicity $d_{\lambda, \rho}$ of $E_\rho$ ($\rho \in M_{j, e_j}$) as a composition factor of $\overline{R}(z_\lambda S_t^G, \varepsilon_t)$ equals the multiplicity of the $\overline{R}A_t$-module $D_\rho$ as a composition factor of $\overline{R}(z_\lambda S_t^G, \varepsilon_t)$.

By [2, 3.7], the latter equals the multiplicity of the $\overline{R}[W_j]$-module $D^\rho$ as a composition factor of $\overline{R}[W_j]$. However $\overline{R}[W_j]$ is a standard Young subalgebra of $\overline{R}[W_t]$. Now [3, 7.4] implies immediately that $\overline{z}_\lambda \overline{R}[W_j] \cong \overline{RS}_t^g \otimes \overline{R}[W_j] \overline{R}[W_t]$ has a Specht series (independent of $q$), which may be calculated by the Littlewood-Richardson rule [8, 16.4].

Denote the $(|M_t| \times |M_{j, e_j}|)$-matrix $(d_{\lambda, \rho})$ by $D_{tj}$ and let $d^{\lambda\mu}$ be the multiplicity of $\overline{RS}_t^\mu$ in a Specht series of $\overline{z}_\lambda \overline{R}[W_j]$ ($\mu \in M_j$).

3.11 Theorem. Let $s$ be a semisimple $r'$-element of $G$, $y \in C_G(s)$, and $t = sy$. Let $j = ht_s(y)$. Then the following hold:

(i) Let $\rho \in M_{j, e_j}$, $1 \leq j' \leq k$, $\lambda \in M_t$ and assume that $E_\rho$ occurs as a composition factor in $\overline{R}(z_\lambda S_t^G, \varepsilon_t)$. Then $d_{\lambda, \rho} < d_{j, j'}$.

(ii) Let $\lambda \in M_t$, and assume that $\overline{RS}_t^\mu$ occurs $d^{\lambda\mu}$ times in a Specht series of $\overline{z}_\lambda \overline{R}[W_j]$. Then the row of $D_{tj}$ corresponding to $\lambda \in M_t$ is $\sum_{\mu \in M_j} d^{\lambda\mu} r_{\mu}$, where $r_{\mu}$ is the row of $D_{j, j'} = D_j$ corresponding to $\mu \in M_j$.

Proof. (i) follows as in [2, 6.12] using (3.10), and (ii) has been proved above. □

Remark. We do not say anything about the multiplicity of $E_\rho$ ($\rho \in M_{j', e_j}$) as a composition factor of $\overline{R}(z_\lambda S_t^G, \varepsilon_t)$ for $j' < j$. The result above indicates that this is given by the Littlewood-Richardson rule as well, i.e. it may be conjectured, that 3.11(ii) holds analogously replacing the matrices $D_{tj}$ and $D_{j, j'}$ by the part of the decomposition matrix of $B_s$ which corresponds to the geometric conjugacy class $(t)^G$, $(t_j)^G$ respectively.

In [9] G. James defined for the special case $s = 1$, $\lambda \in M_t$, an analogue to the Specht module $S^\lambda$ in a characteristic free way for the general linear group. He proved as well an analogue to the Specht series theorem [8, Chapter 17] (compare
This implies in particular that the Littlewood-Richardson rule holds in fact for the unipotent of $G$ and for their reductions modulo $r$. This gives a good reason that the conjecture above is true.

As a final application we get

3.12 Theorem. Let $P$ be a parabolic subgroup of $G$ with Levi complement $L$. Let $V$ be an irreducible $RL$-module, which affords a cuspidal character of $L$, lifted to $P$. Then all indecomposable direct summands of $\overline{R}V^G$ are liftable.

Proof. First of all we may assume that $L = L_t$ for some semisimple $t$ of $G$ and $V = S_t$. Since $\overline{RS_t} = \overline{RS'_t}$ for some reduction stable $t'$ in $G$ by 3.3, we may assume that $t$ is reduction stable. So $t$ satisfies 1.1 by 3.4, in particular $\operatorname{End}_{\overline{R}G}(\overline{RS_t}^G) = \overline{R} \otimes_R \operatorname{End}_{RG}(S_t^G)$. As in the case of permutation modules, this implies immediately the assertion of the theorem. □

Remark. Obviously a direct indecomposable summand of $\overline{RS_t}^G$ may be lifted to a direct summand of $S_t^G$ provided $t$ is reduction stable. This does not hold, in general, if $t$ is not reduction stable.

The indecomposable direct summands of $RS_t^G$ correspond bijectively to the indecomposable direct summands of the endomorphism ring of $RS_t^G$, i.e. to the projective indecomposables of $\overline{R}[W_t]$ ($t$ reduction stable). So, if $\rho \in M_{t,e}$, then the character of $KP$ where $\overline{RP}$, where $\overline{RP}_\rho = P_\rho$ and the $RG$-module $P_\rho$ is defined as in 1.4(iii), is given by the column of the decomposition matrix of $R[W_t]$ corresponding to $\rho \in M_{t,e}$.

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