EXTENSIONS OF A THEOREM OF WINTNER ON SYSTEMS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

BY

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Abstract. A theorem of Wintner concerning sufficient conditions for a system $y' = A(t)y$ to have linear asymptotic equilibrium is extended to a system $x' = A(t)x + f(t, x)$. The integrability conditions imposed on $f$ permit conditional convergence of some of the improper integrals that occur. The results improve on Wintner's even if $f = 0$.

An $n \times n$ system

$$y' = A(t)y, \quad t > 0,$$

is said to have linear asymptotic equilibrium if for each constant vector $c$ there is a solution of (1) such that $\lim_{t \to \infty} y(t) = c$. It is well known that (1) has this property if $A$ is continuous and

$$\int_{0}^{\infty} \| A(t) \| \, dt < \infty.$$

Wintner [5] attributed this result to Bôcher and improved on it as follows.

**Theorem 1 (Wintner).** Let $A$ be continuous on $[a, \infty)$ and suppose the integrals

$$A_j(t) = \int_{t}^{\infty} A_{j-1}(s) A(s) \, ds, \quad 1 \leq j \leq k \quad (A_0 = I),$$

converge, and

$$\int_{0}^{\infty} \| A_k(t) A(t) \| \, dt < \infty.$$

Then (1) has linear asymptotic equilibrium.

Notice that (3) is vacuous and (4) reduces to (2) when $k = 0$.

Here we apply Wintner's idea to the system

$$x' = A(t)x + f(t, x).$$

We give sufficient conditions for (5) to have a solution $x$ such that

$$\lim_{t \to \infty} x(t) = c$$

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for a given constant vector $c$. The assumptions on $f$ in our main theorem apply only “near” the set $\{(t,c) | t \geq a \}$. Our integral smallness conditions permit conditional convergence of some of the improper integrals that occur. This continues a theme developed previously in [3 and 4]. (See also Hallam [1, 2].) The idea of dealing with the iterated integrals (3) is due to Wintner [5].

Throughout the paper $k$ is a nonnegative integer. It is to be understood that conditions stated for $1 \leq j \leq k$ are vacuous if $k = 0$, and that $\Sigma_{j=0}^k = 0$. Our results apply with $x, A$, and $f$ real- or complex-valued. Improper integrals occurring in hypotheses are tacitly assumed to converge, and the convergence may be conditional except when the integrand is obviously nonnegative. We use “$O$” and “$o$” in the standard way to indicate orders of magnitude as $t \to \infty$.

To indicate the direction of proof of our main theorem, it is convenient to state part of its hypotheses here.

**Assumption A.** Let $w$ be continuous and nonincreasing on $[0, \infty)$, $0 < w(t) \leq 1$, and suppose that either $\lim_{t \to \infty} w(t) = 0$ or $w = 1$. Suppose $A$ is continuous on $[0, \infty)$ and $A_j$ in (3) exists for $1 \leq j \leq k$. Let $c$ be a given constant vector, and suppose there is a constant $M > 0$ such that $f$ is continuous and

$$
(7) \quad \|f(t,x) - f(t,c)\| \leq R(t,\|x - c\|)
$$
on $S = \{(t,x) | t \geq a, \|x - c\| \leq Mw(t)\}$,

where $R(t, \lambda)$ is continuous on $\{(t, \lambda) | t \geq a, 0 \leq \lambda \leq Mw(t)\}$ and nondecreasing in $\lambda$ for each $t$.

We define $\Gamma_k = \Sigma_{j=0}^k A_j$, and observe from (3) that

$$
(9) \quad \Gamma_k' = -\Gamma_{k-1} A, \quad k \geq 0 (\Gamma_{-1} = 0).
$$

Moreover, since $\lim_{t \to \infty} \Gamma_k(t) = I$, $\Gamma_k$ is invertible for large $t$, and $\lim_{t \to \infty} \Gamma_k^{-1}(t) = I$.

Now let $t_0 \geq a$ be such that $\Gamma_k^{-1}$ exists on $[t_0, \infty)$. For convenience below, we define

$$
(10) \quad \mu_k(t) = \sup_{s \geq t} \left\{ \|\Gamma_k^{-1}(s)\| \right\} = 1 + o(1),
$$

and

$$
(11) \quad v_k(t) = \mu_k(t) \sup_{s \geq t} \left\{ \|\Gamma_k(s)\| \right\} = 1 + o(1).
$$

Let $C[t_0, \infty)$ be the space of continuous $n$-vector functions (with real or complex components) on $[t_0, \infty)$, with the topology of uniform convergence on finite intervals. Let $V[t_0, \infty)$ be the closed convex subset of $C[t_0, \infty)$ defined by

$$
(12) \quad V[t_0, \infty) = \left\{ x \in C[t_0, \infty) \|x(t) - c\| \leq Mw(t), t \geq t_0 \right\}.
$$
We obtain our results by applying the Schauder-Tychonov theorem to an appropriate transformation $T$ of $V[t_0, \infty)$ (for sufficiently large $t_0$) into itself. To motivate the choice of $T$, we observe that if 

$$x(t) = c - \int_t^\infty [A(s)x(s) + f(s, x(s))] \, ds,$$

where the integral is assumed to converge, then $x$ satisfies (5) and (6). Repeated integration by parts, assuming at each step that $x$ satisfies (5), yields the equation

$$\Gamma_k(t)x(t) = c - \int_t^\infty A_k(s)A(s)x(s) \, ds - \int_t^\infty \Gamma_k(s)f(s, x(s)) \, ds.$$

Although these manipulations are completely formal, (13) suggests the transformation $T$ defined by

$$(14) \quad (Tx)(t) = \Gamma_k^{-1}(t) \left[ c - \int_t^\infty [A_k(s)A(s)x(s) + \Gamma_k(s)f(s, x(s))] \, ds \right].$$

Assumption A implies that the function $F(t) = f(t, x(t))$ is continuous on $[t_0, \infty)$ if $x \in V[t_0, \infty)$. Hence, if the integrals in (14) converge, differentiation yields

$$(15) \quad (Tx)'(t) = \Gamma_k^{-1}(t) \left[ \Gamma_{k-1}(t)A(t)(Tx)(t) + A_k(t)A(t)x(t) \right] + f(t, x(t)),$$

where we have used (9) and the fact that $(\Gamma_k^{-1})' = -\Gamma_k^{-1}\Gamma_k'\Gamma_k^{-1}$. Therefore, if $T$ has a fixed point (function) $x_0$ in $V[t_0, \infty)$, we see on setting $Tx = x = x_0$ in (15) that

$$x'_0(t) = \Gamma_k^{-1}(t) \left[ \Gamma_{k-1}(t)A(t)x_0(t) + f(t, x(t)) \right] + f(t, x(t))$$

(since $\Gamma_{k-1} + A_k = \Gamma_k$); i.e., $x_0$ satisfies (5). Moreover, setting $Tx = x = x_0$ in (14) shows that $x_0$ also satisfies (6).

The following theorem allows the integrals occurring in $Tc$ (the function obtained by setting $x = c$ in (14)) to converge conditionally, and exploits the rapidity with which $Tc - c$ approaches zero for large $t$ to restrict the set $V[t_0, \infty)$ on which $T$ must satisfy the hypotheses of the Schauder-Tychonov theorem. (See Remark 1, below.)

**Theorem 2.** Suppose Assumption A holds. Let

$$(16) \quad h(t) = (\Gamma_k^{-1}(t) - I)c - \Gamma_k^{-1}(t) \int_t^\infty \left[ A_k(s)A(s)c + \Gamma_k(s)f(s, c) \right] \, ds,$$

and suppose that

$$(17) \quad \lim_{t \to \infty} \frac{\|h(t)\|}{w(t)} = \alpha.$$ 

Suppose also that

$$(18) \quad \lim_{t \to \infty} \int_t^\infty \left[ \frac{R(s, Mw(s))}{M} + \|A_k(s)A(s)\|w(s) \right] \, ds = \theta < 1$$

and

$$(19) \quad \alpha < M(1 - \theta).$$
Then, if \( t_0 \) is sufficiently large, there is a solution \( x_0 \) of (5) on \([t_0, \infty)\) such that

\[
\|x_0(t) - c\| \leq Mw(t), \quad t \geq t_0,
\]

and

\[
\lim_{t \to \infty} (w(t))^{-1}\|x_0(t) - c\| \leq \alpha + M\theta;
\]

or, more precisely,

\[
\lim_{t \to \infty} (w(t))^{-1}\|x_0(t) - c - h(t)\| \leq M\theta.
\]

**Proof.** We can rewrite (14) as

\[
(Tx)(t) = c + h(t) - \Gamma_k^{-1}(t) \int_{t_0}^{t} \left[ A_k(s)A(s)(x(s) - c) + \Gamma_k(s)(f(s,x(s)) - f(s,c)) \right] ds,
\]

where the integral converges if \( x \in V[t_0, \infty) \), because of (7), (12), and (18); moreover

\[
\| (Tx)(t) - c \| \leq \|h(t)\| + M\mu_k(t) \int_{t_0}^{t} \|A_k(s)A(s)\| w(s) ds + \nu_k(t) \int_{t_0}^{t} R(s,Mw(s)) ds.
\]

From (10), (11), (18), and (19), we can assume henceforth that \( t_0 \) is so large that the right side of (24) is \( \leq Mw(t) \) if \( t \geq t_0 \). Then

\[
(T(V[t_0, \infty))) \subset V[t_0, \infty).
\]

Now we show that \( T \) is continuous on \( V[t_0, \infty) \). Let \( \{x_j\} \) be a sequence in \( V[t_0, \infty) \) which converges to a limit \( x \) in \( V[t_0, \infty) \). From (10), (11), and (23),

\[
\|(Tx_j)(t) - (Tx)(t)\| \leq \mu_k(t_0) \int_{t_0}^{t} \|A_k(s)A(s)\| \|x_j(s) - x(s)\| ds + \nu_k(t_0) \int_{t_0}^{t} \|f(s,x_j(s)) - f(s,x(s))\| ds, \quad t \geq t_0.
\]

The integrands here converge to zero on \([t_0, \infty)\), and they are dominated by \( 2M\|A_k(s)A(s)\|w(s) \) and \( 2R(s,Mw(s)) \), respectively. (See (7) and (12).) Therefore, (18) and Lebesgue’s dominated convergence theorem imply that the integrals approach zero as \( j \to \infty \). Hence, \( \{Tx_j\} \) converges to \( Tx \) uniformly on \([t_0, \infty)\), and therefore \( T \) is continuous on \( V[t_0, \infty) \).

From (12) and (25), \( T(V[t_0, \infty)) \) is equibounded on finite intervals. This, (15), and Assumption A imply that \( T(V[t_0, \infty)) \) is also equicontinuous on finite intervals. Now we have verified the hypotheses of the Schauder-Tychonov theorem, which implies that \( Tx_0 = x_0 \) for some \( x_0 \) in \( V[t_0, \infty) \). Setting \( Tx = x_0 \) in (24) and invoking (10), (11), (17), and (18) yields (20). Similarly, setting \( x = Tx = x_0 \) in (23) yields (22). This completes the proof.
**Corollary 1.** In addition to the assumptions of Theorem 2, suppose that
\[ R(t, \lambda_1)/R(t, \lambda_2) \leq \lambda_1/\lambda_2, \quad 0 \leq \lambda_1 < \lambda_2. \]
Then (21) can be replaced by
\[ (w(t))^{-1}\|x_0(t) - c\| \leq \alpha/(1 - \theta). \]

**Proof.** Let
\[ \phi(t) = \sup_{s \geq t} \{(w(s))^{-1}\|x_0(s) - c\|\} \]
and
\[ \delta = \lim_{t \to \infty} \phi(t) = \lim_{t \to \infty} (w(t))^{-1}\|x_0(t) - c\|. \]
Setting \( x = Tx = x_0 \) in (23) and using routine estimates yields
\[ \|x_0(t) - c\| \leq \|h(t)\| + v_k(t) \int_t^\infty \|A_k(s)A(s)\|\|x_0(s) - c\| ds \]
\[ + \mu_k(t) \int_t^\infty R(s, \|x_0(s) - c\|) ds. \]
Applying (26) in the second integral and then dividing (29) by \( w(t) \) yields the inequality
\[ (w(t))^{-1}\|x_0(t) - c\| \leq (w(t))^{-1}\|h(t)\| + P(t)\phi(t), \]
where
\[ P(t) = (w(t))^{-1}\left[ v_k(t) \int_t^\infty \|A_k(s)A(s)\|w(s) ds + \frac{\mu_k(t)}{M} \int_t^\infty R(s, Mw(s)) ds \right], \]
so that
\[ \lim_{t \to \infty} P(t) = \theta, \]
from (10), (11), and (18). Letting \( t \to \infty \) in (30) and invoking (17), (28), and (31) shows that \( \delta \leq \alpha + \theta \delta \), which proves (27).

It is worthwhile to state Theorem 2 separately for the case where \( w = 1 \), so that \( \alpha \) and \( \theta \) are necessarily zero and (19) is automatic.

**Theorem 3.** Suppose Assumption A holds with \( w = 1 \). Suppose also that the integral in (16) converges and that
\[ \int_0^\infty R(t, M) dt < \infty, \quad \int_0^\infty \|A_k(t)A(t)\| dt < \infty. \]
Then, if \( t_0 \) is sufficiently large, there is a solution \( x_0 \) of (5) on \([t_0, \infty)\) such that \( \|x_0(t) - c\| \leq M \) for \( t \geq t_0 \) and \( \lim_{t \to \infty} x_0(t) = c \).

**Remark 1.** The continuity assumption on \( f \) is the most stringent when \( w = 1 \), since the set \( S \) in (8) is maximized in that case. More importantly, if \( R \) satisfies (26), then (32) implies (18) with \( \theta = 0 \) for every admissible \( w \neq 1 \), while the converse is obviously false. Nevertheless, the conclusions of Theorem 2 are weakest when
$w = 1$. (See Theorem 3.) The reason for this is that in Theorem 3 it is assumed only
that the integral in (16) converges, while Theorem 2 exploits the rapidity of its
convergence.

The hypotheses of Theorem 2 may hold for some constant vectors $c$ and fail to
hold for others. In the following theorem $c$ may be chosen arbitrarily.

**Theorem 4.** Suppose $f$ is continuous for $t \geq a$ and all $x$, and
\[
\|f(t, x_1) - f(t, x_2)\| \leq R(t, \|x_1 - x_2\|),
\]
where $R(t, \lambda)$ is continuous on $[a, \infty) \times [0, \infty)$ and nondecreasing in $\lambda$. For some
integer $k \geq 0$, suppose the integrals $A_1, \ldots, A_{k+1}$ converge, and
\[
\|A_j(t)\| = O(w(t)), \quad 1 \leq j \leq k + 1.
\]
Suppose also that
\[
\int_a^\infty \Gamma_k(s) f(s, c) \, ds = O(w(t))
\]
for every constant vector $c$, and that (18) holds for all $M > 0$. Then, if $c$ is a given
constant vector, there is a solution $x_0$ of (5) which is defined for $t$ sufficiently large and
satisfies
\[
\|x_0(t) - c\| = O(w(t)).
\]
Moreover, if $\theta = 0$ in (18) and (33) and (34) hold with “$O$” replaced by “$o$” (which
is necessarily true if $w = 1$), then (35) holds with “$O$” replaced by “$o$.”

**Proof.** The hypotheses imply (17) for some $a$ (which may depend upon $c$, but is
zero if (33) and (34) hold with “$o$”). Simply choose $M$ to satisfy (19) and invoke
Theorem 2.

Theorem 4 has the following corollary for the linear system
\[
x' = [A(t) + B(t)] x + g(t).
\]

**Corollary 2.** Let $A, B,$ and $g$ be continuous on $[a, \infty)$. Suppose (33) holds.

\[
\int_t^\infty \Gamma_k(s) g(s) \, ds = O(w(t)), \quad \int_t^\infty \Gamma_k(s) B(s) \, ds = O(w(t)).
\]
and
\[
\lim_{t \to \infty} (w(t))^{-1} \int_t^\infty [\|A_k(s) A(s)\| + \|B(s)\|] w(s) \, ds = \theta < 1.
\]
Then, for any constant $c$, (36) has a solution which satisfies (35); moreover, if $\theta = 0$
and (33) and (37) hold with “$O$” replaced by “$o$,” then so does (35).

The following special case of Corollary 2 extends Theorem 1.

**Corollary 3.** Suppose (33) holds and
\[
\lim_{t \to \infty} (w(t))^{-1} \int_t^\infty \|A_k(s) A(s)\| w(s) \, ds < 1
\]
for some $w$ as in Assumption A. Then (1) has linear asymptotic equilibrium.


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