ON THE DICHOTOMY PROBLEM FOR TENSOR ALGEBRAS

BY

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Abstract. Let \( I, J \) be discrete spaces and \( E \subseteq I \times J \). Then either \( E \) is a \( V \)-Sidon set (in the sense of [2, §1]), or the restriction algebra \( A(E) \) is analytic. The proof is based on probabilistic methods, involving Slépian's lemma.

1. Introduction and definitions. A subset \( E \) of \( I \times J \) is called a \( V \)-Sidon set provided the restriction of \( l^\infty(I) \otimes l^\infty(J) \) coincides with \( l^\infty(E) \). It is known then that \( E \) is obtained as the finite union of "sections" \( F \subseteq I \times J \), meaning that either \( \pi_1 | F \) or \( \pi_2 | F \) is one-to-one (\( \pi_1, \pi_2 \) respective coordinate projections). Our purpose is to show that the algebra \( A(E) \), obtained by restricting \( l^\infty(I) \otimes l^\infty(J) \) to \( E \), is either \( l^\infty(E) \) or analytic. Recall that an algebra is analytic provided that only analytic functions operate on it (see [2] for more details). In view of Malliavin's characterization of analytic algebras, it amounts to showing the following (see [2, p. 102]).

**Theorem.** If \( E \subseteq I \times J \) is not a \( V \)-Sidon set, then for some \( c > 0 \)

\[
\sup_{\|\phi\|_{A(E)} \leq 1} \|e^{it\phi}\|_{A(E)} > e^{ct}, \quad t > 0.
\]

In fact, \( c \) will be an absolute constant.

2. A condition for analyticity. In this section, a criterion is explained which permits us to minorize \( \|e^{it\phi}\|_{A(E)} \). Let \( f_z \) stand for the translate of \( f \) by \( z \).

**Lemma 1.** Let \( G \) be a compact Abelian group and \( E \) be a subset of the dual group \( \Gamma \) of \( G \). Denote by \( C_E \) the space of continuous functions with Fourier transform supported by \( E \). Fix a positive integer \( l \) and assume the existence of a function \( f \) in \( C_E \) and a sequence of points \( x_1, \ldots, x_l \) in \( G \) satisfying

1. \( f(0) = \|f\|_\infty = 1 \),
2. \( \sum_{S \subseteq \{1, \ldots, l\}} |f_{\Sigma_k \in S} x_k| \leq B \) pointwise on \( G \).

(\( \Sigma_k x_k \) refers to the group operation in \( G \).) Then \( (c = \text{numerical}) \)

\[
\sup_{\|\phi\|_{A(E)} \leq 1} \|e^{it\phi}\|_{A(E)} \geq e^{ct} \quad \text{if } B < t \leq l.
\]

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Proof. Define $a(t)$ to be the left member of (3). From the simple estimation, valid in any Banach algebra $A$,

$$\left\| \prod_{k}(1 + u_k) \right\|_A \leq e^{\sum_{k} u_k^2} \sup_S e^{\Sigma u_k} \|, \quad \| u_k \| < 1/2;$$

applied to the elements

$$u_k = \frac{it}{4l} \varepsilon_k \left[ \left( 1 - i\varepsilon_k' \right) \delta_{x_k} | E + (1 + i\varepsilon_k') \delta_{-x_k} | E \right] \quad (i^2 = -1),$$

it follows that

$$\left\| \prod_{1 \leq k \leq l} \left[ 1 + \frac{it}{4l} \varepsilon_k \left[ \left( 1 - i\varepsilon_k' \right) \delta_{x_k} | E + (1 + i\varepsilon_k') \delta_{-x_k} | E \right] \right] \right\|_{A(E)} \leq 2e^{t^2/\sigma(t)}.$$ 

Here $\varepsilon \in \{1, -1\}^l$, $\varepsilon' \in \{1, -1\}^l$ will be used in an averaging argument. Let $\{d_{S(S)} \mid S \subseteq \{1, \ldots, l\}\}$ be elements of the unit disc. From the $C_0 - A(E)$ norm duality and (2), the following minoration for the left member of (4) is valid ($w_{S(S)}$ refers to the usual Walsh system):

$$\frac{1}{B} \int \left\| \prod_{1 \leq k \leq l} \left[ 1 + \frac{it}{4l} \varepsilon_k (\cdots) \right] \right\|_{A(E)} \left\| \sum_{S \subseteq \{1, \ldots, l\}} d_{S(S)}(\varepsilon) f_{\Sigma x_k} \right\|_{L(E)} \geq \frac{1}{B} \left\| \sum_{S \subseteq \{1, \ldots, l\}} d_{S(S)} \left( \frac{it}{4l} \right)^{\left| S \right|} \left( \sum_{k \in S} \left[ \left( 1 - i\varepsilon_k' \right) \delta_{x_k} + (1 + i\varepsilon_k') \delta_{-x_k} \right] \right) \right\|_{L(E)}.$$ 

For an appropriate choice of the $d_{S(S)} = d_{S(S)}(\varepsilon')$, the identity

$$\int \left\{ \sum_{k \in S} \left[ \left( 1 - i\varepsilon_k' \right) \delta_{x_k} + (1 + i\varepsilon_k') \delta_{-x_k} \right] \right\} \left( \prod_{k \in S} \frac{1 + i\varepsilon_k'}{2} \right) d\varepsilon' = \delta_{\Sigma x_k}$$

and integration in $\varepsilon'$ lead to the minoration

$$\sum_{S \subseteq \{1, \ldots, l\}} B^{-1} \left( \frac{t}{2\sqrt{2l}} \right)^{\left| S \right|} \left\| f_{\Sigma x_k} \right\|_{L(E)} \left( 1 + \frac{t}{2\sqrt{2l}} \right)^{l} \frac{1}{B}$$

as a consequence of (1). Hence $\sigma(t) \geq (1/B)e^{-t^2/1} \cdot e^{nt}$, and the result easily follows. \[ \square \]

Remark. To satisfy (1), (2) is possible only if $C_0$ contains $l^\infty_k$-subspaces of arbitrary large dimension $k$ (in the Banach space sense). Hence, a natural question is the “cotype-dichotomy” problem (explained in [4]). This conjecture was recently solved in the affirmative (see [1]), and implies that if $E$ is not a Sidon set, then

$$\sup_{\| \phi \|_{A(E)} \leq 1} \| e^{it\phi} \| > ct, \quad \forall t > 0.$$

3. Verification of the condition in the tensor algebra case. It remains to prove that if $E \subseteq I \times J$ is not a $V$-Sidon set, then (1), (2) of Lemma 1 can be realized. In this case, let $G$ be a Cantor-group $\{1, -1\}^N \times \{1, -1\}^N$ and identify $I$ (resp. $J$) with the
Rademacher sequence $\alpha_i(x)$ (resp. $\beta_j(y)$) on the first (resp. second) factor $(i, j = 1, l, \ldots)$. The following well-known (and easy) combinatorial lemma is applied to $E$ (see [2, 11.8.1]).

**Lemma 2.** If $E \subset I \times J$ is not a $V$-Sidon set, then for arbitrary $K$ there are finite subsets $I_1 \subset I$ and $J_1 \subset J$ (say $|I_1| \geq |J_1|$), and for each $i \in I_1$ a subset $A_i \subset J_1$, $|A_i| = K$, satisfying $\bigcup_{i \in I_1}(\{i\} \times A_i) \subset E$.

With those notations, let

$$f = \sum_{i \in I_1} \sum_{j \in A_i} \alpha_i \otimes \beta_j.$$ 

Thus $f(0) = K|I_1| = \|f\|_\infty$. The realization of (1), (2) above will be clear from

**Lemma 3.** Let $2^l < K^{1/A}$. Then, as for an absolute constant $C$,

$$\left(\int_{G} \left\| \sum_{S \subset \{1, \ldots, l\}} f_{\sum S^d_k} \right\|_{\infty} dz_1 \cdots dz_l \right)^{1/2} \leq CK|I_1|.$$ 

($G^l$ is the $l$-fold product $G \times \cdots \times G$ with normalized measure.)

**Proof.** Write $z \in G = \{1, -1\}^N \times \{1, -1\}^N$ as $z = (u, v)$. Thus

$$f_z = \sum_{i} \sum_{j \in A_i} \alpha_i(u) \beta_j(v) \alpha_i \otimes \beta_j.$$ 

For fixed $(x, y) \in G$, there are 1-bounded scalars $\{c_S | S \subset \{1, \ldots, l\}\}$ satisfying

$$\sum_S |f_{\sum S^d_k}(x, y)| = \left| \sum_{i \in I_1} \alpha_i(x) \sum_{S \subset \{1, \ldots, l\}} c_S \alpha_i \left( \sum_{j \in A_i} \sum_{k} u_k \right) \sum_{j \in A_i} \beta_j \left( \sum_{k} v_k \right) \beta_j(y) \right|$$

$$\leq |I_1|^{1/2} \left\{ \sum_{i \in I_1} \left| \sum_{j \in A_i} c_S \alpha_i \left( \sum_{k} u_k \right) \left( \sum_{j \in A_i} \beta_j \left( \sum_{k} v_k \right) \beta_j(y) \right) \right| \right\}^{1/2}.$$ 

The second factor may be estimated by expanding the inner square as

$$\left(\sum_{S \not= S'} \left| \sum_{i \in I_1} \sum_{j \in A_i} \beta_j \left( \sum_{k} v_k \right) \beta_j(y) \right|^2 \right)^{1/2}$$

$$\left(\sum_{S \not= S'} \left| \sum_{i \in I_1} \alpha_i \left( \sum_{k} u_k \right) \left( \sum_{j \in A_i} \beta_j \left( \sum_{k} v_k \right) \beta_j(y) \right) \right|^2 \right)^{1/2}$$

It remains to take the supremum over $y$. The first and second terms will be treated separately.
First term in (6). Fix \( i \in I_1 \). Linearize the square function by considering scalars \( \{a_S|S \subset \{1, \ldots, l\}\}, \Sigma |a_S|^2 = 1 \) so that

\[
\left( \sum_{S} \left| \sum_{j \in A_i} \beta_j \left( \sum_{k \in S} v_k \right) \beta_j(y) \right|^2 \right)^{1/2} = \left| \sum_{j \in A_i} \beta_j(y) \sum_{S} a_S \beta_j \left( \sum_{k \in S} v_k \right) \right|
\]

\[
\leq |A_i|^{1/2} \left( \sum_{j \in A_i} \left| \sum_{S} a_S \beta_j \left( \sum_{k \in S} v_k \right) \right|^2 \right)^{1/2}
\]

by the Cauchy-Schwarz inequality.

For fixed \( j \in A_i \), expand the square. Reversing the order of summation yields the estimation

\[
|A_i|^{1/2} \left( |A_i| + \sum_{S \neq S'} |a_S||a_{S'}| \left| \sum_{j \in A_i} \beta_j \left( \sum_{k \in S \Delta S'} v_k \right) \right| \right)^{1/2}
\]

\[
\leq |A_i|^{1/2} \left( |A_i| + \left( \sum_{S \neq S'} \left| \sum_{j \in A_i} \beta_j \left( \sum_{k \in S \Delta S'} v_k \right) \right|^2 \right)^{1/2} \right)^{1/2}
\]

This estimation is uniform on \( G \) and depends only on \( z_1, \ldots, z_{l} \). Since for \( S \neq S' \), \((v_1, \ldots, v_{l}) \mapsto \sum_{k \in S \Delta S'} v_k \) gives the Haar measure on \( \{1, -1\}^N \) as the image measure, the integration w.r.t. \( z_1, \ldots, z_{l} \) appearing in (5) yields the estimation for (7)

\[
|A_i|^{1/2} \left( |A_i| + 2 \left\| \sum_{j \in A_i} \beta_j \right\|_2^2 \right)^{1/2} \leq K^{1/2} (K + 2K^{1/2})^{1/2} < 2K,
\]

by hypothesis on \( l \) and \(|A_i| = K\).

Second term in (6). Let \( \{g_i(\omega)|i \in I_1\} \) denote a sequence of independent Gaussian variables on some probability space \( \Omega \), and let \( X_i(y) = \sum_{j \in A_i} \beta_j(y) \) be defined on \( \{1, -1\}^N \). The expressions

\[
\int \sup_{y'} \left| \sum_{i \in I_1} \alpha_i \left( \sum_{S \Delta S'} u_k \right) X_i \left( y \sum_{S} v_k \right) \right| du_1 \cdots du_l
\]

are dominated by

\[
\int \sup_{y, y' \in \{1, -1\}^N} \left| \sum_{i \in I_1} g_i(\omega) X_i(y) X_i(y') \right| d\omega.
\]

It follows from the inequality

\[
\left( \sum_{i \in I_1} \left| X_i(y) X_i(y') - X_i(y_1) X_i(y'_1) \right|^2 \right)^{1/2}
\]

\[
\leq K \left( \sum_{i \in I_1} \left| X_i(y) - X_i(y_1) \right|^2 + \sum_{i \in I_1} \left| X_i(y') - X_i(y'_1) \right|^2 \right)^{1/2}
\]
and Slépian's comparison lemma for Gaussian processes [3] that (8) may be estimated by

\[ CK \int \sup_{y \in \{1, -1\}^N} \left| \sum_{i \in I_1} g_i(\omega) X_i(y) \right| d\omega. \tag{9} \]

Since

\[ \left| \sum_{i \in I_1} g_i(\omega) X_i(y) \right| \leq \sum_{j \in I_1} \left| \sum_{i | j \in A_i} g_i(\omega) \right|, \]

(9) is less than

\[ CK \left( \sum_{j \in I_1} \left| \{i \mid j \in A_i\} \right|^{1/2} \right) \leq CK |I_1|^{1/2} \left( \sum_{i \in I_1} \left| A_i \right| \right)^{1/2} \leq CK^{3/2} |I_1|. \]

Therefore, (6) contributes to \( C2^kK^{3/4}|I_1|^{1/2} \). Collecting estimations, one concludes

\[ \int_{G'} \left\| \sum_{S} \left| f_{x_k \in z_k} \right| \right\|_\infty dz_1 \cdots dz_l \leq C |I_1|^{1/2} (K |I_1|^{1/2} + 2K^{3/4} |I_1|^{1/2}) < CK |I_1| \]

since \( l \) was chosen small enough. Hence (5) is proved.

4. Further remarks. (1) The result stated in the abstract can be generalized as follows: Let \( k \) be a positive integer, \( I_1, \ldots, I_k \) discrete spaces and \( E \subset (I_1 \times \cdots \times I_k) \). Then either \( E \) is a \( K \)-Sidon set, or the restriction algebra

\[ [ l^\infty(I_1) \otimes \cdots \otimes l^\infty(I_k) ] / E \]

is analytic. The argument presented above can indeed be adapted to the case of several factors. This adaptation, however, requires some additional work. (Notice that the role of the factors \( I \) and \( J \) in the previous computation is different.)

(2) Let \( F \) be a finite subset of the dual \( \Gamma \) of a compact abelian group \( G \). According to [5], call the arithmetical diameter \( d(F) \) of \( F \) the smallest number \( d \) for which there exists a subset \( P \) of the unit ball of \( PM(F) \), \( |P| = d \), such that

\[ \| f \|_\infty \leq 2 \sup_{\mu \in P} |\langle f, \mu \rangle| \quad \text{if} \ f \in C_F. \]

The method presented in this note permits us to show that for \( E \subset \Gamma \) the restriction algebra \( A(E) \) is analytic as soon as

\[ \lim_{k} \sup_{F \subset E \atop |F| = k} \frac{(\log |F|)^2}{\log d(F)} = \infty, \]

improving on the sufficient condition obtained in [5]. Details will appear elsewhere.

(3) The verification "at random" of the dichotomy conjecture for Sidon sets [6] is possible by using the criterion presented in §2 of this note.
References


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