THE DISTRIBUTION OF SOLUTIONS TO EQUATIONS OVER FINITE FIELDS

BY

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ABSTRACT. Let $F_q$ be the finite field in $q = p^f$ elements, $F(x)$ be a $k$-tuple of polynomials in $F_q[x_1, \ldots, x_n]$, $V$ be the set of points in $F_q^n$ satisfying $F(x) = 0$ and $S, T$ be any subsets of $F_q^n$. Set $\phi(V, y) = |V| - q^{n-k}$,

$$
\phi(V, y) = \sum_{\lambda \in V} e\left(\frac{2\pi i}{p} \text{Tr}(\lambda \cdot y)\right) \quad \text{for } y \neq 0.
$$

and $\Phi(V) = \max_{\lambda \in V} \phi(V, y)$. We use finite Fourier series to show that $(S + T) \cap V$ is nonempty if $|S| |T| > \Phi^2(V) q^k$. In case $q = p$ we deduce from this, for example, that if $C$ is a convex subset of $\mathbb{R}^n$ symmetric about a point in $\mathbb{Z}^n$, of diameter $< 2p$ (with respect to the sup norm), and $\text{Vol}(C) > 2^{2n} \Phi(V) p^k$, then $C$ contains a solution of $F(x) \equiv 0 \pmod{p^f}$.

We also show that if $B$ is a box of points in $F_q^n$ not contained in any $(n - 1)$-dimensional subspace and $|B| > 4 \cdot 2^n \Phi(V) q^k$, then $B \cap V$ contains $n$ linearly independent points.

1. Introduction. Let $F_q$ be the finite field in $q = p^f$ elements where $p$ is a prime. Let $F(x) = (f_1(x), \ldots, f_k(x))$ be a $k$-tuple of polynomials in $F_q[x_1, \ldots, x_n]$ and $V = V(F)$ be the algebraic subset of $F_q^n$ defined by the equations

$$
f_1(x) = \cdots = f_k(x) = 0.
$$

Considerable attention has been given to the problem of finding solutions of (1.1) in which the variables are restricted to a box of points of the type

$$
B = \left\{ x \in F_q^n : x_i = \sum_{j=1}^f x_{ij} \xi_j, a_{ij} \leq x_{ij} < a_{ij} + m_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq f \right\},
$$

where $\xi_1, \ldots, \xi_f$ is a basis for $F_q$ over $F_p$ and $a_{ij}, m_{ij}$ are integers such that $1 \leq m_{ij} \leq p$ for $1 \leq i \leq n, 1 \leq j \leq f$. (Here we have identified $F_p$ with the set of integers $\{0, 1, \ldots, p - 1\}$.) See for example Mordell [Mo1, Mo2], Chalk [Ch1, Ch2], Chalk and Williams [CW], Tietäväinen [Ti], R. Smith [Sm], Spackman [Sp] and Myerson [My].

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In this work we extend the method of Tietäväinen [Ti] by viewing it in a new way, in terms of the convolution of finite Fourier series. In so doing we obtain solutions of (1.1) in sets of the form \( S + T = \{ s + t: s \in S, t \in T \} \) where \( S \) and \( T \) are subsets of \( \mathbb{F}_q^n \); see Theorem 1.1. We also obtain linearly independent solutions of (1.1) in boxes of sufficiently large cardinality; see Theorem 1.4.

The key ingredient in the investigations mentioned above is a uniform upper bound on the function

\[
\phi(V, y) = \begin{cases} 
    \sum_{x \in V} e(x \cdot y), & \text{for } y \neq 0, \\
    |V| - q^{n-k}, & \text{for } y = 0,
\end{cases}
\]

where \( e(\alpha) = e^{(2\pi i / p) \Tr(\alpha)} \) for any \( \alpha \in \mathbb{F}_q, x \cdot y = \sum_{i=1}^n x_i y_i, \Tr \) \( \alpha \) is the trace of \( \alpha \) from \( \mathbb{F}_q \) to \( \mathbb{F}_p \) and \( |V| \) denotes the cardinality of \( V \). Set \( \Phi(V) = \max_{y \in \mathbb{F}_p^n} |\phi(V, y)| \).

From Deligne’s work on the Riemann Hypothesis, a good bound for \( \Phi(V) \) is available if \( V \) is suitably nonsingular. To be precise we shall say that a polynomial \( f(x) \) over \( \mathbb{F}_q \) is nonsingular at infinity over \( \mathbb{F}_q \) if its maximal homogeneous part is nonsingular as a form over the algebraic closure of \( \mathbb{F}_q \), and that a \( k \)-tuple \( F(x) = (f_1(x), \ldots, f_k(x)) \) is “nonsingular” at infinity over \( \mathbb{F}_q \) if every polynomial in the pencil \( \{ \lambda \cdot F = \sum_{i=1}^k \lambda_i f_i: \lambda \in \mathbb{F}_q^k, \lambda \neq 0 \} \) is of degree \( d \geq 2, \ p + d \), and is nonsingular at infinity.

If \( F(x) \) is “nonsingular” at infinity then it follows from Theorem 8.4 of Deligne [De] and the observation

\[
\phi(V, y) = q^{-k} \sum_{\lambda \in \mathbb{F}_q^k} \sum_{x \in \mathbb{F}_q^n} e(\lambda \cdot F(x) + x \cdot y)
\]

for all \( y \) in \( \mathbb{F}_q^n \), that

\[
\Phi(V) \leq (d - 1)^n q^{n/2},
\]

where \( d \) is the maximum degree of the polynomials in \( F(x) \). In the special case that \( g(x) \) is a quadratic polynomial in an odd number of variables over \( \mathbb{F}_q \) and nonsingular at infinity, one can use estimates for Salié sums to improve on (1.4). In this case \( \Phi(V(g)) \leq 2q^{n/2 - 1/2} \); see e.g. Carlitz [Car].

We can now state our main results.

**Theorem 1.1.** Let \( S \) and \( T \) be subsets of \( \mathbb{F}_q^n \) and \( V \) be an algebraic subset of \( \mathbb{F}_q^n \) as defined by (1.1). Then \( (S + T) \cap V \) is nonempty provided that \( |S||T| > \Phi^2(V)q^{2k} \).

This theorem has interesting geometric consequences. For example if we let \( q = p \), then (1.1) can be viewed as the system of congruences

\[
f_1(x) \equiv \cdots \equiv f_k(x) \equiv 0 \pmod{p},
\]

where now the \( f_i \) are taken as polynomials in \( \mathbb{Z}[x_1, \ldots, x_n] \). Let \( B_p \) be the box in \( \mathbb{R}^n \) given by \( B_p = \{ x \in \mathbb{R}^n: 0 \leq x_i < p, 1 \leq i \leq n \} \), and again let \( V \) be the set of points in \( \mathbb{F}_p^n \) satisfying (1.5). We then have

**Theorem 1.2.** If \( C \) is a convex subset of \( B_p \) containing the origin and the projections of \( C \) onto the coordinate planes and \( \Vol(C) > 2^n \Phi(V) p^k \), then \( C \) contains an integral solution of (1.5).
Of course, since \( \Phi(V) \) is invariant under translations and nonsingular linear transformations (mod \( p \)), Theorem 1.2 can be applied to a wider class of subsets of \( \mathbb{R}^n \). In Corollary 4.1 we state a similar result for any convex subset of \( \mathbb{R}^n \) symmetric about a point in \( \mathbb{Z}^n \).

Another consequence of Theorem 1.1 is the following

**Corollary 1.3.** Let \( B \) be a box of points in \( \mathbb{F}_q^n \) as given by (1.2) and \( V \) be the set of solutions of (1.1). Then \( B \cap V \) is nonempty provided that

\[
|B| > 2^n/\Phi(V)q^k.
\]

The corollary follows by applying Theorem 1.1 with

\[
S = \left\{ x \in \mathbb{F}_q^n : x_i = \sum_{j=1}^{f} a_{ij} \xi_j, 0 \leq x_{ij} < \left( (m_{ij} + 1)/2 \right), 1 \leq i \leq n, 1 \leq j \leq f \right\}
\]

and

\[
T = S + a, \quad a = (\sum_{j=1}^{f} a_{ij} \xi_j, \ldots, \sum_{j=1}^{f} a_{nj} \xi_j),
\]

observing that \( S + T \subset B \) and that \( |S| = |T| \geq 2^{-n/|B|} \). When \( V \) is defined by a set of polynomials “nonsingular” at infinity this corollary is essentially Myerson’s Theorem 2 [My]. However, we have eliminated the hypotheses of his theorem that \( p \) be sufficiently large and that \( V \) be absolutely irreducible over \( \mathbb{F}_p \). R. C. Baker [Ba, Theorem 2] can improve on Corollary 1.3 in the case that \( B \) is centered at the origin, \( p \) is sufficiently large and \( V = V(f) \), where \( f \) is a nonsingular form of degree \( \geq 3 \). He obtains a nontrivial zero \( \bar{x} \) of \( f \) with \( 0 < \max |x_i| \leq p^{1/2+\delta_{n+1}} \) where \( \delta_n = 1/(2n - 2) \) for \( n \geq 4 \) and \( \delta_3 = \frac{1}{6} \).

We shall say that the points \( x_1, \ldots, x_n \) in \( \mathbb{F}_q^n \) are linearly independent if they are linearly independent as vectors over the field \( \mathbb{F}_q \). In order for a subset of \( \mathbb{F}_q^n \) to contain \( n \) linearly independent points it is necessary that it not be contained in any \((n - 1)\)-dimensional subspace of \( \mathbb{F}_q^n \). On the other hand, if the set is a box we have

**Theorem 1.4.** Let \( B \) be a box of points in \( \mathbb{F}_q^n \) as given by (1.2) and \( V \) be the set of solutions of (1.1). If \( B \) is not contained in any \((n - 1)\)-dimensional subspace of \( \mathbb{F}_q^n \) and \( |B| > 4 \cdot 2^n/\Phi(V)q^k \), then \( B \cap V \) contains \( n \) linearly independent points.

Thus, by increasing the cardinality of \( B \) by a factor of 4 we are ensured not only of a solution of (1.1) in \( B \) (see (1.6)) but of \( n \) linearly independent solutions of (1.1) in \( B \). In particular, if \( F(x) \) is “nonsingular” at infinity then there exist \( n \) linearly independent solutions \( x = (x_1, \ldots, x_n) \) of \( F(x) = 0 \) with \( x_i = \sum_{j=1}^{f} x_{ij} \xi_j \) and

\[
\max_{i,j} |x_{ij}| \leq 4^{1/n}(d - 1)^{1/p^{1/2+k/n}},
\]

provided the latter quantity is \( < p/2 \), where \( d \) is the maximum degree of the polynomials in \( F \).

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2. Method of proof, finite Fourier series. Throughout the paper we shall abbreviate “complete” sums $\sum_{x \in \mathbb{F}_q^n}(\cdot)$ by just $\sum_{x}(\cdot)$. Let $S$ be a subset of $\mathbb{F}_q^n$ and $V$ be an algebraic subset of $\mathbb{F}_q^n$ as defined by (1.1). Let $a(x)$ be a real valued function on $\mathbb{F}_q^n$ such that $a(x) \leq 0$ for all $x$ not in $S$, and $\Sigma_{x \in V} a(x) > 0$. In order to show $S \cap V$ is nonempty it suffices to choose $a(x)$ so that $\Sigma_{x \in V} a(x) > 0$. Now $a(x)$ has a finite Fourier expansion $a(x) = \sum_{y} a(y)e(y \cdot x)$, where $a(y) = q^{-n} \Sigma_{x \in V} a(x)e(-y \cdot x)$ for $y \in \mathbb{F}_q^n$. Thus

$$\begin{align*}
\sum_{x \in V} a(x) &= \sum_{x \in V} \sum_{y} a(y)e(y \cdot x) \\
&= a(0)|V| + \sum_{y \neq 0} a(y) \sum_{x \in V} e(y \cdot x) \\
&= a(0)q^{-k} + a(0)(|V| - q^{-k}) + \sum_{y \neq 0} a(y)\phi(V, y)
\end{align*}$$

and so,

$$\tag{2.1} \sum_{x \in V} a(x) = q^{-k}\sum_{x} a(x) + \sum_{y} a(y)\phi(V, y).$$

Equation (2.1) expresses the “incomplete” sum $\sum_{x \in V} a(x)$ as a fraction of the “complete” sum $\sum_{x} a(x)$ plus an error term. In §5 we consider the problem of making an optimal choice of $a(x)$ in order to minimize the error term.

The idea of Tietäväinen [Ti] which has since been used by Chalk [Ch2] and Myerson [My] was to count the number of ways of expressing points in $V$ as the sum of points from subsets $S$ and $T$ of $\mathbb{F}_q^n$. This can be viewed as a special case of (2.1), taking $a(x)$ as the convolution of $\chi_S$ and $\chi_T$, the characteristic functions of $S$ and $T$ respectively. Chalk’s equation (15) [Ch2] is a variation of (2.1) for this choice of $a(x)$. We recall that if $a(x)$ and $\beta(x)$ are complex valued functions on $\mathbb{F}_q^n$, then their convolution, written $\alpha * \beta$, is defined by

$$\alpha * \beta(x) = \sum_{u} \alpha(u)\beta(x - u) = \sum_{u + v = x} \alpha(u)\beta(v) \quad \text{for } x \in \mathbb{F}_q^n.$$

If $H$ is an additive subgroup of $\mathbb{F}_q^n$ we define its orthogonal space $H^\perp$ as follows:

$$H^\perp = \{ x \in \mathbb{F}_q^n : \text{Tr}(x \cdot y) = 0 \text{ for all } y \in H \}.$$

Using the fact that $\mathbb{F}_q^n = H \oplus H^\perp$ one can easily deduce that the Fourier coefficients $a_H(y)$ of $\chi_H$ are given by

$$a_H(y) = \begin{cases} q^{-n}|H| & \text{if } y \in H^\perp, \\
0 & \text{if } y \not\in H^\perp. \end{cases}$$

Thus

$$\tag{2.2} \sum_{y} |a_H(y)| = 1.$$
3. Proofs of Theorems 1.1 and 1.4. Let \( S \) and \( T \) be subsets of \( \mathbb{F}_q^n \) and \( H \) be an additive subgroup of \( \mathbb{F}_q^n \). The proofs of Theorems 1.1 and 1.4 are based on the following identity:

\[
(3.1) \quad \sum_{x \in S \cap H} \chi_S \ast \chi_T(x) = q^{-k} \sum_{x \in H} \chi_S \ast \chi_T(x) + \theta \Phi(V)|S|^{1/2}|T|^{1/2}
\]

for some \( \theta \) with \(|\theta| \leq 1 \). To obtain (3.1) we use equation (2.1) with \( a(x) = (\chi_S \ast \chi_T) \cdot \chi_H(x) \). It suffices to show that the error term in (2.1) is less than \( \Phi(V)|S|^{1/2}|T|^{1/2} \) in absolute value, and so it is enough to show that \( \sum_x |a(x)| \leq |S|^{1/2}|T|^{1/2} \). Let \( a_T(y), a_S(y) \) and \( a_H(y) \) be the Fourier coefficients of \( \chi_H, \chi_S \) and \( \chi_T \) respectively. Then by elementary properties of Fourier coefficients, \( a(y) = q^{-n}(a_S \cdot a_T) \ast a_H(y) \), and so by (2.2) we have

\[
\sum_y |a(y)| \leq q^n \sum_y |(a_S \cdot a_T)(y)| \cdot \sum_y |a_H(y)| = q^n \sum_y |a_S(y)| \cdot |a_T(y)| \\
\leq q^n \left( \sum_y |a_S(y)|^2 \right)^{1/2} \left( \sum_y |a_T(y)|^2 \right)^{1/2}.
\]

Using Parseval’s identity we deduce that

\[
\sum_y |a(y)| \leq q^n \left( \sum_x |\chi_S(x)|^2 \right)^{1/2} \left( \sum_x |\chi_T(x)|^2 \right)^{1/2} = |S|^{1/2}|T|^{1/2}.
\]

To prove Theorem 1.1 we apply (3.1) with \( H = \mathbb{F}_q^n \), yielding

\[
(3.2) \quad \sum_{x \in V} \chi_S \ast \chi_T(x) \geq q^{-k}|S| |T| - \Phi(V)|S|^{1/2}|T|^{1/2}.
\]

The left-hand side of (3.2) is positive provided that \(|S||T| > \Phi^2(V)q^{2k} \).

Theorem 1.4 follows from the following proposition. For any subsets \( S, T \) and \( H \) of \( \mathbb{F}_q^n \) we set

\[
N(H, S, T) = \sum_{x \in H} \chi_S \ast \chi_T(x) = \left| \{(s, t) \in S \times T : s + t \in H \} \right|.
\]

**Proposition 3.1.** Let \( S \) and \( T \) be subsets of \( \mathbb{F}_q^n \) and \( V \) be an algebraic subset of \( \mathbb{F}_q^n \) as given by (1.1). Suppose \( \kappa \) is a number less than one such that for every \((n - 1)\) - dimensional subspace \( H \) of \( \mathbb{F}_q^n \), \( N(H, S, T) \leq \kappa|S||T| \). Then \((S + T) \cap V \) contains \( n \) linearly independent points provided that

\[
|S||T| > \left( \frac{2}{1 - \kappa} \right)^2 \Phi^2(V)q^{2k}.
\]

**Proof.** Suppose that \((S + T) \cap V \) contains no more than \((n - 1)\) linearly independent points. Then there exists an \((n - 1)\)-dimensional subspace \( H \) such that \((S + T) \cap V \subset H \), which implies that

\[
\sum_{x \in H \cap V} \chi_S \ast \chi_T(x) = \sum_{x \in V} \chi_S \ast \chi_T(x).
\]
Therefore, by (3.1) and our assumption on $N(H, S, T)$,
\[
\sum_{x \in V} \chi_S \ast \chi_T(x) \leq \kappa q^{-k} |S| |T| + \Phi(V)|S|^{1/2}|T|^{1/2}.
\]
Hence, by (3.2) we conclude that
\[
|S| |T| \leq \left(\frac{2}{1 - \kappa}\right)^2 \Phi^2(V)q^{2k}.
\]

Proof of Theorem 1.4. We simply apply the proposition to the boxes $S$ and $T$ as defined by (1.7). It suffices to show that $\kappa$ can be taken as $\frac{1}{2}$. Let $H$ be an $(n-1)$-dimensional subset of $F_q^n$. Without loss of generality we may assume that $H$ is the zero set of a linear equation $\sum a_i x_i = 0$, where $1 \leq r \leq n$ and $a_i \neq 0$ for $1 \leq i \leq r$. The quantity $N(H, S, T)$ is the number of $(s, t)$ in $S \times T$ such that $\sum a_i (s_i + t_i) = 0$. Now, $S$ and $T$ can be written as $S = S_1 \times \cdots \times S_n$ and $T = T_1 \times \cdots \times T_n$. If $|S_i| > 1$ or $|T_i| > 1$ for some $i$ with $1 \leq i \leq r$, then on solving for $s_i$ or $t_i$ respectively in the above equation we see that $N(H, S, T) \leq \frac{1}{2}|S||T|$. Thus we may suppose that $S_i = \{s_i\}$ and $T_i = \{t_i\}$ for some $s_i, t_i \in F_q$, $1 \leq i \leq r$. Since $(S + T) \not\subset H$ it follows that $N(H, S, T) = 0$ in this case.

4. Geometric consequences of Theorem 1.1. Let $F(x)$ be a $k$-tuple of polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ and $p$ be a prime. We define $V = V(F)$ and $\Phi(V)$ as in §1, reading the polynomials in $F(x)$ modulo $p$. For any subset $S$ of $\mathbb{Z}^n$ let $|\hat{S}|$ denote the number of distinct points in $S \pmod{p}$, that is $|\hat{S}| = |S + p \mathbb{Z}^n|/p \mathbb{Z}^n$. Theorem 1.1 now says that for any subsets $S$ and $T$ of $\mathbb{Z}^n$, $S + T$ contains a solution of (1.5) provided that $|\hat{S}| |\hat{T}| > \Phi^2(V)p^{2k}$. In particular if we let $C$ be any convex subset of $\mathbb{R}^n$ and let $S = \frac{1}{2}C \cap \mathbb{Z}^n = \{x \in \mathbb{Z}^n: 2x \in C\}$, then $C$ contains an integral solution of (1.5) provided that $|\hat{S}| > \Phi(V)p^k$. This follows by taking $T = S$ and observing that $S + T \subset \frac{1}{2}C + \frac{1}{2}C \subset C$.

Proof of Theorem 1.2. Let $C$ be a convex subset of $B_p$ containing the origin and the projections of $C$ onto the coordinate planes. It is easy to see that for any $x$ in $C$, $C$ contains the set of $y$ in $\mathbb{R}^n$ such that $0 \leq y_i \leq x_i$ for $1 \leq i \leq n$. Let $S = \frac{1}{2}C \cap \mathbb{Z}^n$ and let $D$ be the unit box $D = \{x \in \mathbb{R}^n: 0 \leq x_i < 1, 1 \leq i \leq n\}$. We know $\frac{1}{2}C \subset \bigcup_{x \in S}(x + D)$, for if $x \in \frac{1}{2}C$ then $y = ([x_1],[x_2],\ldots,[x_n]) \in \frac{1}{2}C \cap \mathbb{Z}^n = S$ and $x \in y + D$. Thus $\text{Vol}(\frac{1}{2}C) \leq |S| = |\hat{S}|$ and so it suffices to take $\text{Vol}(C) \geq 2^n \Phi(V)p^k$ in order for $C$ to contain a solution of (1.5).

For any $x \in \mathbb{R}^n$ let $\|x\| = \max_{i=1,\ldots,n}|x_i|$, and for any subset $S$ of $\mathbb{R}^n$ let $\|S\| = \sup_{x,y \in S} \|x - y\|$.

Corollary 4.1. Let $C$ be a convex subset of $\mathbb{R}^n$ symmetric about a point $z$ in $\mathbb{Z}^n$ such that $\|C\| < 2p$ and
\[
\text{Vol}(C) > 2^{2n-1}(\Phi(V)p^k + 1).
\]
Then $C$ contains a solution of (1.5).

Proof. Since $\Phi(V)$ is invariant under translations, we may assume that $z = 0$. Let $S = \frac{1}{2}C \cap \mathbb{Z}^n$ and suppose $\text{Vol}(C)$ satisfies (4.1). Then
\[
\text{Vol}(\frac{1}{2}C) > 2^n(\frac{1}{2}\Phi(V)p^k + \frac{1}{2}),
\]
and so by a generalized version of Minkowski’s fundamental theorem (see [Cas, Theorem II, p. 71]), \( \frac{1}{C} \) contains at least \( \Phi(V)p^k \) distinct lattice points. But as \( \| \frac{1}{C} \| < p \), this implies that \( |\hat{S}| > \Phi(V)p^k \) and so \( C \) contains a solution of (1.5).

5. Best possible choices for \( \alpha(x) \). Let \( S, V \) and \( \alpha(x) \) be as defined in §2, where without loss of generality \( \alpha(x) \) is taken so that \( \sum_x \alpha(x) = 1 \). We now seek the optimal choice of \( \alpha(x) \) in order to show \( S \cap V \) is nonempty, that is, \( \sum_{x \in S} \alpha(x) > 0 \). This amounts to minimizing the error term

\[
E(V, \alpha) = \sum_y a(y) \phi(V, y)
\]

in equation (2.1). If we bound \( E(V, \alpha) \) by saying

\[
|E(V, \alpha)| \leq \Phi(V) \sum_y |a(y)| \tag{5.1}
\]

then the problem becomes one of minimizing \( \sum_y |a(y)| \), a quantity which depends only on the pair \( S, \alpha(x) \) and not on \( V \). The following lemma gives us a lower bound on this quantity.

**Lemma 5.1.** Let \( \alpha(x) \) be a real valued function on \( \mathbb{F}_q^n \) such that \( \alpha(x) \leq 0 \) for \( x \not\in S \), \( \sum_x \alpha(x) = 1 \) and \( \alpha(x) = \sum_y a(y) e(x \cdot y) \). Then \( \sum_y |a(y)| \geq |S|^{-1} \).

**Proof.** For any subset \( W \) of \( \mathbb{F}_q^n \) it follows from the assumption \( \sum_x \alpha(x) = 1 \) that

\[
\sum_{x \in W} \alpha(x) = q^{-n} |W| + \sum_{y \not\in 0} a(y) \phi(W, y), \tag{5.2}
\]

where as before \( \phi(W, y) = \sum_{x \in W} e(x \cdot y) \) for \( y \neq 0 \). If we take \( W \) to be the complement of \( S \) in \( \mathbb{F}_q^n \), then for \( y \neq 0 \), \( \phi(W, y) = \sum_{x \in S} e(x \cdot y) - \sum_{x \in S} e(x \cdot y) = -\sum_{x \in S} e(x \cdot y) \), and so \( |\phi(W, y)| \leq |S| \). Since \( W \cap S = \emptyset \), we deduce from (5.2) that

\[
0 \geq \sum_{x \in W} \alpha(x) - q^{-n} |W| - |S| \sum_{y \neq 0} |a(y)| = 1 - |S| \sum_{y} |a(y)|,
\]

and the conclusion follows.

If \( S \) is a box of points as given by (1.2), then the lower bound in Lemma 5.1 can be obtained, up to a factor of \( 2^{nf} \). This is seen by taking \( \alpha(x) = |T|^{-1/2} |U|^{-1/2} X_T \ast X_U (x) \), where \( U \) and \( T \) are boxes as given by (1.7). As we saw in deriving equation (3.1), \( \sum_y |a(y)| \leq |T|^{-1/2} |U|^{-1/2} \leq 2^{nf} |B|^{-1} \). Thus the only improvement that can be made in Corollary 1.3 if we use a bound of the type (5.1) is a savings of a factor of \( 2^{nf} \) in (1.6).

**References**


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