LOCAL AND GLOBAL ENVELOPES
OF HOLOMORPHY OF DOMAINS IN C^2

BY

ERIC BEDFORD

ABSTRACT. A criterion is given for a smoothly bounded domain D \subset C^2 to be locally extendible to a neighborhood of a point z_0 \in \partial D. (This result may also be formulated in terms of extension of CR functions on \partial D.) This is related to the envelope of holomorphy of the semitubular domain

\[ \Omega(\Phi) = \{ (z, w) \in C^2 : \text{Re } w + r^k \Phi(\theta) < 0 \}, \]

where \( r = |z|, \theta = \text{arg}(z) \). Necessary and sufficient conditions are given for the envelope of holomorphy of \( \Omega(\Phi) \) to be \( C^2 \). These conditions are equivalent to the existence of a subharmonic minorant for \( r^k \Phi(\theta) \).

1. Introduction. Let us consider a smoothly bounded domain D \subset C^2 and ask whether D is locally extendible at p \in \partial D, i.e. for every open set U containing p do all holomorphic functions on D \cap U extend holomorphically through p?

This question has been answered when \( \partial D \) is pseudoconcave and real analytic at p (see [3]) and when \( \partial D \) has so-called "type k" with k odd (see [2, 5, 10]). The question of local extension of holomorphic functions from D is essentially equivalent to the question of local extension of CR functions from \( \partial D \) (see [1, 8]). However, we do not discuss CR functions further since our contribution is to deal with the geometric structure of certain envelopes, and we would like our presentation to be as self-contained as possible.

We may make a holomorphic change of coordinates (z, w) in a neighborhood of p such that p = (0, 0), w = u + iv, and that \( \partial D \) is given near p by the equation

\[ u + p_k(z) + R(z, v) < 0, \]

where \( p_k(z) = \sum_{j=1}^{k-1} a_j z^{i\bar{z}^{k-j}} \)

is a real, homogeneous polynomial of degree k, and the remainder is given by

\[ R(z, v) = O(v^2 + |vz| + |z|^{k+1}) \]
(cf. [3]). With a holomorphic change of variables \( w' = w + \alpha w^2 + \beta zw \), \( z' = z \), \( \partial D \) can be given by
\[
u + cv^2 + R'(z, v) + p_k(z) < 0,
\]
where \( c \in \mathbb{R} \) is arbitrary, and
\[
R'(z, w) = O\left(|v|^3 + |vz^2| + |z|^{k+1}\right).
\]
We will be interested in domains that satisfy the following stronger condition: For any \( \varepsilon > 0 \), there exists \( c > 0 \) such that
\[
(2) \quad |R'(z, v)| = O\left(cv^2 + \epsilon |z|^k\right).
\]
If (2) holds, then for every \( \varepsilon > 0 \), there exists \( \eta > 0 \) and a change of coordinates as above such that
\[
(3) \quad \{(z, w) < \eta\} \cap D \subset \{u + p_k(z) < \varepsilon |z|^k\}.
\]
Writing \( z = re^{i\theta} \) and \( p_k(z) = r^k\Phi(\theta) \), we see that the local study of \( D \) at \( (0,0) \) is related to the domain
\[
\Omega(\Phi) = \{(z, w) \in \mathbb{C}^2: \text{Re } w + r^k\Phi(\theta) < 0\}.
\]
To make the connection between \( D \) and \( \Omega(\Phi) \), we will need to discuss (global) envelopes of holomorphy. The envelope of holomorphy \( E(D) \) of a domain \( D \subset \mathbb{C}^n \) is a Riemann domain \( \pi: E(D) \rightarrow \mathbb{C}^n \) with \( i: D \rightarrow E(D) \) and \( E(D) \) is the minimal domain of holomorphy such that every function \( f \in \mathcal{O}(D) \) extends holomorphically to \( E(D) \). A convenient method for staying within the class of domains in \( \mathbb{C}^n \) while taking envelopes is to consider \( D \) which are starshaped with respect to the origin, i.e., \( \delta_t(D) \subset D \), where \( \delta_t(z) = (tz_1, \ldots, tz_n) \), \( 0 < t < 1 \). If \( D \subset \mathbb{C}^n \) is starshaped, then the envelope is a starshaped domain in \( \mathbb{C}^n \) with \( D \subset E(D) \subset \mathbb{C}^n \). To prove this assertion it suffices to show that the projection \( \pi \) is one-to-one. The mapping \( \delta_t \) has a holomorphic continuation to a map \( \delta_t: E(D) \rightarrow E(D) \). We note that \( \pi\delta_t = \delta_t\pi \), \( \delta_1 \) is the identity map, and \( \delta_0 = \lim_{t \rightarrow 0} \delta_t \) is the constant \( i(0) \). Let \( z_1, z_2 \in E(D) \) be points such that \( \pi(z_1) = \pi(z_2) \), and let \( \gamma_j, j = 1, 2, \) be the path given by \( \gamma_j(t) = \delta_t(z_j), 0 \leq t \leq 1 \).

Now \( \sigma_1 \) and \( \sigma_2 \) project under \( \pi \) to the same path in \( \mathbb{C}^n \), and \( \gamma_1(0) = \gamma_2(0) = i(0) \). Since \( \pi \) is locally invertible, the paths \( \sigma_1 \) and \( \sigma_2 \) coincide, and thus \( z_1 = \gamma_1(1) = \gamma_2(1) = z_2 \).

It follows (e.g. from a result of Docquier and Grauert [7]), that if \( D \) is starshaped, then it is a Runge domain, i.e. every holomorphic function on \( D \) may be uniformly approximated by polynomials on compact subsets.

The domain \( \Omega(\Phi) \) is invariant under the transformations
\[
(4) \quad (z, w) \rightarrow (z, w + \xi), \quad \xi \in \mathbb{C}, \text{Re } \xi < 0,
\]
\[
(5) \quad (z, w) \rightarrow (tz, t^k w), \quad 0 < t < \infty.
\]
The envelope of holomorphy has the same invariance and is thus given by
\[
E(\Omega(\Phi)) = \{(z, w) \in \mathbb{C}^2: \text{Re } w + r^k\Phi(\theta) < 0\} = \Omega(\Phi),
\]
where \( r^k\Phi(\theta) \) is the greatest subharmonic minorant of \( r^k\Phi(\theta) \). (This is a special case of a result on semitubular domains, see [6].)
We may approximate $\Omega(\Phi)$ by the truncated domain
\[
\Omega_\lambda(\Phi) = \Omega(\Phi) \cap \{|z| < \lambda, |u| < \lambda^k, |v| < c\lambda^k\}
\]
for $0 < \lambda < \infty$. Since $\Omega_\lambda(\Phi)$ is starshaped with respect to $(0, -c\lambda^k/2)$ for $c$ sufficiently large, the envelope is again starshaped. Further, $\Omega_\lambda(\Phi)$ is mapped biholomorphically to $\Omega_\lambda(\Phi)$ by the transformation (5), and so $E(\Omega_\lambda(\Phi))$ is also mapped to $E(\Omega_\lambda(\Phi))$. Thus
\[
E(\Omega(\Phi)) = \bigcup \limits_\lambda E(\Omega_\lambda(\Phi)),
\]
and so $(0,0) \in E(\Omega(\Phi))$ if and only if $(0,0) \in E(\Omega_\lambda(\Phi))$ for all $\lambda$.

The question of local extendibility of $D$ at $(0,0)$ is tied to the global question for $\Omega(\Phi)$: Does $(0,0)$ belong to the envelope of holomorphy $E(\Omega(\Phi))$ of $\Omega(\Phi)$? There are two possibilities:

(i) $(0,0) \in E(\Omega(\Phi))$, and in this case $E(\Omega(\Phi)) = \mathbb{C}^2$.

(ii) $(0,0) \notin E(\Omega(\Phi))$, and $E(\Omega(\Phi)) = \Omega(\Phi)$ with $\Phi$ not identically $-\infty$.

**Proposition.** If there exists $\varepsilon > 0$ such that $E(\Omega(\Phi + \varepsilon)) = \mathbb{C}^2$, then for all open $U$ containing $(0,0)$, every analytic function on $U \cap D$ extends analytically to a neighborhood of $(0,0)$.

Conversely, if $D$ satisfies (2), and if $E(\Omega(\Phi - \varepsilon)) \neq \mathbb{C}^2$ for some $\varepsilon > 0$, then there exists $\eta > 0$ and a function
\[
f \in \mathcal{O}(D \cap \{|(z,w)| < \eta\})
\]
which cannot be extended holomorphically past $(0,0)$.

**Proof.** If $(0,0)$ is in the envelope of $E(\Omega(\Phi + \varepsilon))$, there is a compact $K \subset \Omega(\Phi + \varepsilon)$ such that $|f(0,0)| = |f|_K$ for all $f \in \mathcal{O}(\Omega(\Phi + \varepsilon))$. Since $K$ is compact, we may shrink $\varepsilon$ if necessary, so that $K \subset \omega_{\varepsilon}$, where
\[
\omega_{\varepsilon} = \{u + p_\varepsilon(z) + \varepsilon|z|^k + \varepsilon|v|\} < 0.
\]
By (1), we may choose $\eta$ sufficiently small such that $D \supset \{(z,w)| < \eta\} \cap \omega_{\varepsilon}$. Now $\omega_{\varepsilon}$ is invariant under the transformation (5), so we may apply (5) to $K$ with $t$ small to have $K \subset \{(z,w)| < \eta\} \cap \omega_{\varepsilon}$.

Finally, since $D \cap \{(z,w)| < \eta\}$ is starshaped for $\eta$ small, it is Runge. Thus, $f \in \mathcal{O}(D \cap \{(z,w)| < \eta\})$ may be approximated by polynomials uniformly on $K$. Since $(0,0)$ is in the hull of $K$, we may extend $f$ past $(0,0)$.

Now we prove the converse statement. If $D$ satisfies (2), then we have (3), and so for $\Psi = \Phi - \varepsilon$
\[
D \cap \{(z,w)| < \eta\} \subset \Omega(\Phi).
\]
Since $\Omega(\Phi)$ is a domain of holomorphy there exists $f \in \mathcal{O}(\Omega(\Phi))$ which cannot be continued past $(0,0)$.

**Remarks.** The first part of the Proposition can be used to give sufficient conditions for local extension of functions from domains $D \subset \mathbb{C}^n$. For this, let $P$ be a complex 2-plane intersecting $\partial D$ transversally at $z_0 \in \partial D$. If $D \cap P$ satisfies the
first hypotheses of the Proposition in a neighborhood of \( z_0 \) in \( P \), then there is a compact \( K \subset D \cap P \) such that \( z_0 \) is in its polynomial hull. For \( \epsilon > 0 \) sufficiently small, a closed \( \epsilon \)-neighborhood \( K^\epsilon \) of \( K \) is contained in \( D \). Since \( K^\epsilon \) contains all \( \epsilon \)-translates of \( K \), the polynomial hull of \( K^\epsilon \) contains all \( \epsilon \)-translates of \( z_0 \), i.e. an \( \epsilon \)-neighborhood of \( z_0 \). Thus if we have local extension in a 2-dimensional slice of \( D \), we have local extension from \( D \).

By writing the Laplacian in polar coordinates,

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2},
\]

we see that \( \Delta(r^k \Phi(\theta)) \geq 0 \) if and only if \( \mathcal{L} \Phi \geq 0 \), where \( \mathcal{L} = d^2/d\theta^2 + k^2 \). Of course, \( \mathcal{L} \psi = 0 \) if and only if \( \psi(\theta) = c \sin(k\theta) + d \cos(k\theta) \) and in this case \( r^k \psi(\theta) = \text{Re}((d - ic)z^k) \). The intervals in \( \theta \) where \( \Phi \) is positive or negative are of some importance. If \( \mathcal{L} \Phi > 0 \), then the intervals where \( \{ \Phi < 0 \} \) have length \( < \pi/k \) and the intervals where \( \{ \Phi > 0 \} \) have length \( > \pi/k \). This follows from (10) below.

It is also useful to adjoin nearby intervals.

**Definition.** Given an open set \( \mathcal{O} \subset \mathbb{R} \), the **amalgamated component** \( \tilde{I} \) of an interval \( I \subset \mathcal{O} \) is the smallest connected, open interval \( \tilde{I} \supset I \) with the property: If \( J \subset \mathcal{O} \) is an open interval with \( \text{dist}(J, I) < \pi/k \), then \( J \subset \tilde{I} \).

**Definition.** An upper semicontinuous periodic function \( \Phi \) on \( \mathbb{R} \) with period \( 2\pi \) has a **wide (amalgamated) sector** if either

(i) \( 0 < k < 1/2 \), and \( \Phi(\theta) < 0 \) for some \( \theta \), or

(ii) \( k > 1/2 \), and there exist \( c_1, c_2 \in \mathbb{R} \) and \( \epsilon > 0 \) such that an (amalgamated) component of

\[
\mathcal{O}(\epsilon, c_1, c_2) = \{ \theta \in \mathbb{R}: \Phi(\theta) + \epsilon + c_1 \sin(k\theta) + c_2 \cos(k\theta) < 0 \}
\]

has length \( \geq \pi/k \).

Note that the length will be \( > \pi/k \) if we take \( \epsilon > 0 \) smaller. By this same remark we see also that if \( \Phi \) is continuous and has no wide sectors, then for \( 0 < c < \infty \), there exists \( \epsilon_0 > 0 \) such that every connected component of \( \mathcal{O}(\epsilon, c_1, c_2) \) has length \( \leq \pi/k - \epsilon_0 \) if \( |c_1| + |c_2| < c \) and \( 0 < \epsilon \leq \epsilon_0 \).

**Theorem.** Let \( \Phi \) be periodic and u.s.c. on \([0, 2\pi]\). Then the envelope of holomorphy \( E(\Omega(\Phi)) = \mathbb{C}^2 \) if and only if \( \Phi + \epsilon \) has a wide amalgamated sector for some \( \epsilon > 0 \).

**Remark.** The “only if” part of the Theorem is easily seen. If \( E(\Omega(\Phi)) \neq \mathbb{C}^2 \), then there is a subharmonic \( r^k \Phi(\theta) \leq r^k \Phi(\theta) \). Thus each interval of \( \{ \Phi + \epsilon < 0 \} \) lies in an interval of \( \{ \Phi + \epsilon < 0 \} \), which has length \( < \pi/k \), since \( \mathcal{L} \Phi(\theta) > 0 \). Further, since the sectors of \( \{ \Phi + \epsilon < 0 \} \) are separated by a distance \( > \pi/k \), the amalgamated components of \( \{ \Phi + \epsilon < 0 \} \) lie in the components of \( \Phi + \epsilon < 0 \).

**Remark.** The works [2 and 9, 10] use the weaker “sector property”, which is just that \( \Phi \) has a wide sector. We note that if \( \Phi \) does not have the sector property, and if \( I_1 \) and \( I_2 \) are intervals of \( \mathcal{O}(\epsilon, c_1, c_2) \), and if \( \text{dist}(I_1, I_2) < \pi/k \), then \( I_1 \cup I_2 \) is contained in an interval of length \( < \pi/k \).

(To see this, we may assume, to the contrary, that \( 0 \in I_1 \) and \( \pi/k \in I_2 \). Then we make \( c_1 \) very large and negative so that \( [0, \pi/k] \subset \mathcal{O}(\epsilon, c_1, c_2) \).)
From this we conclude that if $\Phi$ has the sector property, and if $\partial(\epsilon, c_1, c_2)$ contains no more than two intervals (for all $\epsilon, c_1, c_2$), then $\Phi$ has a wide amalgamated sector. The case $k = 4$, which was treated in [2], is a special case of this situation.

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2. Construction of the envelope. Since $r^k\Phi(\theta)$ is subharmonic and constant on the sets $\{\theta = \text{const}\}$, it follows that $\Phi$ is bounded. Further, since $\mathcal{L}\Phi \geq 0$, we have $\Phi'' \geq -\text{const}$, and so $\Phi \in C^1$. Thus if the envelope $E(\Omega(\Phi)) \neq C^2$, and if $k > 1$, the boundary $\partial E(\Omega(\Phi))$ is $C^1$ smooth. In general, however, $\Phi \notin C^2$.

We may approximate $\Phi + \delta$ from below by $\tilde{\Phi} + \delta$, where $\tilde{\Phi} = \tilde{\Phi} \chi_\epsilon$ is a usual smoothing in $\theta$, and $0 < \delta_\epsilon < \delta$, $\lim_{\epsilon \to 0} \delta_\epsilon = \delta$. Thus

$$\Phi + \delta)(\theta) = \sup\{h(\theta): h \text{ is of class } C^2, h \leq \Phi + \delta, \mathcal{L}h \geq 0\}.$$

Remark. In terms of the envelope (6) our question is whether the competing family of subsolutions is nonempty. Thus an alternative statement of our Theorem is: $r^k\Phi(\theta)$ has a subharmonic minorant if and only if $\Phi(\theta) + \epsilon$ does not have a wide amalgamated sector for any $\epsilon > 0$.

The envelope formulation (6) also suggests the structure of $\tilde{\Phi}$:

$$\mathcal{L}\tilde{\Phi} = 0 \text{ on } \partial = \{\tilde{\Phi} < \Phi\},$$

$$\Phi = \tilde{\Phi} \text{ and } \nabla\Phi = \nabla\tilde{\Phi} \text{ on } \partial\Omega.$$

We will construct $\Phi$ in the manner suggested by Figure 1. If $E = \{\mathcal{L}\Phi < 0\}$ is the set where the Levi form is negative, we must have $E \subset \{\tilde{\Phi} < \Phi\}$, and $\Phi$ is obtained by patching solutions $\psi_j$ of $\mathcal{L}\psi_j = 0$ onto $\Phi$ so that they satisfy (7) and (8) above.

The last feature of the construction we shall require is

$$\text{each interval in } \partial = \{\Phi < \tilde{\Phi}\} \text{ has length } < \pi/k.$$
Without (9), the solution constructed according to Figure 1 is not unique. For instance, if $\Phi(\theta) = \sin(k\theta) + 1$, then $\mathcal{L}\Phi > 0$, and $\Phi = \tilde{\Phi}$. If we take $\psi_1(\theta) = 0$, $-\pi/2k < \theta < 3\pi/2k$, and equal to $\tilde{\Phi}$ for other values of $\theta$, then the resulting solution $\tilde{\Phi}$ satisfies $\mathcal{L}\tilde{\Phi} > 0$, but $\{\tilde{\Phi} < \tilde{\Phi}\} = (-\pi/2k, \pi/2k)$.

We will use the following version of the Sturm Comparison Theorem (see [4]):

$\psi_1, \psi_2 \in C^2$ and $\mathcal{L}\psi_1 \geq \mathcal{L}\psi_2$, and if

$$\psi_1(\theta_0) = \psi_2(\theta_0), \psi'_1(\theta_0) \geq \psi'_2(\theta_0),$$

then

$$\psi_1(\theta) \geq \psi_2(\theta) \text{ for } \theta_0 < \theta < \theta_0 + \pi/k.$$ (10)

To prove (10), we consider $\psi = \psi_1 - \psi_2$, and we may add $e((\theta - \theta_0) + (\theta - \theta_0)^2)$ so that $\psi'(\theta_0) > 0$ and $\mathcal{L}\psi > 0$ on $(\theta_0, \theta_0 + \pi/k)$. Now we will show that $\psi > 0$ on $(\theta_0, \theta_0 + \pi/k)$. Let $\theta_1 > \theta_0$ be the first point where $\psi(\theta_1) = 0$. We may assume $\psi'(\theta_1) < 0$. We set

$$h(\theta) = \arctan(\psi'(\theta)/\psi(\theta)).$$

Since $h(\theta_0) = +\pi/2$ and $h(\theta_1) = -\pi/2$ we have

$$\int_{\theta_0}^{\theta_1} h'(\theta) d\theta = -\pi.$$

Further, since $\mathcal{L}\psi > 0$, we have $\psi'' > -k^2\psi^2$, and with this we may compute that $h'(\theta) > -k$. Thus we have

$$-\pi = \int_{\theta_0}^{\theta_1} h'(\theta) d\theta > -(\theta_1 - \theta_0)k,$$

and so $\theta_1 - \theta_0 > \pi/k$ which yields (10).

We will use the notation $\psi_p$ for the function

$$\psi_p(\theta) = c \sin(k\theta) + d \cos(k\theta)$$

such that $\psi_p(p) = \Phi(p)$ and $\psi'_p(p) = \Phi'(p)$.

Some properties of $\psi_p$ are formulated in the following lemmas and are illustrated in Figure 2.

**Lemma 1.** If $\mathcal{L}\Phi(p) > 0$, then there exists $\epsilon > 0$ such that $\psi_p(\theta) \leq \Phi(\theta)$ for $\theta \in (p - \epsilon, p + \epsilon)$. If $\mathcal{L}\Phi(p) > 0$ for $p_2 < p < p_1$, then $\psi_{p_2}(\theta) < \psi_{p_1}(\theta)$ for $p_1 < \theta < p_2 + \pi/k$. 

![Figure 2](http://www.ams.org/journal-terms-of-use)
Proof. The first statement is just the comparison (10). The second statement also follows from (10). If we replace \( \psi_p \) by \( \tilde{\psi}_q = \psi_q - \Phi \), then for \( |q - p| \) small

\[
\tilde{\psi}_q(\theta) = -k(q)(\theta - q)^2 + o((\theta - q)^2),
\]

and \( k(q) > 0 \). If \( p_2 < p_1 < p \), and \( |p_2 - p| \) is small, then \( \tilde{\psi}_{p_1} \) and \( \tilde{\psi}_{p_2} \) will intersect at a point \( q \in (p_2, p_1) \). Thus \( \psi_{p_1} \) and \( \psi_{p_2} \) will intersect as in Figure 2, and so by (10) we have \( \psi_{p_1}(\theta) > \psi_{p_2}(\theta) \) for \( \theta \in (q, q + \pi/k) \).

Lemma 2. Let \( \Phi \) have no wide sectors. If \( (a, b) \) is an open interval on which \( \mathcal{L}\Phi < 0 \), then \( \psi_a(\theta) > \Phi(\theta) \) for \( \theta \in (a, b) \).

Proof. By (10), \( \psi_a(\theta) > \Phi(\theta) \) holds for \( a < \theta < \min(a + \pi/k, b) \). Thus the result holds unless \( a + \pi/k < b \). But in this case we have \((a, a + \pi/k) \subset \{ \Phi - \psi_a < 0 \}\) which is a wide sector.

Lemma 3. If \( \Phi - \delta \) has no wide sectors for some \( \delta > 0 \) and if \( E = \{ \mathcal{L}\Phi < 0 \} \) consists of a single interval \( E = (a, b) \), then \( \Phi \) exists.

Proof. Note that if \( E \neq \emptyset \), then by definition \( k > 1/2 \). By Lemma 2, \( \psi_a(\theta) > \Phi(\theta) \) for \( \theta \in E \). And by Lemma 1, \( \psi_p(\theta) < \psi_a(\theta) \) holds for \( p < a \) and \( a < \theta < p + \pi/k \). Further, we claim that there is a wide sector unless \( |q - p| < \pi/k \) holds for all \( p \) (\( q \) is the point where \( \psi_p \) crosses \( \Phi \) from above). First, it is evident that \( |a - q_0| < \pi/k \). Thus for \( p_1 \) near \( a \), it follows that \( |p_1 - r_1| < \pi/k \), where we write \( \{ \Phi < \psi_{p_1} \} \cap (p_1, q_0) = (r_1, q_1) \). Replacing \( \Phi \) by

\[
\Phi_1 = \Phi - \varepsilon \sin(k(\theta - p_1 + \varepsilon))
\]

for \( \varepsilon > 0 \) small, we obtain a small interval \((p_1 - \delta, p_1 + \delta) \subset \{ \Phi_1 < \psi_{p_1} \}\), in addition to \((r_1, q_1) \subset \{ \Phi_1 < \psi_{p_1} \}\). Thus by the Remark at the end of the first section, we have

\[
| (p_1 - \delta) - q_1 | < \pi/k.
\]

Letting \( \varepsilon \) tend to zero, we have \( |q_1 - p_1| < \pi/k \). However, by the remark after the definition of wide sector, we see that \( |q_1 - p_1| < \pi/k \).

We conclude from this that as we slide \( p_2 \) to the left, we must have \( |p_2 - a| < |p_2 - q_2| < \pi/k \) unless the interval \((r_2, q_2) = \{ \Phi < \psi_{p_2} \}\) disappears for some value, say \( p = p_2 \). It is clear, then, that the curve \( \psi_{p_2} \) satisfies (7)–(9).

Proof of the Theorem. Let us start by choosing a sequence \( \Phi_1 \geq \Phi_2 \geq \cdots \) of real analytic functions with \( \Phi_j \rightarrow \Phi \). If there is an envelope \( \tilde{\Phi}_j \) for each \( j = 1, 2, \ldots \), then the sequence of envelopes \( \tilde{\Phi}_1 \geq \tilde{\Phi}_2 \geq \cdots \) is decreasing and will converge to an upper semicontinuous function not identically \(-\infty \), since \( \int \tilde{\Phi}_j \, d\theta \geq 0 \). Clearly \( \tilde{\Phi} := \lim_{j \rightarrow \infty} \tilde{\Phi}_j \) will be our desired function. For the proof we will set \( \Phi = \Phi_j \), and without loss of generality we assume \( k > 1/2 \).

Since we may replace \( \Phi \) by a small \( C^2 \) perturbation, we assume that

\[
\{ \mathcal{L}\Phi < 0 \} = E = E_1 \cup \cdots \cup E_m
\]

is the union of a finite number of connected open intervals with \( \overline{E_1} \cap \overline{E_j} = \emptyset \).

Writing \( E_j = (a_j, b_j) \), we suppose also that \( \cdots < b_2 < a_2 < b_1 < a_1 \). We will also define \( \Phi \) to be a \( C^2 \) function on \( \mathbb{R} \), which is periodic with period \( 2\pi \).

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We start with $\psi_\alpha$ as in the proof of Lemma 3, and we slide $p_3$ to the left. If we obtain a tangency $\psi_{p_3}$ for $b_2 \leq p_3 < a_1$ as in Figure 2, then the interval $E$ has been eliminated. The other possibility is that we arrive at $p = b$ without reaching a tangency. In this case, by the argument of Lemma 3, we have $|q_3 - p_3| < \pi/k$. Thus we may consider

$$\psi(\theta) = \psi_{p_3}(\theta) - \lambda \sin(k(\theta - p_3))$$

and increase $\lambda$ until a tangency $\hat{q} \in (p_3, q_3)$ is obtained (see Figure 3).

In the first case above, we will say that $E_1$ is covered by $\psi_{p_3}$. We will replace $\Phi$ by $\psi_{p_3}$, over the interval $(p_3, q_3)$, and the resulting curve will be $C^1$, and piecewise $C^2$. Since $|p_3 - q_3| < \pi/k$ and $k > 1/2$, we may extend the replacement by $\psi_{p_3}$ to be $2\pi$-periodic on $\mathbb{R}$.

In the second case, we will replace $\Phi$ by the function $\hat{\psi}_1$ on the interval $E_2 = (a_2, \hat{q}_1)$, as in Figure 3. We will call $E_2$ a temporary interval. The new curve we obtain is piecewise $C^2$, with a downward-opening angle at $a_2$. The Sturm Comparison Theorem continues to hold in this nonsmooth case, so we may apply Lemma 3 to conclude that $E_2$ has length $< \pi/k < 2\pi$. Thus we can extend the temporary interval to have period $2\pi$ on $\mathbb{R}$.

Now we proceed by decreasing induction on the number of uncovered intervals. By Lemma 4 below, if there is only one interval left (temporary or not yet touched), it will be covered by the sliding procedure. In Lemma 4 we will show that if we start at a temporary interval and start sliding to the left, then we will produce another temporary interval containing both $E_1$ and $E_2$.

As long as we obtain only temporary intervals, without a covering, we may continue similarly to obtain a temporary interval $E_j$ containing $E_1 \cup \cdots \cup E_{j-1}$. By hypothesis, there is no wide amalgamated sector, so there exist $a \in \mathbb{R}$ and finitely many sectors $E_1 \cup \cdots \cup E_m$ such that

$$\{ \mathcal{L}\Phi < 0 \} \cap (a, a + \pi/k) = E_1 \cup \cdots \cup E_m$$

and

$$\{ \mathcal{L}\Phi \geq 0 \} \supset [a - \pi/k, a].$$
If the next interval $\hat{E}_{m+1}$ is temporary, it must span $[a - \pi/k, a]$ and thus have length $\geq \pi/k$. On the other hand, by Lemma 4, a temporary interval $\hat{E}_{m+1}$ would be forced to have length $< \pi/k$. Thus it follows from Lemma 4 that this sliding procedure must in fact produce intervals that cover $E_j$ for $1 \leq j \leq m$.

Since, at each step, we reduce the total number of uncovered intervals, the proof is completed by Lemma 4.

**Lemma 4.** Let $\hat{E}_2$ be a temporary interval given by $\hat{\psi}_1$. The procedure of sliding $\psi_p$, starting with $p = a_2$ and travelling to the left, will yield either a covering of $\hat{E}_2$ or a new temporary interval $\hat{E}_3$ containing $E_1 \cup E_2$. The interval $\hat{E}_3$, if it exists, will have length $< \pi/k$. Thus if $\mathcal{L} \Phi > 0$ on $[a_2 - \pi/k, a_2]$ then this will yield a covering of $\hat{E}_2$.

**Proof.** As in Lemma 2, we see that $\psi_{a_2}$ lies above $\Phi$ over $E_2$ and above $\hat{\psi}_1$ over $(p_3, q_1)$. Now we slide $p$ to the left and obtain a function $\hat{\psi}_2$ which either covers $\hat{E}_2$ or gives a temporary interval containing $E_2$. If the point $q_2$, where $\hat{\psi}_2$ is tangent to $\Phi$, lies to the right of $E_1$, then $\hat{\psi}_2$ gives a temporary interval $\hat{E}_3$ containing both $E_1$ and $E_2$.

Otherwise, $q_2$ lies between $E_1$ and $E_2$, and so $\hat{\psi}_1$ and $\hat{\psi}_2$ cross at a point $\hat{\theta}$ (see Figure 4). We show that in this case $|b_3 - \nu_2| < \pi/k$. By the construction of the temporary intervals, we have $|b_3 - q_2| < \pi/k$, $|b_2 - q_1| < \pi/k$.

Now for $\delta > 0$ we consider

$$\psi = \hat{\psi}_2 - \delta \sin(k(\theta - \hat{\theta}))$$

and note that for $\epsilon > 0$ sufficiently small, $(\hat{q}_2 - \epsilon, \hat{q}_2) \subset (\Phi < \psi)$. Thus the amalgamated interval of $(\hat{q}_2 - \epsilon, \hat{q}_2)$ in $(\Phi < \psi)$ contains $(b_3, \nu_2 - \epsilon')$. Letting $\delta$ tend to zero, we have $|b_3 - \nu_2| \leq \pi/k$.

Now we may replace $\hat{\psi}_2$ by $\psi'\lambda(\theta) = \hat{\psi}_2(\theta) - \lambda \sin(k(\theta - b_3))$ and lower $\hat{\psi}_2$ until we obtain a function $\psi_3$ with a tangency $\hat{\theta}_3 \in (\hat{q}_2, \nu_2)$. If $\hat{\theta}_3$ lies to the right of $E_1$, then the new temporary interval $\hat{E}_3$ contains $E_1 \cup E_2$, and the proof of the lemma is complete. Otherwise, if $\hat{\theta}_3 \in (\hat{q}_2, b_1)$ then it is evident from Figure 4 that $\hat{\psi}_3$ will intersect $\psi_1$ at a point $\hat{\theta}_3 \in (\hat{\theta}, b_1)$. By the comparison (10), we see that $\hat{\psi}_3(\theta) \geq \hat{\psi}_1(\theta)$ holds for $\hat{\theta}_3 < \theta < \hat{\theta}_3 + \pi/k$. In particular, $\hat{\psi}_3(\hat{\theta}_3) > 0$, and so we may again increase $\lambda$ to find another tangency.
Thus it follows that whenever we reach a tangency \( \hat{p}_j < b_1 \) we have \( \hat{\psi}_j(\hat{q}_1) > 0 \), and we may increase \( \lambda \) further to find another tangency \( \hat{p}_{j+1} \in (\hat{p}_j, v_2) \). Clearly this process must end, i.e., we must have a tangency \( \hat{p}_j \geq b_1 \), since for \( \lambda \) sufficiently large we have \( \psi_\lambda(\hat{q}_1) < 0 \). This completes the proof.

**References**


**Department of Mathematics, Indiana University, Bloomington, Indiana 47405**