

SOLVABILITY OF DIFFERENTIAL EQUATIONS WITH LINEAR COEFFICIENTS OF REAL TYPE

BY
 RAINER FELIX

ABSTRACT. Let L be the infinitesimal generator associated with a flow on a manifold M . Regarding L as an operator on a space of testfunctions we deal with the question if L has closed range. (Questions of this kind are investigated in [4, 1, 2].) We provide conditions under which $L + \mu 1: \mathcal{S}(M) \rightarrow \mathcal{S}(M)$, $\mu \in \mathbb{C}$, has closed range, where $M = \mathbb{R}^n \times K$, K being a compact manifold; here $\mathcal{S}(M)$ is the Schwartz space of rapidly decreasing smooth functions. As a consequence we show that the differential operator $\sum_{i,j} a_{ij} x_j (\partial/\partial x_i) + b$ defines a surjective mapping of the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions onto itself provided that all eigenvalues of the matrix (a_{ij}) are real. (In the case of imaginary eigenvalues this is not true in general [3].)

1. Preliminaries and notations. Let M be a differentiable manifold. We assume that \mathbb{R} acts on M (on the right) by diffeomorphisms; i.e. we have a one-parameter group $(\rho_t)_{t \in \mathbb{R}}$ of transformations (or a global flow) on M . Let L be the infinitesimal generator associated with this flow. We regard L as a differential operator on M given by

$$(1.1) \quad L\varphi(m) = \left. \frac{d}{dt} \varphi(m \cdot t) \right|_{t=0}, \quad m \in M, \varphi \in C^\infty(M).$$

Or, if $\varphi_t := \varphi \circ \rho_t$, $t \in \mathbb{R}$, we have $L\varphi = (d/dt)\varphi_t|_{t=0}$. Furthermore, L is invariant under (ρ_t) , i.e.

$$(1.2) \quad L(\varphi_t) = (L\varphi)_t = \frac{d}{dt} \varphi_t$$

for all $t \in \mathbb{R}$. For $\mu \in \mathbb{C}$ we define the first order differential operator $L_\mu := L - \mu 1$.

We denote by $\mathcal{D}(M)$ the space of C^∞ -functions with compact support on M . Its dual space $\mathcal{D}'(M)$ is the space of distributions on M . A distribution $T \in \mathcal{D}'(M)$ is called *relatively invariant with weight μ* if

$$(1.3) \quad \langle T, \varphi_t \rangle = e^{\mu t} \langle T, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(M)$, $t \in \mathbb{R}$. We write $\mathcal{D}'_\mu(M)$ for the space of relatively invariant distributions with weight μ .

Clearly, L_μ defines a continuous mapping of $\mathcal{D}(M)$ into itself. The aim of this paper is to provide conditions under which this mapping has closed range. By differentiating equation (1.3) it is seen that the closure $L_\mu \mathcal{D}(M)$ of the range of L_μ

Received by the editors February 5, 1985.

1980 *Mathematics Subject Classification.* Primary 58G99, 35A05, 35D05.

Key words and phrases. Divergences, invariant distributions, differential operators with critical points.

©1986 American Mathematical Society
 0002-9947/86 \$1.00 + \$.25 per page

in $\mathcal{D}(M)$ can be characterized as the orthogonal of $\mathcal{D}'_\mu(M)$ in $\mathcal{D}(M)$; we write

$$(1.4) \quad \overline{L_\mu \mathcal{D}(M)} = \mathcal{D}'_\mu(M)^\perp.$$

Let $L'_\mu: \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ be the transpose of $L_\mu: \mathcal{D}(M) \rightarrow \mathcal{D}(M)$. Given a distribution $T \in \mathcal{D}'(M)$, by (1.4) we have

$$(1.5) \quad T \in \mathcal{D}'_\mu(M) \quad \text{iff} \quad L'_\mu T = 0.$$

Let $C^r(M)$ be the space of r -times continuously differentiable functions on M , $r \in \mathbf{N}$. For $\varphi \in C^1(M)$ we have

$$(1.6) \quad \frac{d}{dt}(e^{-\mu t} \varphi_t) = e^{-\mu t} (L_\mu \varphi)_t.$$

Therefore, if $L_\mu \varphi = 0$ we have $\varphi_t = e^{\mu t} \varphi$ for all $t \in \mathbf{R}$.

Furthermore, let $L_\mu \varphi = f$, $\varphi \in \mathcal{D}(M)$, and suppose that, if $m \in M$ is given, $e^{-\mu t} f(m \cdot t)$ is integrable over the interval $-\infty < t < 0$ and that

$$\lim_{t \rightarrow -\infty} e^{-\mu t} \varphi(m \cdot t) = 0,$$

then from (1.6) we derive the solution formula

$$(1.7) \quad \varphi(m) = \int_{-\infty}^0 e^{-\mu t} f(m \cdot t) dt.$$

Moreover, suppose that $e^{-\mu t} \varphi(m \cdot t)$ is integrable over the whole real line $-\infty < t < \infty$ and that $\lim_{t \rightarrow \pm\infty} e^{-\mu t} \varphi(m \cdot t) = 0$ for all $\varphi \in \mathcal{D}(M)$. Then the distribution $\lambda_{\mu,m}: \varphi \mapsto \int_{-\infty}^\infty e^{-\mu t} \varphi(m \cdot t) dt$ is relatively invariant with weight μ , i.e.

$$(1.8) \quad \lambda_{\mu,m} \in \mathcal{D}'_\mu(M).$$

Therefore, if $f \in \overline{L_\mu \mathcal{D}(M)}$ we have the equation

$$(1.9) \quad \int_{-\infty}^0 e^{-\mu t} f(m \cdot t) dt = - \int_0^\infty e^{-\mu t} f(m \cdot t) dt.$$

In this paper we are mainly concerned with the case that our manifold M is a product of \mathbf{R}^n with a d -dimensional compact differentiable manifold K . In this case there is a natural notion of the space $\mathcal{S}(M)$ of Schwartz functions and its dual space $\mathcal{S}'(M)$ of tempered distributions.

Assume that there are d vector fields Z_1, \dots, Z_d on K such that for every $\tau \in K$ the tangent vectors $Z_1(\tau), \dots, Z_d(\tau)$ span the tangent space $T_\tau(K)$ to K at τ . Then $\mathcal{S}(\mathbf{R}^n \times K)$ is the space of all smooth functions φ on $\mathbf{R}^n \times K$ such that the term

$$(1.10) \quad (1 + |x|^2)^{s/2} \partial_x^\alpha Z_\tau^\beta \varphi(x, \tau)$$

is bounded with respect to $(x, \tau) \in \mathbf{R}^n \times K$ for any $s \in \mathbf{N}$ and for any multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_d)$, where $\alpha_j, 1 \leq j \leq n$, and $\beta_k, 1 \leq k \leq d$, belong to the set \mathbf{N}_0 of nonnegative integers and $\partial_x^\alpha := \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ with $|\alpha| := \sum_{j=1}^n \alpha_j$ and $Z_\tau^\beta := Z_1^{\beta_1} \dots Z_d^{\beta_d}$. Sometimes it is convenient to write Y_j for $\partial / \partial x_j, j = 1, \dots, n$, and Y_{n+k} for $Z_k, k = 1, \dots, d$; then we have $\partial_x^\alpha Z_\tau^\beta = Y_1^{\alpha_1} \dots Y_{n+d}^{\alpha_n + \beta_d} =: Y^\gamma$ with $\gamma = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_d)$.

A C^∞ -function $h(x, \tau, t)$ on $\mathbf{R}^n \times K \times \mathbf{R}$ is called of type E (resp. of type P) if for any $r \in \mathbf{N}_0$ and any multi-index γ of length $n + d$ there are $\eta, \theta, \sigma \in \mathbf{N}$ such that

$$(1.11) \quad \left| Y_{x,\tau}^\gamma \left(\frac{\partial}{\partial t} \right)^r h(x, \tau, t) \right| \leq \theta (1 + |x|^2)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } \left| Y_{x,\tau}^\gamma \left(\frac{\partial}{\partial t} \right)^r h(x, \tau, t) \right| \leq \theta (1 + |x|^2)^{\sigma/2} (1 + t^2)^{\eta/2} \right)$$

for all x, τ, t . (Of course, this definition does not depend on the special chosen vector fields Z_1, \dots, Z_d .) It is obvious that sums, products and derivatives of type E functions (resp. type P functions) are of type E (resp. of type P).

Let p and q be the projection of $\mathbf{R}^n \times K$ onto \mathbf{R}^n and K , respectively, and let p_j be the j th component of p . Our one-parameter group (ρ_t) of transformations is called of type E (resp. of type P) if the functions $p_j((x, \tau) \cdot t)$ and $\psi \circ q((x, \tau) \cdot t)$ are of type E (resp. of type P) for all $j = 1, \dots, n$ and for all $\psi \in C^\infty(K)$. In this case we are able to estimate x by $p((x, \tau) \cdot t) =: x'$ for any τ and t . In fact, let $(x', \tau') \cdot (-t) = (x, \tau)$; because

$$|p((x', \tau') \cdot (-t))|^2 \leq \theta (1 + |x'|^2)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } |p((x', \tau') \cdot (-t))|^2 \leq \theta (1 + |x'|^2)^{\sigma/2} (1 + t^2)^{\eta/2} \right)$$

for some $\eta, \theta, \sigma \in \mathbf{N}$, we have

$$(1.12) \quad 1 + |x|^2 \leq (1 + \theta) (1 + |p((x, \tau) \cdot t)|^2)^{\sigma/2} e^{\eta|t|}$$

$$\left(\text{resp. } 1 + |x|^2 \leq (1 + \theta) (1 + |p((x, \tau) \cdot t)|^2)^{\sigma/2} (1 + t^2)^{\eta/2} \right)$$

and therefore

$$(1.13) \quad 1 + |p((x, \tau) \cdot t)|^2 \geq \delta (1 + |x|^2)^{\varepsilon/2} e^{-\zeta|t|}$$

$$\left(\text{resp. } 1 + |p((x, \tau) \cdot t)|^2 \geq \delta (1 + |x|^2)^{\varepsilon/2} (1 + t^2)^{-\zeta/2} \right)$$

for some $\delta, \varepsilon, \zeta > 0$.

Clearly, for each $k \in \{1, \dots, n + d\}$ there are C^∞ -functions a_{ik} on $\mathbf{R}^n \times K \times \mathbf{R}$, $1 \leq i \leq n + d$, such that for any $\varphi \in C^\infty(\mathbf{R}^n \times K)$ we have

$$(1.14) \quad Y_k(\varphi_t)(x, \tau) = \sum_{i=1}^{n+d} a_{ik}(x, \tau, t) (Y_i \varphi)_t(x, \tau)$$

for all x, τ, t . Similarly we have

$$(1.15) \quad \frac{d}{dt} \varphi_t(x, \tau) = \sum_{i=1}^{n+d} b_i(x, \tau, t) (Y_i \varphi)_t(x, \tau)$$

where $b_i, 1 \leq i \leq n + d$, are C^∞ -functions on $\mathbf{R}^n \times K \times \mathbf{R}$.

Now let (ρ_t) be of type E (resp. of type P). Then all the functions a_{ik} and b_i are of type E (resp. of type P). This is evident by inserting p_j and $\psi \circ q$ for φ in (1.14) and (1.15), respectively. Reiterating formula (1.14) we derive that, given $t \in \mathbf{R}$, the

function φ_t belongs to $\mathcal{S}(\mathbf{R}^n \times K)$ for any $\varphi \in \mathcal{S}(\mathbf{R}^n \times K)$ and that the mapping $\varphi \mapsto \varphi_t$ is a continuous mapping of $\mathcal{S}(\mathbf{R}^n \times K)$ into itself. Hereby formula (1.13) is used. Together with (1.15) we derive that the infinitesimal generator L defines a continuous mapping of $\mathcal{S}(\mathbf{R}^n \times K)$ into itself, and our previous considerations concerning $\mathcal{D}(M)$ and $\mathcal{D}'(M)$ remain valid with regard to $\mathcal{S}(M)$ and $\mathcal{S}'(M)$, $M = \mathbf{R}^n \times K$.

2. Lemmata. Let $M = \mathbf{R}^n \times K$ and let our one-parameter group (ρ_t) be of type E . In the whole section we assume that there is $\lambda \in \mathbf{R}$ such that

$$(2.1) \quad p_1(m \cdot t) = e^{-\lambda t} p_1(m)$$

for all $t \in \mathbf{R}$, $m = (x, \tau) \in M$. Then we have

$$(2.2) \quad L_\mu(p_1\varphi) = p_1 L_{\mu+\lambda}\varphi$$

for any continuously differentiable function φ on M .

The submanifold $M^1 := \{m \in M \mid p_1(m) = 0\} \cong \mathbf{R}^{n-1} \times K$ is invariant under (ρ_t) . Let (ρ_t^1) be the restriction of (ρ_t) to M^1 and let L^1 be the associated infinitesimal generator. If φ is a function on M , let φ^1 be its restriction to M^1 . For any continuously differentiable function φ on M we have

$$(2.3) \quad (L\varphi)^1 = L^1\varphi^1.$$

LEMMA 1. *Suppose that $(L_\mu^1)'$: $\mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)$ is surjective. If $p_1 f \in \overline{L_\mu \mathcal{S}(M)}$ for $f \in \mathcal{S}(M)$, then $f \in \overline{L_{\mu+\lambda} \mathcal{S}(M)}$.*

PROOF. By (1.4), the assertion follows from the inclusion $\mathcal{S}'_{\mu+\lambda}(M) \subseteq p_1 \mathcal{S}'_\mu(M)$, which we are going to prove.

Let $S \in \mathcal{S}'_{\mu+\lambda}(M)$. By division of distributions there is $T_1 \in \mathcal{S}'(M)$ such that $p_1 T_1 = S$. By (2.2) and (1.5) we have

$$(2.4) \quad p_1 L'_\mu T_1 = L'_{\mu+\lambda} S = 0;$$

i.e. $L'_\mu T_1$ is the trivial extension of a distribution $W^1 \in \mathcal{S}'(M^1)$. By assumption, $W^1 = (L^1_\mu)' R^1$ with $R^1 \in \mathcal{S}'(M^1)$. Let $R \in \mathcal{S}'(M)$ be the trivial extension of R^1 and let $T := T_1 - R$. Then we have $p_1 T = S$, and $T \in \mathcal{S}'_\mu(M)$ since

$$\langle L'_\mu T, \varphi \rangle = \langle W^1 - (L^1_\mu)' R^1, \varphi^1 \rangle = 0 \quad \text{for all } \varphi \in \mathcal{S}(M). \quad \square$$

For convenience, we define the set $(L_\mu \mathcal{S}(M))_k$, $k \in \mathbf{N}_0$, consisting of all functions $f \in \overline{L_\mu \mathcal{S}(M)}$ for which there are functions $\psi_k \in \mathcal{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda} \mathcal{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. Clearly, $(L_\mu \mathcal{S}(M))_{k+1} \subseteq (L_\mu \mathcal{S}(M))_k$ by (2.2). Put

$$(L_\mu \mathcal{S}(M))_\infty := \bigcap_{k \in \mathbf{N}} (L_\mu \mathcal{S}(M))_k.$$

LEMMA 2. *Suppose that $(L^1_{\mu+\kappa\lambda})'$: $\mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)$ is surjective for all $\kappa = 0, \dots, k-1$. Then $\overline{L_\mu \mathcal{S}(M)} = (L_\mu \mathcal{S}(M))_k$.*

PROOF. We prove the lemma by induction on k . For $k = 0$ the assertion is trivial. Now assume $f = L_\mu \psi_k + p_1^k f_k$ with $\psi_k \in \mathcal{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda} \mathcal{S}(M)}$. Obviously, $f_k^1 \in \overline{L_{\mu+k\lambda}^1 \mathcal{S}(M^1)}$. Since $L_{\mu+k\lambda}^1 \mathcal{S}(M^1)$ is closed by assumption, $f_k^1 = L_{\mu+k\lambda}^1 \psi^1$ for some $\psi^1 \in \mathcal{S}(M^1)$. Select $\psi \in \mathcal{S}(M)$ such that ψ^1 is the restriction of ψ to M^1 . Then $f_k - L_{\mu+k\lambda} \psi$ vanishes on M^1 . Therefore it can be divided by p_1 ; i.e. there is a function $f_{k+1} \in \mathcal{S}(M)$ such that $f_k - L_{\mu+k\lambda} \psi = p_1 f_{k+1}$. By Lemma 1, $f_{k+1} \in \overline{L_{\mu+(k+1)\lambda} \mathcal{S}(M)}$. Put $\psi_{k+1} := \psi_k + p_1^k \psi$. Using (2.2) we get the desired equation for $k + 1$.

LEMMA 3. Let $\lambda \neq 0$. Suppose that $\overline{L_\mu \mathcal{S}(M)} = (L_\mu \mathcal{S}(M))_\infty$. Then $L_\mu \mathcal{S}(M)$ is closed in $\mathcal{S}(M)$.

PROOF. Replacing (ρ_i) and μ by (ρ_{-i}) and $-\mu$ in case of need, we may assume that $\lambda > 0$.

Let $f \in \overline{L_\mu \mathcal{S}(M)}$. For any $k \in \mathbb{N}$, we take $\psi_k \in \mathcal{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda} \mathcal{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. In the course of the proof we determine $k_0 \in \mathbb{N}$ sufficiently large for our need. First of all we assume that the real part $\nu'_k := \operatorname{Re} \nu_k$ of $\nu_k := \mu + k\lambda$ is positive for $k \geq k_0$. For $k \geq k_0$ we put

$$(2.5) \quad \varphi_k(m) := \psi_k(m) - p_1^k(m) \int_0^\infty e^{-\nu_k t} f_k(m \cdot t) dt, \quad m \in M.$$

It is easily seen that the distribution $\lambda_{\nu_k, m}$ (see (1.8)) is well defined for all $k \geq k_0$ and $m \in M \setminus M^1$. In fact, for $t < 0$ we apply (2.1) and get the estimate

$$(2.6) \quad |\varphi(m \cdot t)| \leq \frac{c(\varphi, r)}{|p_1(m)|^r} e^{r\lambda t}, \quad \varphi \in \mathcal{S}(M),$$

where r is an arbitrary integer ≥ 0 and $c(\varphi, r)$ is constant with respect to m and t . Therefore, by (1.9), for $m \in M \setminus M^1$, equation (2.5) can be written in the following form:

$$(2.7) \quad \varphi_k(m) = \psi_k(m) + p_1^k(m) \int_{-\infty}^0 e^{-\nu_k t} f_k(m \cdot t) dt.$$

Now, by induction on $r \in \mathbb{N}$ we see: For any $r \in \mathbb{N}$ there is $k_0 \in \mathbb{N}$ such that $\varphi_k \in C^r(M)$ for $k \geq k_0$; in fact, for any multi-index γ with $|\gamma| \leq r$ the derivative $Y^\gamma(\varphi_k - \psi_k)(m)$ is a finite sum of terms of the form

$$(2.8) \quad cp_1^{s'}(m) \int_a^b e^{-\nu_k t} h(m, t) (Y^\gamma f_k)(m \cdot t) dt,$$

where $c \in \mathbb{C}$, $s' \in \mathbb{N}$ depend on k , $s' \geq k - |\gamma|$, and $h(m, t)$ is a C^∞ -function of type E independent of k ; $|\gamma'| \leq |\gamma|$, $a = 0$, $b = \infty$. Hereby, (1.14) is used.

For $m \in M \setminus M^1$ we can apply (2.6) and, proceeding from (2.7), we can express $Y^\gamma(\varphi_k - \psi_k)(m)$ by a finite sum of terms of the form (2.8) with $a = -\infty$, $b = 0$.

Now let us prove that for any $s \in \mathbb{N}$ and for any multi-index γ there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the term

$$(2.9) \quad (1 + |p(m)|^2)^{s/2} |Y^\gamma \varphi_k(m)|$$

is bounded with respect to $m \in M$. In view of (2.8) and by continuity it is sufficient to prove the boundedness of the terms

$$(2.10) \quad (1 + |p(m)|^2)^{s/2} |p_1^{s'}(m)| \int_a^b e^{-\nu_k t} |h(m, t)| |Y^{\nu_k} f_k(m \cdot t)| dt$$

on the domain $\{0 \leq |p_1(m)| \leq 1\}$ for $a = 0, b = \infty$ and on the domain $\{|p_1(m)| \geq 1\}$ for $a = -\infty, b = 0$. For $a = 0, b = \infty$ we use (1.13); since $f_k \in \mathcal{S}(M)$ we can estimate (2.10) by

$$(2.11) \quad (1 + |p(m)|^2)^{s/2} \int_0^\infty e^{-\nu_k t} |h(m, t)| \frac{C(f_k, N)}{\delta^N (1 + |p(m)|^2)^{\varepsilon N/2} e^{-\xi N t}} dt,$$

where N is a positive integer which satisfies $\varepsilon N \geq s + \sigma$ with σ from (1.11). The boundedness of (2.11) is obvious if k_0 is sufficiently large. For $a = -\infty, b = 0$ we use (2.1). Let $k \geq k_0$ be given, we choose N as above and take a positive integer $r \geq s'$; then we get an estimate of (2.10) by the term

$$(2.12) \quad (1 + |p(m)|^2)^{s/2} |p_1^{s'}(m)| \int_{-\infty}^0 e^{-\nu_k t} |h(m, t)| \frac{C(f_k, N, r)}{\delta^N (1 + |p(m)|^2)^{\varepsilon N/2} e^{-\xi N |t|} e^{-\lambda r |t|} |p_1(m)|^r} dt$$

which is obviously bounded if r is sufficiently large.

Now let k_0 be sufficiently large and let $k \geq k_0$. From (2.5) and (2.2) we get

$$(2.13) \quad L_\mu \varphi_k = L_\mu \psi_k - p_1^k L_{\nu_k} g_k,$$

where

$$g_k(m) := \int_0^\infty e^{-\nu_k t} f_k(m \cdot t) dt.$$

Applying (1.6) with $\varphi = g_k, \mu = \nu_k$ for $t = 0$ we get

$$(2.14) \quad L_{\nu_k} g_k = -f_k$$

and therefore

$$(2.15) \quad L_\mu \varphi_k = f$$

for any $k \geq k_0$.

From (1.6) we derive that $L_\mu \varphi = 0$ implies $\varphi = 0$ for $\varphi \in C^1(M)$ vanishing at infinity; in fact, for $m \in M \setminus M^1$ we have $\varphi(m \cdot t) = e^{\mu t} \varphi(m)$ and therefore $\varphi(m) = 0$ because $m \cdot t \rightarrow \infty$ for $t \rightarrow -\infty$ by (2.1).

Therefore, looking at (2.15), we see that φ_k does not depend on k ; i.e. $\varphi_k =: \varphi$ for all $k \geq k_0$. Thus, by (2.9), $\varphi \in \mathcal{S}(M)$.

LEMMA 4. *Let $\operatorname{Re} \mu \neq 0$. Suppose that (ρ_t^1) is of type P . Then $(L_\mu^1)'$: $\mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)$ is surjective*

PROOF. Replacing (ρ_t) and μ by (ρ_{-t}) and $-\mu$ in case of need, we may assume that $\operatorname{Re} \mu < 0$.

By (1.6), $L_\mu^1: \mathcal{S}(M^1) \rightarrow \mathcal{S}(M^1)$ is injective. To prove that L_μ^1 is also surjective we put

$$(2.16) \quad \varphi^1(m^1) := \int_{-\infty}^0 e^{-\mu t} f^1(m^1 \cdot t) dt, \quad m^1 = (x^1, \tau) \in M^1,$$

for a given $f^1 \in \mathcal{S}(M^1)$ and show that $\varphi^1 \in \mathcal{S}(M^1)$.

In fact, by equation (1.14), for any $s \in \mathbf{N}$ and for any multi-index γ the term $(1 + |x^1|^2)^{s/2} Y^\gamma \varphi^1(m^1)$ is a finite sum of terms of the form

$$\int_{-\infty}^0 e^{-\mu t} h^1(m^1, t) g^1(m^1 \cdot t) dt$$

where $g^1 \in \mathcal{S}(M^1)$ and $h^1(m^1, t)$ is a C^∞ -function of type P . Using (1.13) we see that $|h^1(m^1, t) g^1(m^1 \cdot t)|$ can be estimated by $c(1 + t^2)^{r/2}$ with some $r \in \mathbf{N}$ and some constant $c > 0$.

LEMMA 5. *Let $\lambda \neq 0$. Suppose that (ρ_t^1) is of type P and that $\rho_t(x_1, m^1) = (e^{-\lambda t} x_1, \rho_t^1(m^1))$ for $(x_1, m^1) \in M \cong \mathbf{R} \times M^1$. Then $L_\mu \mathcal{S}(M)$ is closed in $\mathcal{S}(M)$.*

PROOF. If $\text{Re } \mu + k\lambda \neq 0$ for all $k \in \mathbf{N}_0$, the assertion follows by Lemmas 4, 2 and 3.

Assume that $\text{Re } \mu + k\lambda = 0$ for some $k \in \mathbf{N}_0$. Given $f \in \overline{L_\mu \mathcal{S}(M)}$, by Lemmas 4 and 2 there are $\psi_k \in \mathcal{S}(M)$ and $f_k \in \overline{L_{\mu+k\lambda} \mathcal{S}(M)}$ such that $f = L_\mu \psi_k + p_1^k f_k$. Therefore, by (2.2), we have only to prove that $L_{\mu+k\lambda} \mathcal{S}(M)$ is closed; i.e. it remains to prove that $L_\mu \mathcal{S}(M)$ is closed for $\mu \in \mathbf{C}$ with $\text{Re } \mu = 0$.

Let $\text{Re } \mu = 0$. Using the assumption we derive

$$(2.17) \quad \frac{\partial}{\partial x_1} L_\mu \varphi = L_{\mu+\lambda} \frac{\partial \varphi}{\partial x_1}$$

for all $\varphi \in \mathcal{S}(M)$. From the previous considerations we know that $L_{\mu+\lambda} \mathcal{S}(M)$ is closed. It follows that $L_{\mu+\lambda}: \mathcal{S}(M) \rightarrow L_{\mu+\lambda} \mathcal{S}(M)$ is an isomorphism, because $L_{\mu+\lambda}$ is injective by (1.6). Therefore, since

$$\begin{aligned} \mathcal{F}_1 &:= \left\{ \frac{\partial \varphi}{\partial x_1} \mid \varphi \in \mathcal{S}(M) \right\} \\ &= \left\{ \psi \in \mathcal{S}(M) \mid \int_{\mathbf{R}} \psi(x_1, m^1) dx_1 = 0 \text{ for all } m^1 \in M^1 \right\} \end{aligned}$$

is closed, $L_{\mu+\lambda} \mathcal{F}_1$ is closed. Consequently, by (2.17), $(\partial/\partial x_1) L_\mu \mathcal{S}(M)$ is closed. Since $\partial/\partial x_1: \mathcal{S}(M) \rightarrow \mathcal{F}_1$ is an isomorphism, it follows that $L_\mu \mathcal{S}(M)$ is closed.

3. Main results. Let us briefly sum up our assumptions and notations: We deal with a manifold $M = \mathbf{R}^n \times K$, where K is a d -dimensional compact differentiable manifold with the property that there are d vector fields Z_1, \dots, Z_d on K such that for each $\tau \in K$ the tangent space to K at τ is spanned by the tangent vectors $Z_1(\tau), \dots, Z_d(\tau)$. For $(x, \tau) \in \mathbf{R}^n \times K$ we put $p_j(x, \tau) := x_j$ and $q(x, \tau) := \tau$. Let $(\rho_t)_{t \in \mathbf{R}}$ be a one-parameter group of transformations acting on M and let L be the associated infinitesimal transformation (see (1.1)). For $\mu \in \mathbf{C}$ we define the differential operator $L_\mu := L - \mu 1$.

THEOREM. *Let (ρ_i) be of type E. Given $k \in \mathbf{N}$, $1 \leq k \leq n$, let $M^j := \{(x, \tau) \in M \mid x_1 = \dots = x_j = 0\}$ be invariant under (ρ_i) for $j = 1, \dots, k$. We assume that the restriction of (ρ_i) to M^k is of type P and that the projection of $\rho_i(x, \tau)$ onto M^k does not depend on x_1, \dots, x_k . Suppose that there are real numbers λ_j , $1 \leq j \leq n$, $\lambda_j \neq 0$ for $j = 1, \dots, k$, $\lambda_j = 0$ for $j = k + 1, \dots, n$, such that $p_j((x, \tau) \cdot t)$ has the form*

$$(3.1) \quad p_j((x, \tau) \cdot t) = e^{-\lambda_j t} x_j + w_j(x_1, \dots, x_{j-1}, \tau, t), \quad j = 1, \dots, n,$$

where w_j are functions independent of x_j, \dots, x_n .

Then $L_\mu: \mathcal{S}(M) \rightarrow \mathcal{S}(M)$ is injective and its range is closed.

PROOF. First of all, it is easy to see that (2.1) with $\lambda = \lambda_1$ will follow from (3.1). In fact, we have

$$w_1(\tau, t) = p_1((0, \tau) \cdot t) - e^{-\lambda_1 t} 0$$

and $p_1((0, \tau) \cdot t) = 0$ since M^1 is invariant under (ρ_i) by assumption. From (2.1) we conclude that the orbit $\{(x, \tau) \cdot t \mid t \in \mathbf{R}\}$ is unbounded whenever $x_1 \neq 0$. Together with (1.6) we see that L_μ is injective for any $\mu \in \mathbf{C}$.

Now let us prove by induction on k that $L_\mu \mathcal{S}(M)$ is closed in $\mathcal{S}(M)$ for each $\mu \in \mathbf{C}$. For $k = 1$ the assertion follows by Lemma 5. Let $k > 1$. By induction hypothesis, $L_\mu^1 \mathcal{S}(M^1)$ is closed in $\mathcal{S}(M^1)$ and $L_\mu^1: \mathcal{S}(M^1) \rightarrow \mathcal{S}(M^1)$ is injective by the consideration above. Since $\mathcal{S}(M^1)$ is a Fréchet space it follows that the transpose $(L_\mu^1)^\prime: \mathcal{S}'(M^1) \rightarrow \mathcal{S}'(M^1)$ is surjective for all $\mu \in \mathbf{C}$. Thus, by Lemmas 2 and 3, $L_\mu \mathcal{S}(M)$ is closed.

EXAMPLE. On $M = \mathbf{R}^n \times \mathbf{T}^d$ ($\mathbf{T}^d = d$ -dimensional torus), $n, d \in \mathbf{N}_0$, we consider the one-parameter group

$$\rho_t(x, \tau) = (x_1 e^{\lambda_1 t}, \dots, x_n e^{\lambda_n t}, \tau_1 e^{i\alpha_1 t}, \dots, \tau_d e^{i\alpha_d t}),$$

where $\lambda_1, \dots, \lambda_n, \alpha_1, \dots, \alpha_d \in \mathbf{R}$. The infinitesimal generator L associated with (ρ_t) is given by

$$L\varphi(x, \tau) = \sum_{j=1}^n \lambda_j x_j \frac{\partial \varphi}{\partial x_j}(x, \tau) + \sum_{k=1}^d \alpha_k \frac{\partial \varphi}{\partial \tau_k}(x, \tau).$$

By the Theorem, $L_\mu \mathcal{S}(M)$ is closed in $\mathcal{S}(M)$ for any $\mu \in \mathbf{C}$ provided that $n > 0$ and $\lambda_j \neq 0$ at least for one j . (Compare [4, Example 2].) In general, $L\mathcal{S}(M)$ is not closed for $n = 0$ [4, Example 1]. Particularly, the range of the restriction of L to \mathbf{T}^d may be not closed in spite of the fact that L itself has closed range.

Furthermore, putting $d = 0$ and assuming $\lambda_j \neq 0$ for one j we can conclude that $L: \mathcal{S}'(\mathbf{R}^n) \rightarrow \mathcal{S}'(\mathbf{R}^n)$ is surjective. This should be compared with Miwa's result [5] affirming that $L: \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{B}(\mathbf{R}^n)$ is surjective if additionally it is supposed that $|\lambda_j| \leq 1$ for all $j = 1, \dots, n$, where $\mathcal{B}(\mathbf{R}^n)$ is the set of hyperfunctions on \mathbf{R}^n .

COROLLARY. *Given a first-order differential operator $\neq 0$ on \mathbf{R}^n with linear coefficients*

$$D = \sum_{i,j=1}^n a_{ij} x_j \frac{\partial}{\partial x_i} + b, \quad a_{ij}, b \in \mathbf{R}.$$

Suppose that all eigenvalues of the matrix (a_{ij}) are real.

Then $D: \mathcal{S}'(\mathbf{R}^n) \rightarrow \mathcal{S}'(\mathbf{R}^n)$ is surjective.

PROOF. After change of basis we may assume that the matrix $A = (a_{ij})$ has Jordan form

$$\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

with Jordan boxes

$$J_\rho = \begin{pmatrix} \lambda_\rho & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & & 1 & \lambda_\rho \end{pmatrix}, \quad \lambda_\rho \in \mathbf{R}, 1 \leq \rho \leq r,$$

which are arranged in such a manner that $\lambda_\rho \neq 0$ for $\rho = 1, \dots, k$ and $\lambda_\rho = 0$ for $\rho = k + 1, \dots, r$, where $0 \leq k \leq r$. It is easily seen that $D = L'_\mu$, where L is the infinitesimal generator associated with the one-parameter group $\rho_t(x) = e^{-tA}x$ and $\mu = \text{trace}(A) - b$. Therefore it is sufficient to show that $L'_\mu: \mathcal{S}(\mathbf{R}^n) \rightarrow \mathcal{S}(\mathbf{R}^n)$ is injective and has closed range.

Let $k = 0$. If $\mu \neq 0$, the assertion follows by Lemma 4. If $\mu = 0$, the assertion follows by [3].

Now let $k > 0$. Then we can apply the Theorem, where K is assumed to be trivial.

REFERENCES

1. R. Barrà, *Fonctions divergences et distributions invariantes*, Bull. Sci. Math. **105** (1981), 49–71.
2. _____, *Fonctions divergences et distributions invariantes*. II, Bull. Sci. Math. **107** (1983), 209–217.
3. R. Felix, *Solvability of differential equations with linear coefficients of nilpotent type*, Proc. Amer. Math. Soc. **94** (1985), 161–166.
4. C. S. Herz, *Functions which are divergences*, Amer. J. Math. **92** (1970), 641–656.
5. T. Miwa, *On the existence of hyperfunctions solutions of linear differential equations of the first order with degenerate real principal symbols*, Proc. Japan Acad. **49** (1973), 88–93.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, D-4800 BIELEFELD, FEDERAL REPUBLIC OF GERMANY