LP ESTIMATES FOR SCHRÖDINGER EVOLUTION EQUATIONS

M. BALABANE AND H. A. EMAMI-RAD

ABSTRACT. We prove that for Cauchy data in $L^1(\mathbb{R}^n)$, the solution of a Schrödinger evolution equation with constant coefficients of order $2m$ is uniformly bounded for $t \neq 0$, with bound $(1 + |t|^{-c})$, where $c$ is an integer, $c > n/2m - 1$. Moreover it belongs to $L^q(\mathbb{R}^n)$ if $q > q(m, n)$, with its $L^q$ norm bounded by $(|t|^{-c'} + |t|^{-c})$, where $c'$ is an integer, $c' > n/q$. A maximal local decay result is proved. Interpolating between $L^1$ and $L^2$, we derive $(L^p, L^q)$ estimates.

On the other hand, we prove that for Cauchy data in $L^p(\mathbb{R}^n)$, such a Cauchy problem is well posed as a distribution in the $t$-variable with values in $L^p(\mathbb{R}^n)$, and we compute the order of the distribution. We apply these two results to the study of Schrödinger equations with potential in $L^p(\mathbb{R}^n)$. We give an estimate of the resolvent operator in that case, and prove an asymptotic boundedness for the solution when the Cauchy data belongs to a subspace of $L^p(\mathbb{R}^n)$.

1. Introduction. In this paper we study Schrödinger evolution equations

(*) $\frac{\partial U}{\partial t} = (iP(D) + V(x))U, \quad U(0, x) = U_0(x) \in L^p(\mathbb{R}^n),$

where

$$D = \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_n} \right)$$

and $P(\xi)$ is an elliptic polynomial of order $2m$ with $\text{Im} P_{2m}(\xi) = 0$. It is well known that in the case where $\text{Im} P_{2m}(\xi) > 0$ (Heat equation), the solution belongs to $L^p(\mathbb{R}^n)$ in the $x$-variables for $t > 0$. In the case where $\text{Im} P_{2m}(\xi) < 0$ (backward Heat equation), there are no $L^p$ estimates of the solution by the $L^p$ norm of the Cauchy data. We are concerned with the limiting case of the Schrödinger equation

where $\text{Im} P_{2m}(\xi) = 0$.

Hörmander [7] proved that even in the simplest case (where $P$ is the Laplacian of $\mathbb{R}^n$) the problem is not well posed in the usual sense in $L^p(\mathbb{R}^n)$ if $p \neq 2$: $e^{-it\xi^2}$ is not a multiplier of $L^p$ $(p \neq 2)$. This implies that the Hille-Yosida estimates of the resolvent of $(iP(D) + V(x))$ viewed as an operator in $L^p(\mathbb{R}^n)$ are not fulfilled. In Balabane and Emami-Rad [1] we proved that for the usual Schrödinger equation $(P(D) = \Delta, V = 0)$ the problem (*) is well posed in $L^p(\mathbb{R}^n)$, for any $p$, in the distribution sense in the $t$-variable. In [2] we proved that the result remains true with $V = 0$ and $P(D)$ a homogeneous system with constant coefficients. For that
aim, the abstract tool of Smooth Distribution Semigroups was introduced and a Hille-Yosida theorem proved for such semigroups.

In this paper we study the equation including a potential \( V(x) \in L^r(\mathbb{R}^n) \) and we drop the homogeneity condition. We prove that in this case, the problem (*) is still well posed in the distribution sense in the \( t \)-variable with values in \( L^p(\mathbb{R}^n) \), and we compute the order of the distribution. We derive precise estimates of the \( L^p(\mathbb{R}^n) \) norm of the resolvent operator \( (\lambda I - (iP(D) + V))^{-1} \). We prove uniform boundedness for the solution when Cauchy data belongs to a subspace of \( L^p(\mathbb{R}^n) \).

Another related problem is the uniform boundedness of the solution of the Cauchy problem (*) when \( U_0 \) belongs to \( L^1(\mathbb{R}^n) \) and for \( t \neq 0 \). We prove that in the constant coefficients case, if the restriction of \( P(\xi) \) to the unit sphere fulfills a nondegeneracy condition, then the solution belongs to \( L^\infty(\mathbb{R}^n) \) for \( t \neq 0 \). For \( n > 3 + 2/m - 1 \) it belongs to \( L^q(\mathbb{R}^n) \) for \( q > q(m,n) \). It decays locally as \( |t|^{-n/2} \).

This is done using the foliation of \( \mathbb{R}^n \) by the wave surfaces \( P(\xi) = cte \), and applying the stationary phase method to estimate the integrals involved. As a corollary we prove \( (L^p, L^{p'}) \) and \( (L^p, L^q) \) estimates for the Cauchy problem (*). (Estimates of this type have been given by Brenner [4] for the wave equation using hyperbolicity.)

In §2 \( (L^1, L^q) \) estimates are proved for \( V = 0 \). In §3 \( (L^p, L^{p'}) \) estimates are derived, and \( (L^p, L^q) \) estimates are established. The Cauchy problem (*) in \( L^p \) with \( V = 0 \) is studied in §4. Smooth distribution semigroups are introduced. In §5 the Cauchy problem (*) is solved and \( (L^p, L^p) \) estimates are given for the solution and for the resolvent operator.

REMARK. As usual the notation \( (L^p, L^q) \) means estimates of the \( L^q \) norm of the solution of (*) when the Cauchy data belongs to \( L^p \). \( p' \) is the conjugate index of \( p \).

2. Behaviour of the solutions for Cauchy data in \( L^1(\mathbb{R}^n) \). We consider the Cauchy problem (with constant coefficients)

\[
(\star\star) \quad \partial U/\partial t = iP(D)U, \quad U(0,x) = U_0(x),
\]

where \( U_0(x) \in S(\mathbb{R}^n) \); the solution is given by

\[
U(t,x) = \mathcal{F}(e^{itP(\xi)}) \ast U_0
\]

(where \( \mathcal{F} \) denotes the usual Fourier transform, \( \mathcal{F} \) its inverse, and \( \mathcal{F}(e^{itP(\xi)}) \) is defined as an oscillatory integral).

The aim of this section is to estimate the \( L^q \) norm in the \( x \)-variable of \( U(t,x) \) by the \( L^1 \) norm of \( U_0 \). So what we have to prove is that \( \mathcal{F}(e^{itP(\xi)}) \) belongs to \( L^q \) for any fixed \( t \neq 0 \). This will be done using foliation of the exterior of a compact set of \( \mathbb{R}^n \) by the wave surfaces of \( P \). The Stationary Phase Method (Duistermaat [5]) then gives the behaviour of the integrals defining \( \mathcal{F}(e^{itP(\xi)}) \).

Suitable hypotheses for proving the estimates are:

(H1) \( P(\xi) \) is a real valued elliptic polynomial, with principal part \( p(\xi) \) of degree \( 2m \).

(H2) For \( u \in S^{n-1} \) (the unit sphere of \( \mathbb{R}^n \)), the restriction to \( S^{n-1} \) of \( \psi(\xi) = \langle u, \xi \rangle p^{-1/2m}(\xi) \) is nondegenerate at its critical points (i.e. \( d_{\omega(\xi)}^2(\langle u, \omega \rangle p^{-1/2m}(\omega)) = 0 \)).
(H3) \( m \geq 1 \) and \( n \geq 3 \), or
(H3') \( m \geq 2 \) and \( n > 3 + 2/(m - 1) \).

Let \( c \) and \( c' \) be integers with \( c > n/2m - 1 \) and \( c' > n/q \).

Let
\[
\frac{1}{q(m,n)} = \frac{(m-1)(n-3)}{(2m-1)n} - \frac{2}{(2m-1)n}.
\]

The estimates are

**Theorem 1.** (a) If (H1), (H2) and (H3) are fulfilled, the solution \( U(t, \cdot) \) of the
Cauchy problem (***) with Cauchy data in \( L^1(\mathbb{R}^n) \) belongs to \( L^\infty(\mathbb{R}^n) \) for \( t \neq 0 \).
The bound is
\[
\|U(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_\infty (1 + |t|^{-c}) \|U_0\|_{L^1(\mathbb{R}^n)}.
\]

(b) If (H1), (H2) and (H3') are fulfilled, then \( U(t, \cdot) \) belongs to \( L^q(\mathbb{R}^n) \) for
\( q(m,n) < q \leq \infty \). The estimate is
\[
\|U(t, \cdot)\|_{L^q(\mathbb{R}^n)} \leq C_q (|t|^{-c'} + |t|^{-c}) \|U_0\|_{L^1(\mathbb{R}^n)}.
\]

\( C_q \) and \( C_\infty \) are absolute constants.

**Remark 1.** (i) If \( P \) is homogeneous, the estimates can be trivially improved to
\[
\|U(t, x)\|_{L^q} \leq C_q |t|^{-n/2m} \|U_0\|_{L^1} \quad \text{for } q(m,n) < q \leq \infty.
\]

(ii) Without any change to the bounds and to the proof, these estimates can be
proved with \( W^{s,q} \) norm in place of \( L^q \) norm if \( q > \tilde{q}(m,n,s) \) with
\( \tilde{q}^{-1}(m,n,s) = q^{-1}(m,n) - s/(2m-1)n \).

(iii) Without any change to the bounds and to the proof, (H1) could be replaced
by \( P(\xi) = \rho^{2m}\rho(\omega) + Q(\rho, \omega) \) with a symbol \( Q \in S^{2m-1}_{1,0}(S^{n-1} \times \mathbb{R}_+) \).

The proof of Theorem 1 will follow the lemmas below. We will assume \( m \geq 2 \),
the case \( m = 1 \) can be solved by direct computation.

**A.** A foliation of \( \mathbb{R}^n \setminus \{P(\xi) \leq a\} \). Let \( (\rho, \omega) \in \mathbb{R}_+ \times S^{n-1} \) be the spherical
coordinates in \( \mathbb{R}^n \). In these variables, \( P(\rho, \omega) = \rho^{2m}\rho(\omega) + Q(\rho, \omega) \) with degree of
\( Q \) strictly less than \( 2m \). Ellipticity of \( P \) implies \( |\rho(\omega)| \geq c > 0 \) for \( \omega \in S^{n-1} \).
Since \( \rho \) is real valued, we can assume that \( \rho(\omega) \geq c > 0 \) for \( \omega \in S^{n-1} \).

**Lemma 1.** There exist two positive constants \( a \) and \( b \), and a function \( \rho(s, \omega) \in C^\infty([a, \infty[ \times S^{n-1}) \) such that for \( (s, \omega) \in ]a, \infty[ \times S^{n-1} \) we have \( P(\rho(s, \omega), \omega) = s \) and
\( \rho(s, \omega) > b \).

**Proof.** Let
\[
b' = \sup_{\rho \geq 1} \left( (2m\rho^{2m-2}\rho(\omega))^{-1} \partial Q/\partial \rho \right).
\]
Let \( b = \max(b',1) \). For fixed \( \omega \in S^{n-1} \), \( P(\rho, \omega) \) is a strictly increasing function
of the \( \rho \) variable for \( \rho > b \), and goes to infinity when \( \rho \) does. It is then bijective
from \([b, \infty[\) onto \([b, \infty[\). Let \( \rho(s, \omega) \) be the inverse mapping, and let \( a = \sup_{\omega \in S^{n-1}} P(b, \omega) \). \( \rho(s, \omega) \) is then a mapping from \([a, \infty[ \times S^{n-1} \) to \( \mathbb{R}_+ \), which
verifies the identity quoted in the lemma. Moreover \( \rho \) is infinitely differentiable as
the implicit function theorem asserts. Actually the mapping \( (\rho, \omega) \mapsto (s, \omega) \) is a
\( C^\infty \)-diffeomorphism from the open set \( \mathbb{R}^n \setminus \{P(\xi) < a\} \) onto \([a, \infty[ \times S^{n-1} \). Q.E.D.
Using Hörmander’s definition of the symbols classes (Duistermaat [5]), the function just defined has the following behaviour:

**Lemma 2.**
\[
\rho(s, \omega) = (s/p(\omega))^{1/2m} + \sigma(\omega, s), \quad \text{where } \sigma(\omega, s) \text{ belongs to } S^0(1_0, S^{n-1} \times |a, \infty|).
\]

**Proof.** If we let \( \partial_s = \partial/\partial s \), \( \partial_\omega = \partial/\partial \omega \) and \( H(\omega, s) = (1+p^{-1}p^{-2m}Q)^{1/2m} \) we have to prove that \( \rho(1 - H) \in S^0(1_0, S^{n-1} \times |a, \infty|) \) and \( 1 - H \in S^{1/2m} \). First note that in the identity
\[
(E) \quad s = p \cdot \rho^{2m} + Q,
\]
s goes to infinity whenever \( \rho \) does. Since the degree of \( Q \) is strictly less than \( 2m \), we have, uniformly in \( \omega \),
\[
(S1) \quad \rho \sim p^{-1/2m} s^{1/2m} \quad \text{as } s \to \infty.
\]
Then differentiating \( (E) \) with respect to \( s \) gives
\[
(S2) \quad \partial \rho/\partial s = O(s^{-1+1/2m}).
\]
By induction on \( \alpha \), we prove the formulas:

\[
(F1) \quad \partial^\alpha s^{2m} = 2m \rho^{2m-1} \partial^\alpha s + \sum_{i \in N_\alpha} C_i \rho_i^{\partial} (\partial s^\rho)^{i_1} \cdot \cdots \cdot (\partial s^{-\partial})^{i_{\alpha-1}},
\]

\[
(F2) \quad \partial^\alpha s Q = \partial \rho Q \cdot \partial^\alpha s + \sum_{j \in M_\alpha} D_j (\partial \rho Q)(\partial s^\rho)^{j_1} \cdot \cdots \cdot (\partial s^{-\partial})^{j_{\alpha-1}}
\]

for any \( \alpha \in N^* \), where
\[
N_\alpha = \left\{ i = (i_0, \ldots, i_{\alpha-1}) \in N^\alpha \text{ with } \sum_{\gamma=0}^{\alpha-1} i_{\gamma} = 2m \text{ and } \sum_{\gamma=1}^{\alpha-1} \gamma i_{\gamma} = \alpha \right\},
\]

\[
M_\alpha = \left\{ j = (k, j_1, \ldots, j_{\alpha-1}) \in N^\alpha \text{ with } k = \sum_{v=1}^{\alpha-1} j_v \text{ and } \sum_{v=1}^{\alpha-1} v j_v = \alpha \right\},
\]

and \( C_i \) and \( D_j \) are absolute constants.

Then, applying \( \partial^\alpha \) to \( (E) \) gives inductively
\[
(Sa) \quad \partial^\alpha s \rho = O(s^{-\alpha+1/2m}).
\]
In order to estimate \( s \)-derivatives of \( (1 - H) \), we note that by definition
\[
(T1) \quad 1 - H = O(s^{-1/2m}).
\]
By induction on \( \alpha \), we prove
\[
(F3) \quad \partial^\alpha s H = \sum_{K \in L_\alpha} E_K(\omega) H^c \rho^{-2ma-b} \prod_{\gamma=1}^{\alpha} (\partial^\gamma \rho)^{k_\gamma} \prod_{\nu=0}^{\alpha} (\partial^\nu \rho Q)^{l_\nu},
\]
where \( E_K(\omega) \in C^\alpha(S^{n-1}) \) and \( L_\alpha \) is the finite set of elements \( (a, b, k, l) \in N \times N \times N^\alpha \times N_\alpha \) satisfying
\[
\alpha \geq a \geq 0; \quad \alpha \geq b \geq 0; \quad \sum_{1}^{\alpha} \gamma k_\gamma = \alpha; \quad \sum_{0}^{\alpha} l_\nu = a; \quad \sum_{1}^{\alpha} k_\gamma - \sum_{0}^{\alpha} (\nu+1) l_\nu \leq b-1.
\]
Then using the previous estimate \((S\alpha)\) on \(\partial^\alpha_s \rho\) leads to
\[(T\alpha) \quad \partial^\alpha_s H = O(s^{-\alpha-1/2m}).\]

In order to prove the same estimates for the derivatives \(\partial^\alpha_s \partial^\beta_{\omega}\), we first prove formulas similar to \((F1)\) and \((F2)\) by induction on \(\beta\):
\[(F'1) \quad \partial^\alpha_s \partial^\beta_{\omega} 2m = 2m p^{2m-1} \partial^\alpha_s \partial^\beta_{\omega} \rho + \sum_{i \in N_\alpha} C_i \prod_{\gamma = 0}^{\alpha} \prod_{\gamma' \leq \beta} (\partial_{\rho}^\gamma \partial_{\omega}^\gamma \rho)^{\gamma, \gamma'},\]
where \(\alpha' < \beta\), \(i_\gamma = \sum_{\gamma'} i_{\gamma', \gamma'}\) for \(\gamma = 0, \ldots, \alpha\) and
\[N_\alpha = \left\{ i = (i_0, \ldots, i_\alpha) \text{ with } \sum_{\gamma} i_\gamma = 2m \text{ and } \sum_{\gamma} \gamma i_\gamma = \alpha \right\};\]

\[(F'2) \quad \partial^\alpha_s \partial^\beta_{\omega} Q = \partial_{\rho}^\gamma Q \cdot \partial^\alpha_s \partial^\beta_{\omega} \rho + \sum_{j \in M_\alpha} D_j (\partial_{\rho}^k \partial_{\omega}^l Q) \prod_{\nu = 0}^{\alpha} \prod_{\nu' \leq \beta} (\partial_{\rho}^\nu \partial_{\omega}^\nu \rho)^{\nu, \nu'},\]
where \(k \leq \alpha\), \(l \leq \beta\), \(\alpha' < \beta\), \(j_\nu = \sum_{\nu'} j_{\nu', \nu'}\) for \(\nu = 0, \ldots, \alpha\) and
\[M_\alpha = \left\{ j = (j_0, \ldots, j_\alpha) \text{ with } \sum_{\nu = 1}^{\alpha} j_\nu = k - j_0 \text{ and } \sum_{\nu = 1}^{\alpha} \nu j_\nu = \alpha \right\}.\]

Applying \(\partial^\alpha_s \partial^\beta_{\omega}\) to equation \((E)\) shows that \(\rho \in S^{1/2m}_{1,0}\):
\[(S'\alpha) \quad \partial^\alpha_s \partial^\beta_{\omega} \rho = O(s^{-\alpha+1/2m}).\]

At last, we show inductively on \(\beta\)
\[(F'3) \quad \partial^\alpha_s \partial^\beta_{\omega} H = \sum E_K (\omega) H^c \prod_{\gamma \leq \alpha} \prod_{\gamma' \leq \beta} (\partial^\gamma_s \partial^\gamma_{\omega} \rho)^{\gamma, \gamma'} \prod_{\nu \leq \alpha} \prod_{\nu' \leq \beta} (\partial^\nu_{\rho} \partial^\nu_{\omega} Q)^{\nu, \nu'},\]
where, if we denote by \(k_\gamma = \sum_{\gamma'} k_{\gamma', \gamma'}\) and \(l_\nu = \sum_{\nu'} l_{\nu', \nu'}\), then \(\sum k_\gamma = \alpha\) and \(\sum k_\gamma + \sum (2m - \nu - 1) l_\nu \leq -1\). This shows, using \((S\alpha)\), that
\[(T'\alpha) \quad \partial^\alpha_s \partial^\beta_{\omega} H = O(s^{-\alpha-1/2m});\]
which means, added to \((T1)\), that \(1 - H \in S^{1/2m}_{1,0}\).

(\text{It is straightforward to check that all the estimates given are uniform in the } \omega\text{-variables.}) \quad \text{Q.E.D.}\

We are going to use this foliation of \(\mathbb{R}^n \setminus \{P(\xi) \leq a\}\) to estimate integrals involved in \(\int e^{itP(\xi)}\).

\text{B. Estimating integrals on } S^{n-1}. \text{ Let } \beta(s) \in C^\infty(\mathbb{R}), \beta \equiv 1 \text{ for } s > a' + 1 \beta \equiv 0 \text{ for } s < a', \text{ where } a' > a \text{ (the constant } a \text{ is defined in Lemma 1). Let}
\begin{align*}
x &= ru, \quad r > 0, \ u \in S^{n-1}, \quad \lambda = rs^{1/2m}, \\
\phi(s, \omega) &= s^{-1/2m} \rho(s, \omega) \langle u, \omega \rangle, \quad a_p(s, \omega) = \beta(s) \rho^{n-1} \frac{\partial \rho}{\partial s}, \\
J_p(s, \lambda) &= \int_{S^{n-1}} e^{i\lambda \phi(s, \omega)} a_p(s, \omega) \, dw.
\end{align*}
The following lemma analyses the phase function $\phi$:

**Lemma 3.** There exists a finite number of open sets $\Omega_i \subset S^{n-1}$ ($i = 1, \ldots, N$) and a constant $d > a$ such that, for $s > d$:

1. On the complementary set of $\bigcup_i \Omega_i$ in $S^{n-1}$, $\phi(s, \omega)$ has no critical points in the $\omega$-variables and $\|d_\omega \phi\| \geq C > 0$.

2. On each $\Omega_i$, $\phi(s, \omega)$ has only one critical point $\omega^i(s)$, $\omega^i(s) \in \Omega_i \subset \Omega_i$. At that point, $\phi(s, \omega)$ is nondegenerate: the eigenvalues of the Hessian matrix of $\phi$ in the $\omega$-variables at $(s, \omega^i(s))$ have their modulus bounded from below, i.e. let $\bar{H}(s) = \text{Hess}_\omega(\phi)(s, \omega^i(s)) = d_{\omega\omega}^2(\phi(s, \omega^i(s)))$, then $\|\bar{H}^{-1}(s)\| \leq C'$.

3. The estimates are uniform in $s$, i.e. $\Omega_i, \Omega_i'$, $C$ and $C'$ do not depend on $s > d$. $\omega^i \in C^\infty([d, \infty[; S^{n-1})$.

**Proof.**

\[
\phi(s, \omega) = p^{-1/2m}(\omega)\langle u, \omega \rangle + s^{-1/2m}\sigma(s, \omega)\langle u, \omega \rangle
\]

uniformly on $S^{n-1}$, and this remains true under $\omega$-differentiation, as shown in Lemma 2.

Hypothesis (H2) asserts that $\phi(\infty, \omega)$ has only nondegenerate critical points. Compactness of $S^{n-1}$ and the fact that such points are isolated imply that this set of points is finite: $(\omega^i(\infty); i = 1, \ldots, N)$. The assumption (H2) asserts the nondegeneracy of $\phi(\infty, \omega)$ at these points: $d_{\omega\omega}^2(\phi(\infty, \omega^i(\infty)))$ invertible. This implies, by the implicit function theorem, the existence of $d' > 0$, $O_i$ a neighborhood of $\omega^i(\infty)$ in $S^{n-1}$ and $\omega^i(s) \in C^\infty([d', \infty[; O_i)$ with

\[
(\omega^i(s) = 0 \iff \omega = \omega^i(s)) \quad \text{for } s > d' \text{ and } \omega \in O_i.
\]

Moreover, invertibility of $d_{\omega\omega}^2(\phi(\infty, \omega^i(\infty)))$ implies the existence of open sets $\Omega_i \subset O_i$, where $d_{\omega\omega}^2(\phi(\infty, \omega))$ is invertible and we have

\[
\omega \in \Omega_i \Rightarrow \|(d_{\omega\omega}^2(\phi(\infty, \omega)))^{-1}\| < C'.
\]

Lemma 2 asserts that

\[
d_{\omega\omega}^2(\phi(s, \omega)) = d_{\omega\omega}^2(\phi(\infty, \omega)) + O(s^{-1/2m})
\]

uniformly for $\omega \in S^{n-1}$. This implies the existence of $d'' > 0$ such that

\[
(s > d'' \text{ and } \omega \in \Omega_i) \Rightarrow \|(d_{\omega\omega}^2(\phi(s, \omega)))^{-1}\| < 2C'.
\]

This is uniform nondegeneracy of the phase function $\phi$ on $([\bigcup_i \Omega_i] \times [d'', \infty[$. Putting $\omega = \omega^i(s)$ gives

\[
\|\bar{H}^{-1}(s)\| < 2C' \quad \text{for } s > d''.
\]

Let $\Omega_i' \subset \Omega_i$ with $\omega^i(\infty) \in \Omega_i'$. On the complementary set of $\bigcup_i \Omega_i'$ in $S^{n-1}$, $d_\omega \phi(\infty, \omega) \neq 0$. Therefore there exists $d''' > 0$ such that

\[
(s > d''' \text{ and } \omega \in \bigcap_i C\Omega_i') \Rightarrow \|d_\omega \phi(s, \omega)\| > C.
\]

We take $d = \max(d', d'', d''', a)$, where $a$ is the constant defined in Lemma 1. Q.E.D.

We can therefore apply the Stationary Phase Theorem with parameters (Duistermatt [5]).

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Lemma 4. For $s > d$, there exist absolute constants $D_l$ and $D'_l$ such that

(a) for $r > 0$
\[
\left| \frac{\partial^l}{\partial s^l} \left( e^{i\lambda \phi} a_\beta \right) \right| < D_l r^l s^{-1+(n/2m)-(l(2m-1)/2m)},
\]

(b) for $r > 1$
\[
\left| \frac{\partial^l}{\partial s^l} \int_{S^{n-1}} e^{i\lambda \phi} a_\beta \, d\omega \right| < D'_l s^{-1+((n+1)/4m)-(l(2m-1)/2m)} r^{-l+(n-1)/2}.
\]

Proof. The first estimate is straightforward using the Leibniz rule to differentiate $(e^{i\lambda \phi} a_\beta)$, then Lemma 2 gives $\phi \in S^0_{1,0}$, $a_\beta \in S^{-1+n/2m}_{1,0}$ and $\lambda = rs^{1/2m}$ which implies inequality $a$.

To prove the second estimate, let $\alpha_i(\omega)$ ($i = 0, \ldots, N$) be a $C^\infty$-partition of unity on $S^{n-1}$ fitting the following covering: $(\Omega_0 = \bigcap_i C_\Omega, \Omega_1, \ldots, \Omega_N)$. The integral on $S^{n-1}$ is then a sum of $N+1$ parts.

The integral over $\Omega_0$ is rapidly decreasing when $\lambda \to \infty$, the phase $\phi$ being uniformly nonstationary (Lemma 3): As shown previously, we prove

\[
(R) \quad \frac{\partial^l}{\partial s^l} \left( e^{i\lambda \phi} a_\beta \right) = r^l e^{i\lambda \phi} a_\beta, l
\]

with $a_\beta, l \in S^{-1+(n/2m)-(l(2m-1)/2m)}_{1,0}$. Then we use the fact that $\phi$ is nonstationary on $\Omega_0$ to have a first order differential operator $L$ in the $\omega$-variables, with coefficients in $S^0_{0,0}(S^{n-1} \times \mathbb{R}^+)$ satisfying $L\phi = 1$. We use $L$ $k$-times to integrate by parts in the $\omega$-variables and $(R)$ shows for $\lambda > h_1$ and any $k \in \mathbb{N}$:

\[
\left| \frac{\partial^l}{\partial s^l} \int_{\Omega_0} e^{i\lambda \phi} a_\beta \, d\omega \right| < D_{0,k} r^l \lambda^{-k} s^{-1+(n/2m)-(l(2m-1)/2m)}.
\]

To estimate integrals over $\Omega_i$ ($1 \leq i \leq N$) we use the fact that, on $\Omega_i$, the phase is stationary at a single point and nondegenerate at this point uniformly in $s$ (Lemma 3). We apply the Stationary Phase Theorem with Parameters (Duistermaat [5]) and $(R)$ to prove for $\lambda > h_2$:

\[
\left| \frac{\partial^l}{\partial s^l} \int_{\Omega_i} e^{i\lambda \phi} a_\beta \, d\omega \right| < D_l r^l (-n+1)/2 s^{-1+(n+1)/4m-l(2m-1)/2m}.
\]

Taking $k \geq (n-1)/2$, $D = \max(D_l, D_{0,k})$ proves the estimate. Q.E.D.

C. Proof of Theorem 1. Let $\beta$ be the previously defined function with $a' = d$. Let $\alpha(\xi) = 1 - \beta(P(\xi))$. We have

\[
\mathcal{F}(e^{itP(\xi)}) = \int_{\mathbb{R}^n} e^{i(x,\xi)} e^{itP(\xi)} \alpha(\xi) \, d\xi + \int_0^\infty e^{its} \left( \int_{S^{n-1}} e^{i\lambda \phi} a_\beta \, d\omega \right) \, ds
\]

(a) $I_1(t, x)$ is the Fourier transform of a function in $\mathcal{D}(\mathbb{R}^n)$. It is rapidly decreasing in the $x$-variables and we have

\[
\forall k \in \mathbb{N}, \quad |I_1(t, x)| < K_k |t|^{k-r-k},
\]

where $K_k$ are absolute constants.

Taking $k = 0$ shows that $I_1(t, x) \in L^{\infty}_{loc}$ uniformly in $t$. 

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Taking $k$ an integer, $k > n/q$, shows that $I_1(t, x) \in L^q(\mathbb{R}^n)$.

Let $c' > n/q$, $c' \in \mathbb{N}$. Then

\[(J1) \quad \|I_1(t, \cdot)\|_{L^q(\mathbb{R}^n)} < K(1 + |t|^{c'}).\]

(b) In order to estimate $I_2(t, x)$ locally, we integrate by parts in the $s$-variable $l$-times, and use Lemma 4(a) with $l > n/2m - 1$ to prove

\[(J2) \quad I_2(t, x) \in L^\infty_{loc} \text{ with bound less than } K'|t|^{-c} \text{ where } c \text{ is an integer, } c > n/2m - 1.\]

(c) In order to estimate $I_2(t, x)$ for large $x$, we first integrate by parts $l'$-times in the $s$-variable, as previously. Then apply Lemma 4(b) for $l' > (n + 1)/2(2m - 1)$ to obtain

\[(J3) \quad |I_2(t, x)| < K' |t|^{-l'} c', \quad c > n/2m - 1.\]

Assumptions (H3) and $m > 2$ prove that there exists $l' \in \mathbb{N}$ with

\[(n + 1)/2(2m - 1) < l' \leq (n - 1)/2.\]

Taking this value for $l'$ shows that $I_2(t, \cdot) \in L^\infty(\mathbb{R}^n)$ with

\[(J4) \quad \|I_2(t, \cdot)\|_{L^\infty} < K'|t|^{-l'} + K'|t|^{-c}.\]

Moreover, (J3) shows that $I_2(t, x) \in L^q(\mathbb{R}^n)$ for

\[(n - 1)/2 - n/q > l'' > (n + 1)/2(2m - 1).\]

Assumption (H3') implies that such an integer $l''$ exists for $q > q(m, n)$, and we have

\[(J5) \quad \|I_2(t, \cdot)\|_{L^q} < K'|t|^{-c} + K''|t|^{-l''}.\]

(d) We finally notice that we can choose $l'$ and $l''$ less or equal to $c$ to rewrite (J1), (J2), (J4) and (J5):

\[
\|\mathcal{F}(e^{itP(\xi)})\|_{L^\infty} < C_\infty (1 + |t|^{-c}), \\
\|\mathcal{F}(e^{itP(\xi)})\|_{L^q} < C_q (|t|^{c'} + |t|^{-c}). \quad \text{ Q.E.D.}
\]

**REMARK.** This computation shows that $I_2(t, x)$ is rapidly decreasing as $|t| \to \infty$. But in the general case, $I_1(t, x)$ does not decay as $|t| \to \infty$. Under an additional assumption, this will be the case locally.

**COROLLARY.** **Local decay.** If (H1), (H2) and (H3) are fulfilled and if we have

\[(H4) \quad \text{for } \|\xi\| < a, \text{ } P(\xi) \text{ is nondegenerate at its critical points,}
\quad \text{then, if } B \text{ is a bounded set in } \mathbb{R}^n, \text{ we have for large } |t| \]

\[
\forall x \in B \quad |\mathcal{F}(e^{itP(\xi)})(x)| < C|t|^{-n/2}. \quad \text{ Q.E.D.}
\]

**PROOF.** Using Lemma 4(a), we prove the rapid decay of $I_2(t, x)$ for bounded $x$. To estimate $I_1(t, x)$, we consider it as an oscillatory integral on a compact set with phase $P(\xi)$ and the parameter $|t| \to \infty$. (H4) enables us to apply the Stationary Phase Theorem which gives an estimate by $|t|^{-n/2}$. Q.E.D.
3. \((L^p, L^p')\) and \((L^p, L^q)\) estimates \((1 \leq p \leq 2)\). Let \((e^{itP(D)}U_0)(x)\) be the solution \(U(t, x)\) of the Cauchy problem (**) . We have the following \((L^p, L^p')\) and \((L^p, L^q)\) estimates for \(e^{itP(D)}\).

**THEOREM 2.** Assume \(P\) satisfies (H1), (H2) and (H3). Then for any \(t \neq 0\) and any \(p, 1 \leq p \leq 2\), \(e^{itP(D)}\) maps continuously \(L^p(\mathbb{R}^n)\) into \(L^{p'}(\mathbb{R}^n)\) and we have the estimate

\[
\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^{p'})} < C(1 + |t|^{-c_0}),
\]

where \(\theta = p^{-1} - p'^{-1}, \ p^{-1} + p'^{-1} = 1, c\) is an integer, \(c > n/2m - 1\), and \(C\) does not depend on \(t\).

**THEOREM 2'.** Assume \(P\) satisfies (H1), (H2) and (H3'). If \(1 < p \leq 2\) and \(q(m, n, p) < q < p_1\), then \(e^{itP(D)}\) continuously maps \(L^p(\mathbb{R}^n)\) into \(L^q(\mathbb{R}^n)\) for \(t \neq 0\), and we have the estimate

\[
\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^q)} < C(|t|^{c_0} + |t|^{-c_0}).
\]

Here
\[
\theta = p^{-1} - p'^{-1},
\]
\(c\) is an integer with \(c > n/2m - 1\),
\(c'\) is an integer with \(c' > n(p' - q)/q(p' - 2)\), and
\((q(m, n, p))^{-1} = p'^{-1} + \theta(q(m, n))^{-1}\), where \(q(m, n)\) is given in Theorem 1.

**PROOF.** Since \(e^{itP(\xi)}\) is of modulus one, \(e^{itP(D)}\) is continuous in \(L^2(\mathbb{R}^n)\) with norm equal to one. Theorem 1 gives the continuity of \(e^{itP(D)}\) from \(L^1(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\), \(q(m, n) < q < \infty\). Then the Riesz Thorin interpolation theorem (Stein [11], Lions and Peetre [9]) proves the estimate. Q.E.D.

**REMARK 2.** With \(p\) and \(q\) fulfilling the same assumptions, if \(P(\xi)\) is homogeneous, the estimates can be improved to

\[
\|e^{itP(D)}\|_{\mathcal{L}(L^p, L^q)} < C_{p, q} t^{-(n/2m)(1/p - 1/q)}.
\]

We now come to \((L^p, L^p')\) and \((L^p, L^q)\) estimates of the resolvent operator of \(iP(D)\):

**THEOREM 3.** Assume \(P\) satisfies (H1), (H2) and (H3). Then for \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda \neq 0\), and for \(p, 2c/c + 1 < p \leq 2\), we have

\[
\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^{p'})} < C|\text{Re} \lambda|^{-1} (1 + |\text{Re} \lambda|^{c_0}),
\]

where \(\theta = p^{-1} - p'^{-1}, c\) is an integer, \(c > n/2m - 1\), and \(C\) does not depend on \(\lambda\).

**THEOREM 3'.** Assume \(P\) satisfies (H1), (H2) and (H3'). Then for \(\lambda \in \mathbb{C}\), \(\text{Re} \lambda \neq 0\), for \(p, 2c/c + 1 < p \leq 2\), and for \(q, q(m, n, p) < q < p'\), we have

\[
\|(\lambda - iP(D))^{-1}\|_{\mathcal{L}(L^p, L^q)} < C|\text{Re} \lambda|^{-1} (|\text{Re} \lambda|^{-c_0} + |\text{Re} \lambda|^{c_0}),
\]

where the parameters involved have the same values as in Theorem 2'.

**REMARK 3.** If \(2m > n + 1\), then \(c = 1\) and these estimates are valid for any \(p, 1 < p \leq 2\).

**PROOF.** With the same notation as before, we consider for \(\text{Re} \lambda > 0\)

\[
F(\lambda, x) = \int_0^\infty e^{-\lambda t} U(t, x) \, dt.
\]
For $U_0 \in L^p(\mathbb{R}^n)$ with $2 \geq p > 2c/c + 1$, the integral is convergent in $L^q$ for any $q$, $q(m,n,p) < q \leq p'$, in view of Theorems 2 and 2'. This leads to

$$\|F(\lambda, x)\|_{L^q(\mathbb{R}^n)} \leq C|\text{Re} \lambda|^{-1}(|\text{Re} \lambda|^{-c'} + |\text{Re} \lambda|^{c'})\|U_0\|_{L^p(\mathbb{R}^n)}.$$ 

Then we take $U_0 \in S(\mathbb{R}^n)$, we compute $iP(D)F(\lambda, x)$ by passing the operator under the integral sign, and then integrate by parts to prove

$$F(\lambda, x) = (\lambda - iP(D))^{-1}U_0.$$ 

A density argument ends the proof for $\text{Re} \lambda > 0$.

For $\text{Re} \lambda < 0$, the same proof is valid with

$$F(\lambda, x) = - \int_{-\infty}^{0} e^{\lambda t}U(t, x) dt. \quad \text{Q.E.D.}$$

REMARK 4. For homogeneous $P$ we can take $1 < p \leq 2$, and the bound can be improved to

$$\| (\lambda - iP(D))^{-1} \|_{L^p(L^q)} < C|\text{Re} \lambda|^{-1+(n/2m)(1/p-1/q)}.$$ 

4. $(L^p, L^p)$ estimates and smooth distribution groups $(1 < p < \infty)$.

A. $e^{\lambda P(D)}$ as a distribution with values in $L^p$. In order to prove $(L^p, L^p)$ estimates for $e^{\lambda P(D)}$, we first recall that $e^{\lambda P(D)}$ is not a continuous mapping from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ unless $p = 2$, or $t = 0$, or $P(D)$ is a first order differential operator (Hörmander [7] and Brenner [4]). We will prove that for any $\phi \in \mathcal{D}(\mathbb{R})$ (the Schwartz space), the following operator is continuous in $L^p(\mathbb{R}^n)$:

$$\mathcal{G}(\phi) = \int_{-\infty}^{+\infty} \phi(t)e^{itP(D)} dt.$$ 

(We will denote it by $\mathcal{G}_p(\phi)$ when it will operate in $L^p(\mathbb{R}^n)$.) So we consider $e^{itP(D)}$ as a distribution in the $t$-variable with values in $L^p(\mathbb{R}^n)$, and estimate its order.

The only hypothesis we will assume for $P(D)$ throughout §4 will be

(HP) $P(D)$ is a real valued, elliptic polynomial of order $2m$, $P(\xi) \neq 0$ for $\xi \neq 0$.

REMARK. This last assumption about the zeros of $P(\xi)$ is unnecessary: one can always add to any real elliptic polynomial a constant $c$ such that $P(\xi) + c$ fulfills (HP). Adding this constant changes in an obvious way the subsequent estimates. We adopt it in order to simplify notations.

DEFINITION 1. For $l \in \mathbb{N}$, let $p_l$ be the following norm on $\mathcal{D}(\mathbb{R})$:

$$\forall \phi \in \mathcal{D}(\mathbb{R}) \quad p_l(\phi) = \sum_{0 < k < l} \left| \frac{d^k \phi}{dt^k} \right|_{L^1(\mathbb{R})}.$$ 

Let $T_k$ denote the completion of $\mathcal{D}(\mathbb{R})$ for $p_k$, and $T_{\infty}$ the completion of $\mathcal{D}(\mathbb{R})$ for the family $(p_l)_{l \in \mathbb{N}}$. We will denote by $T_k^+$ and $p_k^+$ the same objects with $\mathcal{D}(\mathbb{R})$ replaced by $\mathcal{D}(\mathbb{R}^+)$. 

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THEOREM 4. Let $P(D)$ satisfy (HP). Then for any $p$, $1 < p < \infty$, and any $k \in \mathbb{N}$, $k > n/2$, $\mathcal{G}_p$ is a continuous linear mapping from $T_k$ to $L^p(\mathbb{R}^n)$. 

PROOF. For any $U_0 \in S(\mathbb{R}^n)$, the Cauchy problem (**) has a unique solution $U(t, x) = (e^{itP(D)}U_0)(x)$ which belongs to $S(\mathbb{R}^n)$ for any fixed $t$. Let $\phi(t) \in \mathcal{D}(\mathbb{R})$. The following computation is obvious in $S(\mathbb{R}^n)$:

$$
\mathcal{G}(\phi)U_0 = \int_{-\infty}^{\infty} \phi(t)e^{itP(D)}U_0 dt = \int_{-\infty}^{\infty} \phi(t)\mathcal{F}(e^{itP(\xi)}\mathcal{F}U_0(\xi)) dt
$$

$$
= \mathcal{F}\left(\int_{-\infty}^{\infty} \phi(t)e^{itP(\xi)} dt \cdot \mathcal{F}U_0(\xi)\right) = \mathcal{F}(\hat{\phi}(P(\xi)) \cdot \mathcal{F}U_0(\xi)),
$$

where $\hat{\phi}$ denotes the inverse Fourier transform of $\phi$ in the variable $t$. To prove Proposition 1, we have to prove that $\hat{\phi}(P(\xi)) \in M_p$, the space of Fourier multipliers in $L^p(\mathbb{R}^n)$, and the $M_p$ norm of $\hat{\phi}(P(\xi))$ is bounded by the $T_k$ norm of $\phi$. A density argument will conclude the proof. We will use a sufficient condition for a function to belong to $M_p$ given by Stein [11]: for every differential monomial $\partial^\alpha_\xi$ with $|\alpha| \leq k$ ($k > n/2$), $|\partial^\alpha_\xi \hat{\phi}(P(\xi))|$ must be bounded by $p_{|\alpha|}(\phi)\|\xi\|^{-|\alpha|}$. This is done inductively. For $|\alpha| = 0$, we have

$$
\forall \phi \in \mathcal{D}(\mathbb{R}), \forall \xi \in \mathbb{R}^n, \ |\hat{\phi}(P(\xi))| \leq \|\phi\|_{L^\infty} \leq \|\phi\|_{L^1} = p_0(\phi).
$$

Assume that for any $\beta \in \mathbb{N}^n$ with $|\beta| < |\alpha|$, we have the following estimate for the derivative of order $\beta$:

$$
\forall \phi \in \mathcal{D}(\mathbb{R}), \forall \xi \in \mathbb{R}^n, \ |\partial^\beta_\xi \hat{\phi}(P(\xi))| \leq C p_{|\beta|}(\phi)\|\xi\|^{-|\beta|}.
$$

Then, with a little abuse of notation, and using the classical formulas of derivatives of Fourier transforms and Fourier transforms of derivatives (taken at the point $P(\xi)$), we compute the derivative of order $\alpha$:

$$
\partial^\alpha_\xi \hat{\phi}(P(\xi)) = \partial^\alpha_\xi^{-1}[i(\partial_\xi P) \cdot \hat{\phi}(P(\xi))]
$$

$$
= \partial^\alpha_\xi^{-1}[(P^{-1} \cdot \partial_\xi P) \cdot (\hat{\phi} + t\hat{\phi}')(P(\xi))]
$$

$$
= \sum C^\dagger_{\alpha-1}(P^{-1} \cdot \partial_\xi P) \cdot \partial^\alpha_\xi^{-1-j}(\hat{\phi} + t\hat{\phi}')(P(\xi))
$$

by the Leibniz rule. Using the assumed estimates we have

$$
|\partial^\alpha_\xi \hat{\phi}(P(\xi))| \leq \sum C^\dagger_{\alpha-1} |\partial^\alpha_\xi(P^{-1} \cdot \partial_\xi P)| \cdot p_{|\alpha|-j-1}(\phi + t\phi') \cdot \|\xi\|^{-|\alpha|-j}
$$

$$
\leq \|\xi\|^{-|\alpha|} \sum C^\dagger_{\alpha-1} \sup_{\xi \in \mathbb{R}^n} (\|\xi\|^{j+1} \cdot |\partial^\alpha_\xi(P^{-1} \cdot \partial_\xi P)|) \cdot p_{|\alpha|-j-1}(\phi + t\phi').
$$

Boundedness of $\|\xi\|^{j+1} \cdot |\partial^\alpha_\xi(P^{-1} \cdot \partial_\xi P)|$ follows from (HP). Using the inequality $p_{|\alpha|-1}(t\phi') < D_{\gamma} p_{|\gamma|}(\phi) < D_{\gamma} p_{|\alpha|}(\phi)$ for $|\alpha| \geq |\gamma|$ we have

$$
|\partial^\alpha_\xi \hat{\phi}(P(\xi))| \leq C' p_{|\alpha|}(\phi)\|\xi\|^{-|\alpha|}. \quad \text{Q.E.D.}
$$

REMARK 5. Following [2], if $P$ is homogeneous, $\mathcal{G}_p$ is a continuous mapping from $T_k$ to $L^p(\mathbb{R}^n)$ if $k > n|1/p - 1/2|$. 

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COROLLARY. Assume $P(D)$ satisfies (HP). Let $iP_p(D)$ be the densely defined, closed operator in $L^p(\mathbb{R}^n)$ defined as $iP(D)$ with domain $W^{2m,p}(\mathbb{R}^n)$. For any $\lambda \in \mathbb{C}$, $\Re \lambda \neq 0$, $(\lambda - iP_p(D))$ has a bounded inverse in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and we have the estimate

$$
\|(\lambda - iP_p(D))^{-1}\|_{L^p(\mathbb{R}^n)} < C|\Re \lambda|^{-1} |\lambda(\Re \lambda)^{-1} |^{(n+3)/(p-1/2)}.
$$

PROOF. We first notice that for any $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$, $Y(t)e^{-\lambda t}$ belongs to $T_k$ with its $p_k$ norm bounded by $C_k|\lambda|^{k} |\Re \lambda|^{-k-1}$. So, for any positive $\varepsilon$, $\mathcal{G}(Y(t)e^{-\lambda t})$ extends to a bounded operator in $L^{1+\varepsilon}(\mathbb{R}^n)$ with norm less than $C_k|\lambda|^k |\Re \lambda|^{-k-1}$, $k > n/2$.

On the other hand, $\mathcal{G}(Y(t)e^{-\lambda t})$ is obviously bounded in $L^2(\mathbb{R}^n)$ with norm less than $|\Re \lambda|^{-1}$. Interpolation between $L^{1+\varepsilon}$ and $L^2$ gives $\mathcal{G}(Y(t)e^{-\lambda t})$ bounded in $L^p(\mathbb{R}^n)$, $1 + \varepsilon \leq p \leq 2$, with norm less than

$$
|\Re \lambda|^{-1} |\lambda(\Re \lambda)^{-1}|^{(E(n/2)+1)(1+\varepsilon)(p-1-p'\varepsilon)/(1-\varepsilon)}
$$

which is always less than $|\Re \lambda|^{-1} |\lambda(\Re \lambda)^{-1}|^{(n+3)(p^{-1}-2^{-1})}$. An adjointness argument gives the same result for $1 + \varepsilon \leq p' \leq 2$. On $S(R^m)$, we have

$$(\lambda - iP(D))\mathcal{G}(Y(t)e^{-\lambda t}) = \mathcal{G}(Y(t)e^{-\lambda t})(\lambda - iP(D)) = I.$$ 

This proves that $\mathcal{G}_p(Y(t)e^{-\lambda t})$ is the inverse operator of the operator $(\lambda - iP(D))$ in $L^p(\mathbb{R}^n)$ with domain

$$\{U_0 \in L^p(\mathbb{R}^n) \text{ with } P(D)U_0 \in L^p(\mathbb{R}^n)\}.$$

But this is exactly $W^{2m,p}(\mathbb{R}^n)$, following the fact that

$$\partial^\alpha_y U_0 = \mathcal{F}(\xi^\alpha \mathcal{F}(U_0)) = \mathcal{F}(\xi^\alpha P(\xi)^{-1} \mathcal{F}(P(D)U_0))$$

and using hypothesis (HP) and the multipliers theorem of Stein [11] quoted above to prove that $\xi^\alpha P(\xi)^{-1}$ is a multiplier of $L^p(\mathbb{R}^n)$ if $|\alpha| \leq 2m$. For $\Re \lambda < 0$, the same proof is valid with $\mathcal{G}(-Y(-t)e^{\lambda t})$ in place of $\mathcal{G}(Y(t)e^{-\lambda t})$.

REMARK. For $p$ close to 1 or to infinity, and small values of $n$, a better estimate could be proved directly using Theorem 4 in $L^p(\mathbb{R}^n)$ to get

$$
\|(\lambda - iP(D))^{-1}\|_{L^p(\mathbb{R}^n)} < C|\Re \lambda|^{-1} |\lambda(\Re \lambda)^{-1}|^{k} \quad \text{with } k > n/2.
$$

In order to study the case $iP(D) + V(x)$, we will need an inverse result. We now introduce the abstract framework which will allow us to prove it.

C. Smooth distribution semigroups on a Banach space. Smooth distribution groups are a special case of distribution semigroups introduced by Lions [8]. They turn out to be the right tool to analyse differential operators in $L^p(\mathbb{R}^n)$. A particular class of these distributions was studied by us in [2]. We will not give those proofs here which are slight modifications of proofs given in [2].

DEFINITION 2. Let $X$ be a Banach space. A smooth distribution semigroup of order $k \in \mathbb{N}$ and exponential growth $\delta > 0$ is a linear mapping $\mathcal{G}_+$ from $D(\mathbb{R}^+)$ to $\mathcal{L}(X)$ such that

1. $e^{-\delta t} \mathcal{G}_+$ extends continuously to $T_k^+$: $\forall \phi \in D(\mathbb{R}^+), \|\mathcal{G}_+(e^{-\delta t} \phi)\|_{\mathcal{L}(X)} < C \rho_k(\phi)$.
2. $\forall \phi \in T_k^+, \forall \psi \in T_k^+, \mathcal{G}_+(\phi \ast \psi) = \mathcal{G}_+(\phi) \mathcal{G}_+(\psi)$.
3. There exists an everywhere dense subspace $D$ of $X$ such that for every $x \in D$ the distribution $\mathcal{G}_+ \otimes x$ is a continuous function on $\mathbb{R}_+$, with value $x$ at the origin.
NOTATION. The class of smooth distribution semigroups of order \( k \) and exponential growth \( \delta \) will be denoted by \( \sigma_+(k, \delta) \).

REMARK 6. (a) (i) makes sense because \( T_k^+ \) is an algebra for the (additive) convolution.

(b) We notice that for any \( \phi \in \mathcal{D}(\mathbb{R}_+) \), \( (s^{-1}\phi(s^{-1} \cdot))_{s > 0} \) is a bounded subset of \( T_k^+ \). The Ascoli theorem then shows that \( e^{-\delta t} \mathcal{G}_+(s^{-1}\phi(s^{-1} \cdot)) \) converges strongly to the identity. This implies

\[
N = \bigcap_{\phi \in T_k^+} \ker \mathcal{G}_+(\phi) = \{0\},
\]

\[
R = \bigcup_{\phi \in T_k^+} \text{Im} \mathcal{G}_+(\phi) \text{ is everywhere dense in } X.
\]

Let \( \mathcal{G}_+(-\delta') \) be the operator defined on \( R \) by

\[
\mathcal{G}_+(-\delta')\mathcal{G}_+(\phi)x = \mathcal{G}_+(-\phi')x \text{ for } x \in X, \phi \in T_k^+.
\]

The properties quoted in Remark 6 show that this definition is consistent and that \( \mathcal{G}_+(-\delta') \) is closable.

DEFINITION 3. The infinitesimal generator of \( \mathcal{G}_+ \in \sigma_+(k, \delta) \) is the closure of \( \mathcal{G}_+(-\delta') \).

Spectral properties of generators of smooth distribution semigroups are summarized in the following:

PROPOSITION 1. If \( A \) generates \( \mathcal{G}_+ \in \sigma_+(k, \delta) \) on a Banach space \( X \), then \( A \) is densely defined, closed, and for any \( \lambda \in \mathbb{C} \), \( \Re \lambda > \delta \), \( \lambda \) belongs to the resolvent set of \( A \). For any \( \mu \in \mathbb{C} \) with \( \Re \mu > 0 \), we have

\[
\| (\lambda I - A)^{-\mu} \|_{\mathcal{L}(X)} < C(\mu) |\lambda - \delta|^k (\Re \lambda - \delta)^{-k - \Re \mu},
\]

where \( C(\mu) = C\Gamma(\Re \mu)|\Gamma(\mu)|^{-1} |\mu|^k \).

PROOF. We will sketch it for \( \mu \) a positive integer. The proof for \( \mu \in \mathbb{C} \) is a slight modification of that given in [2]. We first notice that for \( \Re \lambda > \delta \), \( Y(t)t^{\mu-1}e^{-(\lambda-\delta)t} \) belongs to \( T_k^+ \) and it is easy to see that

\[
T(\mu)(\lambda I - A)^{-\mu} = \mathcal{G}_+(Y(t)t^{\mu-1}e^{-\lambda t}).
\]

Then, \( e^{-\delta t} \mathcal{G}_+ \) being bounded on \( T_k^+ \) gives the estimate if we compute

\[
p_k(Y(t)t^{\mu-1}e^{-(\lambda-\delta)t}) < C|\mu|^k |\lambda - \delta|^k (\Re \lambda - \delta)^{-k - \mu}. \quad \text{Q.E.D.}
\]

The inverse result is the following Hille-Yosida type estimate.

PROPOSITION 2. Let \( A \) be a closed, densely defined operator in a Banach space \( X \). If for any \( \lambda \in \mathbb{C} \), \( \Re \lambda > \delta \), \( \lambda \) belongs to the resolvent set of \( A \) and we have the estimate

\[
\| (\lambda I - A)^{-1} \|_{\mathcal{L}(X)} < C|\lambda - \delta|^k (\Re \lambda - \delta)^{-k - 1},
\]

then \( A \) generates a smooth distribution semigroup \( \mathcal{G}_+ \) of order \( k + 2 \) and exponential growth \( \delta \).

PROOF. For any \( \phi \in \mathcal{D}(\mathbb{R}_+) \) let \( \tilde{\phi} \) denote its Laplace transform

\[
\tilde{\phi}(\lambda) = \int_0^{\infty} e^{\lambda t} \phi(t) \, dt.
\]
Let Γ(c) be the line Re λ = c positively oriented in the direction of increasing Im λ. We define $\mathcal{G}_+$ by

$$\mathcal{G}_+(\phi) = \frac{1}{2\pi i} \int_{\Gamma(c+\epsilon)} \hat{\phi}(\lambda) (\lambda - A)^{-1} d\lambda.$$ 

This integral is obviously convergent in $\mathcal{L}(X)$. Standard holomorphic calculus shows point (i) of Definition 2. The resolvent identity to order $k + 2$ shows that $D(A^{k+2})$ can be taken for the dense subspace of point (ii) of Definition 2. It is straightforward to check that $A$ is the infinitesimal generator of $\mathcal{G}_+$. It remains to show that $e^{-\delta t} \mathcal{G}_+$ extends continuously to $T_k^+$. Using Fubini’s Theorem, one has

$$\mathcal{G}_+(e^{-\delta t}\phi) = \frac{1}{2\pi i} \int_{\Gamma(c)} \hat{\phi}(\lambda) (\lambda + \delta - A)^{-1} d\lambda$$

$$= \int_0^\infty t^{k+2} \phi^{(k+2)}(t) \left( \frac{1}{2\pi i} \int_{\Gamma(c)} \left( t \lambda^{-k-2} e^{\lambda t} (\lambda + \delta - A)^{-1} d\lambda \right) dt \right) dt$$

$$= \int_0^\infty t^{k+2} \phi^{(k+2)}(t) \left( \frac{1}{2\pi i} \int_{\Gamma(c)} \lambda^{-k-2} e^{\lambda (t^{-1} \lambda + \delta - A)^{-1} t^{-1} d\lambda} \right) dt,$$

and we only need to show that the norm of the path integral in $\mathcal{L}(X)$ is finite and does not depend on $t$. We change the integration path to $\Gamma(1)$ and use the estimate assumed on the resolvent operator to end the proof. Q.E.D.

We have the following regularity result for $\mathcal{G}_+ \in \sigma_+(k, \delta)$ on $D(A^k)$. Its proof is similar to that given in [2].

**PROPOSITION 3.** Let $\mathcal{G}_+ \in \sigma_+(k, \delta)$ on a Banach space $X$. Let $A$ be its infinitesimal generator. For any $x \in D(A^k)$, the distribution $\mathcal{G}_+ \otimes x$ is a function on $\mathbb{R}^+$, denoted by $e^{tA}x$, and we have the estimate

$$\forall t \geq 0, \quad \|e^{tA}x\| \leq (\|x\| + \|A^kx\|)(1 + tk)e^{\delta t}.$$

**DEFINITION 4.** Let $X$ be a Banach space and $A$ a linear operator in $X$. $A$ generates a smooth distribution group $\mathcal{G}$ of order $k$ and exponential growth $\delta$ (in short $\mathcal{G} \in \sigma(k, \delta)$) if $A$ and $-A$ generate elements of $\sigma_+(k, \delta)$.

**D. Constant coefficients evolution equations in $L^p(\mathbb{R}^n)$ (1 < p < $\infty$).**

**THEOREM 5.** Let $P(D)$ be a differential operator satisfying (HP) and $\mathcal{G}_p$ the distribution associated to $e^{tP(D)}$ in $L^p(\mathbb{R}^n)$. $\mathcal{G}_p$ is a smooth distribution group of order $k > n/2$ and exponential growth 0 in $L^p(\mathbb{R}^n)$. Its infinitesimal generator is $iP(D)$ with domain $W^{2m,p}(\mathbb{R}^n)$.

**PROOF.** This is a rewriting of Theorem 4. The only thing to be computed is the infinitesimal generator $A_p$ of $\mathcal{G}_p$. Following the proof of the corollary of Theorem 4, it is enough to show that the domain of $A_p$ is $\{U_0 \in L^p(\mathbb{R}^n) \text{ with } iP(D)U_0 \in L^p(\mathbb{R}^n)\}$ and that, on $R$, we have $A_p = iP(D)$.

For $\phi \in T_k^+$ and $U_0 \in L^p(\mathbb{R}^n)$ we have

$$A_p \mathcal{G}_p(\phi)U_0 = \mathcal{G}_p(-\phi')U_0 = \mathcal{F}(-\hat{\phi}'(P(\xi))\mathcal{F}U_0)$$

$$= \mathcal{F}(iP(\xi)\hat{\phi}(P(\xi))\mathcal{F}U_0) = iP(D)\mathcal{G}_p(\phi)U_0.$$
If \( U_0 \in L^p(\mathbb{R}^n) \) and \( iP(D)U_0 \in L^p(\mathbb{R}^n) \), let \( \phi \in \mathcal{D}(\mathbb{R}_+) \) with \( \int \phi = 1 \) and let \( \phi_s = s^{-1}\phi(s^{-1}) \). Then \( \mathcal{G}_p(\phi_s)U_0 \) converges to \( U_0 \) in \( L^p(\mathbb{R}^n) \) because \( \phi_s(\mathcal{G}_p(\xi)) \) converges to one in \( M_p \), applying Stein [11], and \( \mathcal{G}_p(-\phi_s')U_0 = \mathcal{G}_p(\phi_s)(iP(D)U_0) \) converges in \( L^p(\mathbb{R}^n) \) to \( iP(D)U_0 \). So \( D(A_p) \supset W^{2m,p}(\mathbb{R}^n) \). For the inverse inclusion we just note that \( \mathcal{G}_p(-\phi_s')U_0 \) converges to \( iP(D)U_0 \) in \( \mathcal{D}'(\mathbb{R}^n) \) if \( U_0 \in D(A_p) \). So \( iP(D)U_0 \in L^p(\mathbb{R}^n) \). Q.E.D.

REMARK 7. (a) Using the corollary of Theorem 4 and Proposition 2, we can improve the order of \( \mathcal{G}_p \) in \( L^p(\mathbb{R}^n) \) to \( k > 2 + (n + 3)|1/p - 1/2| \).

(b) Theorem 5 and Proposition 1 improve the estimate of the corollary of Theorem 4. They give an estimate for the powers of the resolvent of \( iP_p(D) \).

(c) If \( P \) is homogeneous, the order of \( \mathcal{G}_p \) can be improved to \( k > n|1/p - 1/2| \).

PROPOSITION 4. For \( U_0 \in W^{2mk,p}(\mathbb{R}^n), k > n/2, \) the solution of the Cauchy problem (**) is a continuous function in the \( t \)-variable with values in \( L^p(\mathbb{R}^n) \). We have the estimate

\[
\|e^{itP(D)}U_0\|_{L^p(\mathbb{R}^n)} < C(1 + |t|^k)\|U_0\|_{W^{2mk,p}(\mathbb{R}^n)}.
\]

PROOF. This is a translation of Proposition 3 using Theorem 5.

5. The Cauchy problem \( \partial_t - iP(D) - V(x) \): \( (L^p, L^p) \) estimates. We consider the Cauchy problem

\[
(*) \quad \partial U/\partial t = (iP(D) + V(x))U; \quad U(0, x) = U_0(x) \in L^p(\mathbb{R}^n).
\]

We are now in position to prove that under the subsequent assumptions on \( P \) and \( V \), the solution is a distribution in the \( t \)-variable with values in \( L^p(\mathbb{R}^n) \). The order of this distribution is any integer \( k \) with \( k > (n + 3)|p^{-1} - 2^{-1}| + 4 \). This will imply a precise estimate in \( \mathcal{L}(L^p(\mathbb{R}^n)) \) of the resolvent operator of \( iP(D) + V(x) \).

Here \( iP(D) + V(x) \) will mean this differential operator with domain \( \{U_0 \in W^{2mk,p}(\mathbb{R}^n)\} \) with \( VU_0 \in L^p(\mathbb{R}^n) \).

Let \( c \) be an integer with \( c > n/2m - 1 \).

Let \( q(m, n) = n(2m - 1)/((m - 1)(n - 3) - 2) \).

Let \( q'(m, n) \) be the conjugate index \( q(m, n)^{-1} + (q'(m, n))^{-1} = 1 \).

Let \( q = q'(m, n) \).

THEOREM 6. Assume that \( (i) \ 2c/(c + 1) < p < 2c/(c - 1) \).

(\( ii \)) \( P(D) \) satisfies (H1), (H2), (H3') and (HP).

(\( iii \)) \( V = V_1 + V_2 \) with \( V_1 \in L^{\alpha}(\mathbb{R}^n), \ r_1^{-1} = |p^{-1} - p'^{-1}|, \ V_2 \in L^{\gamma^2}(\mathbb{R}^n), \ \tilde{q}^{-1}|p^{-1} - p'^{-1}| < r_2^{-1} \leq |p^{-1} - p'^{-1}| \).

Then for any integer \( k \), \( k > (n + 3)|p^{-1} - 2^{-1}| + 2 \), and some \( \delta > 0 \), \( iP(D) + V(x) \) generates a smooth distribution group in \( L^p(\mathbb{R}^n) \) of order \( k \) and exponential growth \( \delta \).

PROOF. (a) First we note that \( iP(D) + V \) and \( -(iP(D) + V) \) satisfy the same assumptions. So we just have to prove that \( iP(D) + V \) generates a smooth distribution semigroup of order \( k \) and exponential growth \( \delta \).

(b) For \( p \leq 2 \), we write the resolvent operator, whenever it exists, in the form

\[
(\lambda - (iP(D) + V))^{-1} = (I - V(\lambda - iP(D))^{-1})^{-1} (\lambda - iP(D))^{-1}.
\]

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By the Neumann series, \((I - V(A - iP(D))^{-1})^{-1}\) will exist and be bounded if 
\[\|V(A - iP(D))^{-1}\|_{L^p} < 1.\]
But
\[
\|V(A - iP(D))^{-1}\|_{L^p} \leq \|V_1\|_{L^{p'}_1} \|V_1\|_{L^{p'}_1} \|\lambda - iP(D)\|_{L^{p'}}^{-1} \|A - iP(D)\|_{L^p}^{-1}
+ \|V_2\|_{L^{p'}_2} \|\lambda - iP(D)\|_{L^{p'}}^{-1} \|A - iP(D)\|_{L^p}^{-1}
\leq C \|V_1\|_{L^{p_1}} |\Re \lambda|^{-1} (1 + |\Re \lambda|^\theta)
+ C \|V_2\|_{L^{p_2}} |\Re \lambda|^{-1} (|\Re \lambda|^{-\epsilon} + |\Re \lambda|^\epsilon)
\]
by Theorems 3 and 3'. Following these theorems, we must have
\[2c/(c + 1) < p \leq 2 \text{ and } r_1 = p^{-1} - p_1^{-1},\]
\[q(m, n, p) < s_2 \leq p' \text{ and } r_2^{-1} = p^{-1} - s_2^{-1} \text{ so } p^{-1} - (q(m, n, p))^{-1} < r_2^{-1} \leq p^{-1} - p_1^{-1},\]
\[\theta = p^{-1} - p_1^{-1},\]
\[c \text{ is an integer with } c > n/2m - 1, \text{ and}\]
\[c' \text{ is an integer with } c' > n(p' - s_2)/s_2(p' - 2).\]
Finally we have for \(|\Re \lambda| > 1\)
\[\|V(A - iP(D))^{-1}\|_{L^p} < C(\|V_1\|_{L^{p_1}} + \|V_2\|_{L^{p_2}}) (1 + |\Re \lambda|^\theta) |\Re \lambda|^{-1}.\]
This must be strictly less than one for large \(|\Re \lambda|\): we must add the condition \(c\theta < 1\), and this is true when \(p > 2c/(c + 1)\). Thus there exists some positive \(\delta\) such that for \(|\Re \lambda| > \delta\), 
\[\|V(A - iP(D))^{-1}\|_{L^p} < 1/2.\]
In this case, we have the following estimate for the resolvent by the corollary of Theorem 4:
\[\|\lambda - (iP(D) + V)\|_{L^p} < 2C |\Re \lambda|^{-1} |\lambda|^{-1} |\lambda|^{-1} (|n + 3|p^{-1} - 2^{-1}|).\]
This implies, by the abstract Proposition 2, that \(iP(D) + V\) generates a smooth distribution semigroup of order any integer \(k\) with \(k > (n + 3)(p^{-1} - 2^{-1}) + 2\) and of exponential growth \(\delta\). All encountered assumptions are the assumptions given in the statement of the theorem for \(p \leq 2\), but the assumption on \(r_2\). The condition \(p^{-1} - (q(m, n, p))^{-1} < r_2^{-1}\) is equivalent to \((p^{-1} - p_1^{-1})\bar{q}^{-1} < r_2^{-1}\) by definition of \(q(m, n, p)\)
\[p^{-1} - (q(m, n, p))^{-1} = p^{-1} - p_1^{-1} - \theta(q(m, n))^{-1}
= (p^{-1} - p_1^{-1})(1 - (q(m, n))^{-1}) = (p^{-1} - p_1^{-1})\bar{q}^{-1}.\]
(c) For \(p \geq 2\), we use an adjointness argument to obtain
\[\|\lambda - (iP(D) + V)\|_{L^p} = \|\lambda - (iP(D) + \bar{V})\|_{L^{p'}}.\]
By the previous computation, this leads to the same estimate as in the case \(p < 2\).
But the conditions are to be written on \(p'\) in place of \(p\): \(p' > 2c/(c + 1)\) is equivalent to \(p < 2c/(c - 1)\).
The conditions on \(r_1\) and \(r_2\) remain unchanged: they depend on \(|p^{-1} - p_1^{-1}|.\)
The condition on \(c'\) is changed, but we only need \(c' \geq 0\), which remains the case. Q.E.D.

**Remark.** If \(V_2 = 0\), then assumption (H3') in Theorem 1 can be replaced by (H3).

**Remark.** If \(P(D)\) is homogeneous, then under the assumptions of Theorem 6, 
\((iP(D) + V)\) generates a smooth distribution group in \(L^p(\mathbb{R}^n)\) of order any integer \(k\) with \(k > n|1/p - 1/2| + 2\) and of exponential growth \(\delta\).
COROLLARY 1. Assume $P$, $V$ and $p$ satisfy the assumptions of Theorem 6. Then there exists $\delta > 0$ such that for any $\lambda$ with $|\text{Re}\, \lambda| > \delta$ and any $\mu$, $\text{Re}\, \mu > 1$, we have the estimate

$$\|(\lambda - (iP(D) + V))^{-\mu}\|_{L^p(R^n)} < C(\mu)|\text{Re}\, \lambda - \varepsilon\delta|^{-\text{Re}\, \mu}(|\lambda - \varepsilon\delta| |\text{Re}\, \lambda - \varepsilon\delta|^{-1})^{k},$$

where $\varepsilon$ is the sign of $\text{Re}\, \lambda$, $k$ is any integer with $k > 2 + (n + 3)|p^{-1} - 2^{-1}|$ and $C(\mu) = C\Gamma(\text{Re}\, \mu)|\Gamma(\mu)|^{-1}|\mu|^k$.

PROOF. It is a consequence of Theorem 6 and Proposition 1.

COROLLARY 2. Assume $P$, $V$ and $p$ satisfy the assumptions of Theorem 6. Let $k$ be any integer with $k > (n + 3)|p^{-1} - 2^{-1}| + 2$. Assume $U_0 \in L^p(R^n)$ with $(iP(D) + V)^kU_0 \in L^p(R^n)$. Then the solution $U(t, x)$ of the Cauchy problem (\#) with Cauchy data $U_0$ is a continuous function of the $t$-variable with values in $L^p(R^n)$ and for any $t \in R$ we have the estimate

$$\|U(t, \cdot)\|_{L^p(R^n)} \leq C(\|U_0\|_{L^p} + \|(iP(D) + V)^kU_0\|_{L^p})(1 + |t|^k)e^{\delta t}.$$

PROOF. It is a consequence of Theorem 6 and Proposition 3.

REFERENCES