HYPERREFLEXIVITY AND A DUAL PRODUCT CONSTRUCTION\textsuperscript{1}

BY

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Abstract. We show that an example of a nonhyperreflexive CSL algebra recently constructed by Davidson and Power is a special case of a general and natural reflexive subspace construction. Completely different techniques of proof are needed because of absence of symmetry. It is proven that if $\mathcal{S}$ and $\mathcal{T}$ are reflexive proper linear subspaces of operators acting on a separable Hilbert space, then the hyperreflexivity constant of $(\mathcal{S} \perp \mathcal{T})^\perp$ is at least as great as the product of the constants of $\mathcal{S}$ and $\mathcal{T}$.

This paper was inspired by the interesting “key example” in the recent paper [2] by Davidson and Power in which a nonhyperreflexive CSL algebra was constructed. In an attempt to completely understand this result we obtained a distance constant inequality of a more general nature, which we present here.

Let $H$, $K$ be separable Hilbert spaces—finite or infinite dimensional—and let $\mathcal{S}$, $\mathcal{T}$ be linear subspaces of $L(H)$, $L(K)$, which are reflexive in the Loginov-Shulman sense. ($\mathcal{S}$ is reflexive iff whenever $T \in L(H)$ is such that $Tx \in [Sx]$, $x \in H$, then $T \in \mathcal{S}$, where $[\cdot]$ means closure.) Let $\mathcal{X}(\mathcal{S})$, $\mathcal{X}(\mathcal{T})$ be the constants of hyperreflexivity of $\mathcal{S}$ and $\mathcal{T}$ as defined in [4]. We recall that a subspace $\mathcal{S}$ of $L(H)$ is hyperreflexive if there is a constant $C$ such that for operators $T$ in $L(H)$,

$$d(T, \mathcal{S}) \leq C \sup \{ \| P \perp TQ \| : P, Q \text{ are projections with } P \perp \mathcal{S} Q = 0 \},$$

and the optimal constant is denoted $\mathcal{X}(\mathcal{S})$. If $\mathcal{S}$ is reflexive but not hyperreflexive, then we define $\mathcal{X}(\mathcal{S}) = + \infty$. We make use of preannihilator techniques, and refer the reader to [1, 4, 5, 7] for details. As shown in [4], the reflexive subspaces of $L(H)$ are precisely those for which the preannihilator in $\mathcal{S} \perp = C_1$ is the $\| \cdot \|_1$-closed linear span of rank $\leq 1$ operators, where $\| \cdot \|_1$ denotes trace-class norm. Since $\mathcal{S} \perp$, $\mathcal{T} \perp$ are generated by rank $\leq 1$ operators, so is the tensor product of preannihilators $\mathcal{S} \perp \otimes \mathcal{T} \perp$. By this we mean the $\| \cdot \|_1$-closed linear subspace of the ideal of trace-class operators on $L(H \otimes K)$ generated by the elementary tensors $\{ f \otimes g: f \in \mathcal{S} \perp, g \in \mathcal{T} \perp \}$, where $H \otimes K$ denotes the usual tensor product Hilbert space. (When we write $\mathcal{S} \otimes \mathcal{T}$, we will mean the $\sigma$-weakly closed linear subspace of $L(H \otimes K)$ generated by $\{ S \otimes T: S \in \mathcal{S}, T \in \mathcal{T} \}$.) Thus the annihilator

$$(\mathcal{S} \perp \otimes \mathcal{T} \perp) = \{ A \in L(H \otimes K): \text{Tr}(Ah) = 0, h \in \mathcal{S} \perp \otimes \mathcal{T} \perp \}$$

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is a reflexive subspace of $L(H \otimes K)$. We will call this the dual product of $\mathcal{S}$ and $\mathcal{T}$ since it is in a sense dual to the usual tensor product, and will adopt the notation $\mathcal{S} \ast \mathcal{T} = (\mathcal{S}_\perp \otimes \mathcal{T}_\perp)^\perp$. We extend this term, and notation, to arbitrary $\sigma$-weakly closed subspaces.

For reflexive $\mathcal{S}$ and $\mathcal{T}$, Theorem 8 states that $\mathcal{S} \ast \mathcal{T}$ is the smallest reflexive subspace containing $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. In special cases (and perhaps in general) this coincides with the $\sigma$-weak closure of $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. This is the case in finite dimensions (Proposition 1).

It is clear that for subspaces $\mathcal{S}_i$ we have $(\mathcal{S}_1 \ast \mathcal{S}_2) \ast \mathcal{S}_3 = \mathcal{S}_1 \ast (\mathcal{S}_2 \ast \mathcal{S}_3)$ (or rather, equivalence), so we may drop parentheses with no ambiguity. An $n$-fold dual product $\mathcal{S}_1 \ast \cdots \ast \mathcal{S}_n$ will have the $n$-fold tensor product $(\mathcal{S}_1)_\perp \otimes \cdots \otimes (\mathcal{S}_n)_\perp$ as preannihilator.

The main result of this paper, Theorem 9, states that the inequality $\mathcal{H}(\mathcal{S} \ast \mathcal{T}) \geq \mathcal{H}(\mathcal{S}) \cdot \mathcal{H}(\mathcal{T})$ always holds for reflexive proper subspaces $\mathcal{S}$, $\mathcal{T}$. (If either $\mathcal{S} = L(H)$ or $\mathcal{T} = L(K)$, then $\mathcal{S} \ast \mathcal{T} = L(H \otimes K)$, so the inequality need not hold. These are the only exceptions.)

A special case arises when $\mathcal{D}$ is the algebra of $3 \times 3$ diagonal operators acting on a 3-dimensional Hilbert space. Then, since it is known (M. D. Choi, unpublished) that $\mathcal{H}(\mathcal{D}) \geq \sqrt{9/8}$, the $n$-fold dual product $\mathcal{D} \ast \cdots \ast \mathcal{D}$ has constant $\geq (9/8)^{n/2}$. This is seen to be the Davidson-Power example. The subspace $\mathcal{D}_1 \ast \cdots \ast \mathcal{D}_n$ is a bimodule over the $n$-fold tensor product $\mathcal{D} \otimes \cdots \otimes \mathcal{D}$, a m.a.s.a.; hence

$$
\begin{pmatrix}
\mathcal{D} \otimes \cdots \otimes \mathcal{D} \\
0 \\
\mathcal{D} \otimes \cdots \otimes \mathcal{D}
\end{pmatrix}
$$

is a CSL algebra. It can be shown directly, as in [2], or via duality, as in Theorem 12, that the hyperreflexivity constant for this algebra is at least as great as that of $\mathcal{D} \ast \cdots \ast \mathcal{D}$. Theorem 9 can be viewed as a generalization of the induction step in the Davidson-Power construction. Since averaging techniques utilizing symmetry do not apply, proofs are necessarily different. Prior to their example, inequalities of this nature were not suspected.

We note that while $\mathcal{D}$ is an algebra, $\mathcal{D} \ast \mathcal{D}$ is not. Hence, analysis of multi-dual-products such as $\mathcal{D} \ast \cdots \ast \mathcal{D}$ requires reflexive subspace theory. Also, we note that Propositions 1 and 3 are not used in the proofs of our main results, but are given for perspective on these.

The next lemma will be used repeatedly.

**Lemma 0.** Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be linear subspaces. Then $\mathcal{S} \ast \mathcal{T} \subseteq \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.

**Proof.** If $f \in \mathcal{S}_\perp$, $g \in \mathcal{T}_\perp$, then for each $S \in \mathcal{S}$, $T \in \mathcal{T}$, $A \in L(H)$, $B \in L(K)$ we have

$$
\text{Tr}[(S \otimes B + A \otimes T)(f \otimes g)] = \text{Tr}[(sf) \otimes (Bg)] + \text{Tr}[(Af) \otimes (Tg)]
$$

$$
= \text{Tr}(Sf) \cdot \text{Tr}(Bg) + \text{Tr}(Af) \cdot \text{Tr}(Tg) = 0
$$

since $\text{Tr}(Sf) = 0$ and $\text{Tr}(Tg) = 0$. Since the operators $S \otimes B + A \otimes T$ generate $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, and the operators $f \otimes g$ generate $\mathcal{S}_\perp \otimes \mathcal{T}_\perp = (\mathcal{S} \ast \mathcal{T})_\perp$, we conclude that $(\mathcal{S} \ast \mathcal{T})_\perp \subseteq (\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T})_\perp$, and hence that $\mathcal{S} \otimes \mathcal{T} \supseteq \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. \qed
Proposition 1. Let $H, K$ be finite dimensional Hilbert spaces, and let $\mathcal{F} \subseteq L(H)$, $\mathcal{I} \subseteq L(K)$ be linear subspaces. Then $\mathcal{F} \ast \mathcal{I} = \mathcal{F} \otimes L(K) + L(H) \otimes \mathcal{I}$.

Proof. Let $n_1, n_2$ be the dimensions of $H, K$, respectively, and let $m_1, m_2$ be the vector space dimensions of $\mathcal{F}, \mathcal{I}$, respectively. Then $\dim(L(H)) = n_1^2$, $\dim(L(K)) = n_2^2$, $\dim(\mathcal{F} \perp) = n_1^2 - m_1$, and $\dim(\mathcal{I} \perp) = n_2^2 - m_2$. So $\dim(\mathcal{F} \otimes \mathcal{I} \perp) = (n_1^2 - m_1)(n_2^2 - m_2)$, and thus

$$\dim(\mathcal{F} \otimes \mathcal{I}) = n_1^2 n_2^2 - (n_1^2 - m_1)(n_2^2 - m_2) = n_1^2 m_2 + m_1 n_2^2 - m_1 m_2.$$

If $X, Y$ are finite dimensional vector spaces over $\mathbb{C}$, and if $X_1 \subseteq X, Y_1 \subseteq Y$ are linear subspaces, then $(X_1 \otimes Y) \cap (X \otimes Y_1) = X_1 \otimes Y_1$. Thus $(\mathcal{F} \otimes L(K)) \cap (L(H) \otimes \mathcal{I}) = \mathcal{F} \otimes \mathcal{I}$. So

$$\dim(\mathcal{F} \otimes L(K) + L(H) \otimes \mathcal{I}) = \dim(\mathcal{F} \otimes L(K)) + \dim(L(H) \otimes \mathcal{I}) - \dim(\mathcal{F} \otimes \mathcal{I}) = m_1 n_2^2 + n_1^2 m_2 - m_1 m_2 = \dim(\mathcal{F} \ast \mathcal{I}).$$

So by Lemma 0 we must have $\mathcal{F} \ast \mathcal{I} = \mathcal{F} \otimes L(K) + L(H) \otimes \mathcal{I}$. □

If $\mathcal{A} \subseteq L(H)$ is a reflexive algebra and $T \in L(H)$, the Arveson estimate for the distance from $T$ to $\mathcal{A}$ is $\alpha(T, \mathcal{A}) = \sup\{\|P^\bot TP\|: P \in \text{lat } \mathcal{A}\}$. For reflexive subspaces $\mathcal{F} \subseteq L(H)$ the estimate is defined [4] by $\alpha(T, \mathcal{F}) = \sup\{\|P^\bot TQ\|: P, Q$ are projections with $P^\bot Q = 0\}$. This agrees with the “$P^\bot P^\bot$” formula when $I \in \mathcal{F}$. There is a “projection free” characterization of this estimate which proves useful. Let $d(\cdot, \cdot)$ denote distance.

Lemma 2. Let $\mathcal{F} \subseteq L(H)$ be a reflexive subspace. Then $\alpha(T, \mathcal{F}) = \sup\{d(Tx, \mathcal{F} x): x \in H, \|x\| = 1\}, T \in L(H)$.

Proof. We have $d(Tx, \mathcal{F} x) = \|P^\bot T x\|$, where $P$ is the orthogonal projection onto $[\mathcal{F} x]$. Let $\|x\| = 1$ and let $Q$ be the projection onto $C x$. Then $\|P^\bot T x\| = \|P^\bot T Q\|$. Since $P^\bot Q = 0$, the inequality “$\geq$” follows.

Conversely, suppose $P, Q$ are projections with $P^\bot \mathcal{F} Q = 0$. Let $\epsilon > 0$ be given, and choose $x \in QH, \|x\| = 1$, such that $\|P^\bot T x\| \geq \|P^\bot T Q\| - \epsilon$. Then

$$d(Tx, \mathcal{F} x) \geq d(Tx, PH) = \|P^\bot T x\| \geq \|P^\bot T Q\| - \epsilon.$$

Since $\epsilon$ was arbitrary, we have $\|P^\bot T Q\| \leq \sup\{d(T, \mathcal{F} x): x \in H, \|x\| = 1\}$. Taking the supremum over all pairs $\{P, Q\}$ with $P^\bot \mathcal{F} Q = 0$ completes the proof. □

Lemma 2 points out that only cyclic projections $P$ need be considered in distance estimate computations. Also, taking this as the definition permits natural extension of the concept to general normed linear spaces.

If $\mathcal{F} \neq L(H)$ is reflexive and $T \notin \mathcal{F}$, we write $\mathcal{K}(T, \mathcal{F}) = d(T, \mathcal{F})/\alpha(T, \mathcal{F})$. So $\mathcal{K}(\mathcal{F}) = \sup\{\mathcal{K}(T, \mathcal{F}): T \in L(H), T \notin \mathcal{F}\}$. By convention $\mathcal{K}(L(H)) = 1$. $\mathcal{F}$ is hyperreflexive if $\mathcal{K}(\mathcal{F}) < \infty$, and is nonhyperreflexive otherwise.

We first give an initial generalization of the Davidson-Power induction step in which use is made of symmetry. The proof is more direct than that of our general result, so is included for perspective.
Proposition 3. Let $\mathcal{S}$ be a reflexive subspace of $L(H)$, with $\mathcal{S} \neq L(H)$. Let

$$
\mathcal{G} = \begin{pmatrix}
* & \mathcal{S} & \mathcal{S} \\
\mathcal{S} & * & \mathcal{S} \\
\mathcal{S} & \mathcal{S} & *
\end{pmatrix}
$$

be the subspace of all $3 \times 3$ operator matrices with diagonal elements arbitrary and off-diagonal elements in $\mathcal{S}$. Then $\mathcal{K} (\mathcal{G}) \geq \sqrt{9/8} \cdot \mathcal{K} (\mathcal{S})$.

Proof. First, suppose $\mathcal{K} (\mathcal{S})$ is finite. Fix $\epsilon > 0$. Choose $T \in L(H)$ for which $\mathcal{K} (T, \mathcal{S}) = \mathcal{K} (\mathcal{S}) - \epsilon$. Let

$$
T = \begin{pmatrix}
T & T & T \\
T & T & T \\
T & T & T
\end{pmatrix}.
$$

The averaging technique used in the proof of Theorem 1.1 in [2] yields without modification that $d(T, \mathcal{S}) = \frac{1}{\sqrt{2}} d(T, \mathcal{S})$. We need only show that $d(T, \mathcal{S}) < \sqrt{2} d(T, \mathcal{S})$, for then

$$
\mathcal{K} (\mathcal{S}) = \mathcal{K} (T, \mathcal{S}) \geq \sqrt{9/8} \mathcal{K} (T, \mathcal{S}) > \sqrt{9/8} \mathcal{K} (\mathcal{S}) - \epsilon,
$$

and since $\epsilon$ was arbitrary the desired inequality would follow.

To show that $d(T, \mathcal{S}) < \sqrt{2} d(T, \mathcal{S})$, it is useful to use Lemma 2. Let

$$
\tilde{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
$$

be an arbitrary unit vector in $H \otimes H_3$. Then

$$
\tilde{T} \tilde{x} = \begin{pmatrix}
Tz_1 \\
Tz_2 \\
Tz_3
\end{pmatrix},
$$

where $z = x_1 + x_2 + x_3$. Descriptively, we have

$$
\tilde{\mathcal{G}} \tilde{x} = \begin{pmatrix}
* & \mathcal{S} & \mathcal{S} \\
\mathcal{S} & * & \mathcal{S} \\
\mathcal{S} & \mathcal{S} & *
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
L(H)x_1 + \mathcal{S}x_2 + \mathcal{S}x_3 \\
\mathcal{S}x_1 + L(H)x_2 + \mathcal{S}x_3 \\
\mathcal{S}x_1 + \mathcal{S}x_2 + L(H)x_3
\end{pmatrix}.
$$

We consider three cases:

1. If neither $x_1$, $x_2$ nor $x_3 = 0$, then $\tilde{\mathcal{G}} \tilde{x} = H \otimes H_3$, so $d(\tilde{T} \tilde{x}, \tilde{\mathcal{G}} \tilde{x}) = 0$.

2. If precisely one of $x_1$, $x_2$, $x_3$ is 0, then without loss of generality we may assume $x_1 = 0$ by noting that $\tilde{\mathcal{G}}$ is invariant under the group of unitary transformations corresponding to permutation of basis vectors in $H_3$. We have $z = x_2 + x_3$, and

$$
\tilde{\mathcal{G}} \tilde{x} = \begin{pmatrix}
\mathcal{S}x_2 + \mathcal{S}x_3 \\
L(H) \\
L(H)
\end{pmatrix},
$$

so

$$
d(\tilde{T} \tilde{x}, \tilde{\mathcal{G}} \tilde{x}) = d(Tz, \mathcal{S}x_2 + \mathcal{S}x_3) \leq d(Tz, \mathcal{S}z).
$$
We have $||z|| \leq \sqrt{2}$. If $z = 0$, then $d(\tilde{T}x, \tilde{y}x) = 0$. If $z \neq 0$, let $w = z/||z||$. Then
\[
d(\tilde{T}x, \tilde{y}x) \leq \sqrt{2} d(Tw, \mathcal{S} w) \leq \sqrt{2} \alpha(T, \mathcal{S}),
\]
as desired.

(3) If precisely two of $x_1, x_2, x_3$ are 0, via permutation as above, we may assume $x_1 = x_2 = 0$. Then $z = x_3$, so $||z|| = 1$. We have
\[
\tilde{\mathcal{F}} = \begin{pmatrix}
\mathcal{S} z \\
\mathcal{S} z \\
L(H)
\end{pmatrix},
\]
so
\[
d(\tilde{T}x, \tilde{y}x) = \sqrt{2} d(Tz, \mathcal{S} z) \leq \sqrt{2} \alpha(T, \mathcal{S}).
\]

Now from cases (1)–(3) we have
\[
\alpha(\tilde{T}, \tilde{\mathcal{F}}) = \sup \left\{ d(\tilde{T}x, \tilde{y}x) : \tilde{x} \in H \otimes H_3, ||\tilde{x}|| = 1 \right\}
\leq \sqrt{2} \alpha(T, \mathcal{S}),
\]
as required. For the case $\mathcal{K}(\mathcal{S}) = \infty$, let $n \geq 1$ be arbitrary and choose $T$ with $\mathcal{K}(T, \mathcal{S}) \geq n$. The same argument as above shows that $\mathcal{K}(\tilde{T}, \tilde{\mathcal{F}}) \geq \sqrt{9/8} n$. Hence $\mathcal{K}(\tilde{\mathcal{F}}) = +\infty$. □

A simple duality computation shows that the preannihilator of $\tilde{\mathcal{F}}$ in Proposition 3 has the form
\[
\tilde{\mathcal{P}} = \begin{pmatrix}
0 & \mathcal{S} \perp \\
\mathcal{S} \perp & 0 & \mathcal{S} \perp \\
\mathcal{S} \perp & \mathcal{S} \perp & 0
\end{pmatrix},
\]
where $\mathcal{S} \perp$ is the preannihilator of $\mathcal{S}$. The preannihilator of $\mathcal{D}_3$ has the form
\[
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix},
\]
so $\tilde{\mathcal{P}} = \mathcal{S} \perp \otimes (\mathcal{D}_3) \perp$, and hence $\tilde{\mathcal{P}} = (\mathcal{S} \perp \otimes (\mathcal{D}_3) \perp) \perp = \mathcal{S} \ast \mathcal{D}_3$. This suggests that a generalization is possible. The next proposition is used in place of an averaging technique.

**Proposition 4.** Let $\mathcal{S} \subseteq L(H)$, $\mathcal{I} \subseteq L(K)$ be $\sigma$-weakly closed subspaces. If $A \in L(H)$ and $B \in L(K)$ are arbitrary, then $d(A \otimes B, \mathcal{S} \ast \mathcal{I}) = d(A, \mathcal{S}) \cdot d(B, \mathcal{I})$.

**Proof.** Let $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{I}$. By Lemma 0 we have $\mathcal{R} \subseteq \mathcal{S} \ast \mathcal{I}$, so for each $S \in \mathcal{S}$ and $T \in \mathcal{I}$ we have $(A - S) \otimes (B - T) - A \otimes B \in \mathcal{R} \subseteq \mathcal{S} \ast \mathcal{I}$. Thus
\[
d(A \otimes B, \mathcal{S} \ast \mathcal{I}) = d((A - S) \otimes (B - T), \mathcal{S} \ast \mathcal{I}) \leq ||A - S|| \cdot ||B - T||.
\]
It follows that $d(A \otimes B, \mathcal{S} \ast \mathcal{I}) \leq d(A, \mathcal{S}) \cdot d(B, \mathcal{I})$. 

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For the reverse inequality, let \( \varepsilon > 0 \) be given and choose \( f \in \mathcal{S} \), \( g \in \mathcal{T} \) with \( \|f\|_1 = \|g\|_1 = 1 \), such that \( \text{Tr}(Af) > d(A, \mathcal{S}) - \varepsilon \), and \( \text{Tr}(Bg) > d(B, \mathcal{T}) - \varepsilon \). Let \( h = f \otimes g \). We have

\[
|\text{Tr}[(A \otimes B)h]| = |\text{Tr}(Af \otimes Bg)| = |\text{Tr}(Af)| \cdot |\text{Tr}(Bg)|
\>
>(d(A, \mathcal{S}) - \varepsilon) \cdot (d(B, \mathcal{T}) - \varepsilon).
\]

Since \( h \) is a norm 1 operator in \( \mathcal{S} \otimes \mathcal{T} \), and this is the preannihilator of \( \mathcal{S} \otimes \mathcal{T} \) by definition, this implies that \( d(A \otimes B, \mathcal{S} \otimes \mathcal{T}) > (d(A, \mathcal{S}) - \varepsilon) \cdot (d(B, \mathcal{T}) - \varepsilon) \). Since \( \varepsilon \) was arbitrary, the proof is complete. \( \Box \)

**Lemma 5.** Let \( \mathcal{S} \subseteq L(H) \) be a linear subspace, and let \( x \) be a vector in \( H \otimes K \). Let \( F \) be the smallest projection in \( L(H) \) such that \( (F \otimes I)x = x \). Let \( P \) be the orthogonal projection onto \( [\mathcal{S} F H] \). Then \( P \otimes I \) is the orthogonal projection onto \( [(\mathcal{S} \otimes L(K))x] \).

**Proof.** Let \( \{e_1, e_2, \ldots\} \) be any orthonormal basis for \( K \). Then there is a sequence \( \{x_j\} \) of vectors in \( H \) with \( \sum \|x_j\|^2 = \|x\|^2 \) such that \( x = \sum x_j \otimes e_j \). Let \( E_i \) be the projection onto \( Ce_i \). If \( S \in \mathcal{S}, T \in L(K) \), then \( (S \otimes T E_i)x = S x_i \otimes T e_j \). Hence \( [(\mathcal{S} \otimes L(K))x] \) contains all vectors of the form \( S x_i \otimes y \) for arbitrary \( S \in \mathcal{S}, y \in K \), for each \( i \). Let \( F' \) be the projection onto the closed span of vectors \( \{x_j: i = 1, 2, \ldots\} \). Then \( F' \geq F \), and we have

\[
[(\mathcal{S} \otimes L(K))x] \supseteq [(\mathcal{S} F' H) \otimes K] \supseteq [(\mathcal{S} F H) \otimes K]
\]

so

\[
[(\mathcal{S} \otimes L(K))x] = [(\mathcal{S} F H) \otimes K] = (P \otimes I)(H \otimes K). \quad \Box
\]

**Lemma 6.** Let \( H \) be a Hilbert space, let \( \mathcal{S} \subseteq L(H) \) be a linear subspace, and let \( h \in \mathcal{S} \perp \) be a rank-1 operator. Then \( hP = 0 \), where \( P \) is the orthogonal projection onto \( [\mathcal{S} h H] \).

**Proof.** Write \( h = v \otimes u \), where \( u, v \) are vectors such that \( hv = (w, v)u, w \in H \). Then \( [\mathcal{S} h H] = [\mathcal{S} u] \). If \( S \in \mathcal{S} \) we have \( 0 = \text{Tr}(Sh) = (Su, v) \), so \( [\mathcal{S} u] \perp v \); hence \( P v = 0 \). Then \( hP = (P v) \otimes u = 0 \). \( \Box \)

**Lemma 7.** Let \( \mathcal{S} \subseteq L(H) \), \( \mathcal{F} \subseteq L(K) \) be linear subspaces, and let \( \mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{F} \). Let \( x \in H \otimes K \). Let \( F \in L(H) \), \( E \in L(K) \) be the smallest projections such that \( F \otimes I \) and \( I \otimes E \) contain \( x \) in their range, and let \( P \) be the projection onto \( [\mathcal{S} F H] \) and \( Q \) the projection onto \( [\mathcal{F} E K] \). The projection onto \( [\mathcal{R} x] \) is then \( (P \perp Q) \perp \perp \).

**Proof.** We have \( [\mathcal{R} x] = [(\mathcal{S} \otimes L(K))x] \vee [(L(H) \otimes \mathcal{F})x] \). By Lemma 5, the projections onto \( [(\mathcal{S} \otimes L(K))x] \) and \( [(L(H) \otimes \mathcal{F})x] \) are \( P \otimes I \) and \( I \otimes Q \), respectively. The projection onto \( [\mathcal{R} x] \) is then \( (P \otimes I) \vee (I \otimes Q) \), and since \( P \otimes I \) and \( I \otimes Q \) commute this reduces to \( P \otimes I + I \otimes Q - P \otimes Q \). The orthogonal complement is then

\[
I \otimes I - P \otimes I - I \otimes Q + P \otimes Q = P \perp \otimes I - P \perp \otimes Q = P \perp \otimes Q \perp.
\]

so \( \text{proj}[\mathcal{R} x] = (P \perp \otimes Q \perp) \perp \). \( \Box \)
If $\mathcal{S}$ is a linear subspace of $L(H)$, we adopt the notation $ref(\mathcal{S})$ to mean the smallest reflexive subspace of $L(H)$ containing $\mathcal{S}$. Thus $ref(\mathcal{S}) = \{ T \in L(H) : Tx \in [\mathcal{S}x], x \in H \}$.

**Theorem 8.** Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be reflexive subspaces. Then $\mathcal{S} \ast \mathcal{T}$ is the smallest reflexive subspace containing $\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.

**Proof.** Let $\mathcal{A} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, and let $\mathcal{A} = ref(\mathcal{A})$. By definition, $\mathcal{A}$ and $\mathcal{A}$ have the same closed cyclic subspaces. Since $\mathcal{S} \ast \mathcal{T}$ is reflexive and contains $\mathcal{A}$ it contains $\mathcal{A}$. To show equality, it will suffice to show that every rank-1 operator in $\mathcal{A} \ast$ is in $(\mathcal{S} \ast \mathcal{T}) \ast = \mathcal{S} \ast \mathcal{T} \ast$. Let $h$ be a rank-1 operator in $\mathcal{A} \ast$, and let $x$ be a nonzero vector in the range of $h$. By Lemma 6, $h = hG \perp$, where $G$ is the projection onto $[\mathcal{A}x] = [\mathcal{S}x]$. Let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = x = (I \otimes E)x$, and let $P = proj[\mathcal{S}FH]$, $Q = proj[\mathcal{T}EK]$. Then by Theorem 7, $G \perp = P \perp \otimes Q \perp$. We have $(F \otimes E)x = x$; hence

$$h = (F \otimes E)h = (F \otimes E)(P \perp \otimes Q \perp) \in (F \otimes E)(\mathcal{L}_*(H \otimes K))(P \perp \otimes Q \perp),$$

where $\mathcal{L}_*(H \otimes K)$ denotes the ideal of trace-class operators on $H \otimes K$. Since $\mathcal{L}_*(H \otimes K)$ is the trace-class norm closed span of elementary tensors $\{ f \otimes g : f \in \mathcal{L}_*(H), g \in \mathcal{L}_*(K) \}$, the space $(F \otimes E)(\mathcal{L}_*(H \otimes K))(P \perp \otimes Q \perp)$ is the closed span of elementary tensors $\{ (FP \perp) \otimes (EQ \perp) : f \in \mathcal{L}_*(H), g \in \mathcal{L}_*(K) \}$.

But for $f$ arbitrary and $S \in \mathcal{S}$ we have $Tr(SFP \perp) = Tr(P \perp SF) = 0$ since $P \perp \mathcal{S}F = 0$. So $FP \perp \in \mathcal{S} \perp$. Similarly, $EQ \perp \in \mathcal{T} \perp$ for all $g \in \mathcal{L}_*(K)$. So each $(FP \perp) \otimes (EQ \perp) \in \mathcal{S} \perp \otimes \mathcal{T} \perp$, and hence $h \in \mathcal{S} \perp \otimes \mathcal{T} \perp$. $\square$

**Theorem 9.** Let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be reflexive proper subspaces. Then $\mathcal{X}(\mathcal{S} \ast \mathcal{T}) \geq \mathcal{X}(\mathcal{S}) \ast \mathcal{X}(\mathcal{T})$.

**Proof.** Let $A \in L(H)$, $B \in L(K)$ be arbitrary. By Proposition 4 we have $d(A \otimes B, \mathcal{S} \ast \mathcal{T}) = d(A, \mathcal{S}) \cdot d(B, \mathcal{T})$. We will show that in general $\alpha(A \otimes B, \mathcal{S} \ast \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$, and hence if $A \notin \mathcal{S}$ and $B \notin \mathcal{T}$ then $\mathcal{X}(A \otimes B, \mathcal{S} \ast \mathcal{T}) \geq \mathcal{X}(A, \mathcal{S}) \ast \mathcal{X}(B, \mathcal{T})$. Taking of suprema over all such $A$, $B$ then yields $K(\mathcal{S} \ast \mathcal{T}) \geq \mathcal{X}(\mathcal{S}) \ast \mathcal{X}(\mathcal{T})$, since by hypothesis $\mathcal{S} \neq L(H)$ and $\mathcal{T} \neq L(K)$.

We utilize Lemma 2. Let $x$ be a unit vector in $H \otimes K$. Let $\mathcal{A} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. By Theorem 8, $\mathcal{S} \ast \mathcal{T} = ref(\mathcal{A})$, so $\mathcal{S} \ast \mathcal{T}$ and $\mathcal{A}$ have the same cyclic subspaces. As in the proof of Theorem 8, let $G$ be the projection onto $[\mathcal{A}x] = [(\mathcal{S} \ast \mathcal{T})x]$, and let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = (I \otimes E)x$. Let $P = proj[\mathcal{S}FH]$, $Q = proj[\mathcal{T}EK]$. By Lemma 7, $G = (P \perp \otimes Q \perp)$. Since $(F \otimes E)x = x$ we have

$$d \left[ (A \otimes B)x, (\mathcal{S} \ast \mathcal{T})x \right] = \left\| G \perp (A \otimes B)x \right\| = \left\| (P \perp \otimes Q \perp)(A \otimes B)(F \otimes E)x \right\|$$

$$= \left\| (P \perp AF) \otimes (Q \perp BE) \right\| \leq \left\| P \perp AF \right\| \cdot \left\| Q \perp BE \right\| \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$$

since $P \perp \mathcal{S}F = 0$ and $Q \perp \mathcal{T}E = 0$.

Since $x$ was an arbitrary unit vector, we have $\alpha(A \otimes B, \mathcal{S} \ast \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$, as required. $\square$
The proof of Theorem 9 can be improved slightly to show that for arbitrary $A \in L(H)$, $B \in L(K)$ the inequality $\alpha(A \otimes B, \mathcal{S} \otimes \mathcal{T}) \leq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$ is in fact an equality. We capture this fact.

**Corollary 10.** Let $\mathcal{S} \subseteq L(H)$ and $\mathcal{T} \subseteq L(K)$ be reflexive subspaces, and let $A \in L(H)$, $B \in L(K)$ be arbitrary. Then $\alpha(A \otimes B, \mathcal{S} \otimes \mathcal{T}) = \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$.

**Proof.** The inequality “$\leq$” is contained in the proof of Theorem 9. For the converse, let $F$, $P \in L(H)$ and $E$, $Q \in L(K)$ be arbitrary projections satisfying $P \perp \mathcal{S}F = 0$ and $Q \perp \mathcal{T}E = 0$. Then if $\mathcal{R} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$, we have $(P \perp Q \perp) \mathcal{R}(F \otimes E) = 0$, so since $\mathcal{S} \otimes \mathcal{T} = \text{ref}(\mathcal{R})$ by Theorem 8, we have $(P \perp Q \perp)(\mathcal{S} \otimes \mathcal{T})(F \otimes E) = 0$. Since $\| (P \perp Q \perp)(A \otimes B)(F \otimes E) \| = \| P \perp AF \| \cdot \| Q \perp BE \|$, we have

$$\alpha(A \otimes B, \mathcal{S} \otimes \mathcal{T}) = \sup \{ \| L(A \otimes B)M \| : L, M \text{ are projections} \}
$$

in $L(H \otimes K)$ with $L(\mathcal{S} \otimes \mathcal{T})M = 0$)

$$\geq \| P \perp AF \| \cdot \| Q \perp BE \|.$$

So since $P$, $F$, $Q$, $E$ were arbitrary, we conclude that $\alpha(A \otimes B, \mathcal{S} \otimes \mathcal{T}) \geq \alpha(A, \mathcal{S}) \cdot \alpha(B, \mathcal{T})$. Finally, we note that equality is trivially true if either $\mathcal{S} = L(H)$ or $\mathcal{T} = L(K)$. □

From Corollary 10 and Proposition 4 we conclude that $\mathcal{K}(A \otimes B, \mathcal{S} \otimes \mathcal{T}) = \mathcal{K}(A, \mathcal{S}) \cdot \mathcal{K}(B, \mathcal{T})$ whenever $\mathcal{S}$, $\mathcal{T}$ are reflexive proper subspaces with $A \in \mathcal{S}$, $B \in \mathcal{T}$. That is, the basic inequality is an equality when restricted to the class of elementary tensors. It can happen, however, that for some operator $T \in L(H \otimes K)$, which is not an elementary tensor, we have $\mathcal{K}(T, \mathcal{S} \otimes \mathcal{T}) > \mathcal{K}(\mathcal{S}) \cdot \mathcal{K}(\mathcal{T})$, and thus the inequality in Theorem 9 may be strict. The following simple example shows this.

**Example 11.** Let

$$S = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\}$$

be regarded as a subspace of operators acting on 2-dimensional Hilbert space. An elementary computation shows that $S$ is reflexive. An application of [6, Lemma 3.3] after interchanging rows, and either [4, Proposition 3 or 5, Theorems 1.1 or 1.2] to the preannihilator

$$S_\perp = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix},$$

shows that $\mathcal{K}(S) = 1$. Since

$$\mathcal{S}_\perp \otimes \mathcal{S}_\perp = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & 0 & * \\ * & * & * & * \end{pmatrix},$$

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we have

\[
S \ast S = \begin{pmatrix}
* & * & * & 0 \\
* & * & 0 & 0 \\
* & 0 & * & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

acting on 4-dimensional Hilbert space. Let

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

Then \(P(S \ast S) \subseteq S \ast S\), and thus by [5, Lemma 1.3] the compression \(P(S \ast S)|_{pH}\) is reflexive with hyperreflexivity constant no greater than that of \(S \ast S\). But this compression has diagram

\[
\begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\]

and by [5, Example 4.7] this has constant \(\geq \sqrt{9/8}\). Thus \(\mathcal{K}(S \ast S) \geq \sqrt{9/8} > 1 = \mathcal{K}(S) \cdot \mathcal{K}(S)\). So in this case the inequality of Theorem 9 is strict. \(\Box\)

**Theorem 12.** Let \(n\) be a positive integer, and for \(1 \leq i \leq n\) let \(\mathcal{A}_i\) be a reflexive proper subalgebra of \(\mathcal{L}(H_i)\) for \(H_i\) a separable Hilbert space. Suppose the tensor product \(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n\), acting on \(H = H_1 \otimes \cdots \otimes H_n\), is reflexive. Let

\[
\mathcal{A} = \begin{pmatrix}
\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n & \mathcal{A}_1 \ast \cdots \ast \mathcal{A}_n \\
0 & \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n
\end{pmatrix}.
\]

Then \(\mathcal{A}\) is a reflexive subalgebra of \(\mathcal{L}(H \otimes H)\), and \(\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{A}_1) \cdots \mathcal{K}(\mathcal{A}_2) \cdots \mathcal{K}(\mathcal{A}_n)\).

**Proof.** \(\mathcal{A}\) is an algebra since \(\mathcal{A}_1 \ast \cdots \ast \mathcal{A}_n\) is a bimodule over \(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n\). A simple calculation shows that

\[
\mathcal{A}_\perp = \begin{pmatrix}
(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp & L_\ast(H) \\
(\mathcal{A}_1 \ast \cdots \ast \mathcal{A}_n)_\perp & (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp
\end{pmatrix}.
\]

Since \(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n\) is reflexive, \((\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)_\perp\) is generated by rank-1 operators. Since \(\mathcal{A}_1 \cdots \mathcal{A}_n\) are reflexive, \((\mathcal{A}_1 \ast \cdots \ast \mathcal{A}_n)_\perp = (\mathcal{A}_1)_\perp \otimes \cdots \otimes (\mathcal{A}_n)_\perp\) is also generated by rank-1 operators. Hence \(\mathcal{A}_\perp\) is generated by rank-1 operators, so \(\mathcal{A}\) is reflexive.

To show \(\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{A}_1) \cdots \mathcal{K}(\mathcal{A}_n)\) we utilize [4, Proposition 3]. Let \(P\) be the orthogonal projection from \(H \oplus H\) onto \(H\). Let \(S = \mathcal{A}_1 \ast \cdots \ast \mathcal{A}_n\). By Theorem 9, \(\mathcal{K}(S) = \mathcal{K}(\mathcal{A}_1) \cdots \mathcal{K}(\mathcal{A}_n)\). Let \(\mathcal{G}_1(\mathcal{A}_i), \mathcal{G}_1(S)\) denote the closed convex hulls of the rank \(\leq 1\) operators in the unit balls of \(\mathcal{A}_\perp, S\), respectively. Then clearly

\[
P \perp \mathcal{G}_1(\mathcal{A}) P = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]
Let $R(\mathcal{A})$, $R(S)$ be the largest radii such that \{ $f \in \mathcal{A}_\perp : \|f\|_1 \leq R(\mathcal{A})$ \} $\subseteq \mathcal{E}_1(\mathcal{A})$ and \{ $g \in S_\perp : \|g\|_1 \leq R(S)$ \} $\subseteq \mathcal{E}_1(S)$. It follows that $R(\mathcal{A}) \leq R(S)$. By [4, Proposition 3] we have $\mathcal{K}(\mathcal{A}) = 1/R(\mathcal{A})$ and $\mathcal{K}(S) = 1/R(S)$, so $\mathcal{K}(\mathcal{A}) \geq \mathcal{K}(S)$, as required. □

Remarks. The requirement that $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ be reflexive will be met if each $\mathcal{A}_i$ is finite dimensional, and more generally, if each $\mathcal{A}_i$ has property $S_n$ (Kraus [3]). (It is, of course, an open question whether the tensor product of reflexive algebras is necessarily reflexive.) As in the special case of the “key example” in [2], Theorem 12 gives a means of constructing reflexive algebras of arbitrarily large distance constant. If each $\mathcal{A}_i$ is a CSL algebra and so contains a m.a.s.a., then $\mathcal{A}$ will also contain a m.a.s.a., so will be a CSL algebra. A direct sum of such algebras, with increasing constants, will be nonhyperreflexive.

REFERENCES


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