TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 294, Number 1, March 1986

ON SIEVED ORTHOGONAL POLYNOMIALS. III: ORTHOGONALITY ON SEVERAL INTERVALS[1]

BY

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ABSTRACT. We introduce two generalizations of Chebyshev polynomials. The continuous spectrum of either is $\{x: -2\sqrt{c}/(1+c) \leqslant T_k(x) \leqslant 2\sqrt{c}/(1+c)\}$, where c is a positive parameter. The weight function of the polynomials of the second kind is $\{1-((1+c)^2/4c)T_k^2(x)\}^{1/2}/|U_{k-1}(x)|$ when $c \geqslant 1$. When $c < 1$ we pick up discrete masses located at the zeros of $U_{k-1}(x)$. The weight function of the polynomials of the first kind is also included. Sieved generalizations of the symmetric Pollaczek polynomials and their q-analogues are also treated. Their continuous spectra are also the above mentioned set. The q-analogues include a sieved version of the Rogers q-ultraspherical polynomials and another set of q-ultraspherical polynomials discovered by Askey and Ismail. Generating functions and explicit formulas are also derived.

1. Introduction. The theory of polynomials orthogonal on an interval is very rich but very little seems to be known about the theory of polynomials orthogonal on several intervals. The classical Szegö class consists of polynomials orthogonal on $[-1,1]$ with respect to an absolutely continuous measure $w(x)dx$ satisfying

$$\text{(1.1)} \quad \text{(i)} \quad w(x) > 0 \quad \text{on } (-1,1) \qquad \text{(ii)} \quad \int_{-1}^{1} |\ln(w(x))| \frac{dx}{\sqrt{1-x^2}} < \infty.$$

The prototypes of the polynomials in the Szegö class are the Chebyshev polynomials

$$\text{(1.2)} \qquad T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \sin[(n+1)\theta]/\sin\theta.$$

Recently, Al-Salam, Allaway and Askey [3] introduced two sets of sieved ultraspherical polynomials. The polynomials of the first kind $\{c_n^\lambda(x;k)\}$ are generated by

$$\text{(1.3)} \quad \begin{cases} c_0^\lambda(x;k) = 1, \qquad c_1^\lambda(x;k) = x, \\ 2xc_n^\lambda(x;k) = c_{n+1}^\lambda(x;k) + c_{n+1}^\lambda(x;k), \qquad k \nmid n, \\ 2(n+\lambda)xc_{nk}^\lambda(x;k) = (n+2\lambda)c_{nk+1}^\lambda(x;k) + nc_{nk-1}^\lambda(x;k), \qquad n > 0, \end{cases}$$

Received by the editors February 5, 1985 and, in revised form, April 15, 1985.

1980 *Mathematics Subject Classification*. Primary 45C05; Secondary 33A65.

Key words and phrases. Chebyshev polynomials, ultraspherical polynomials, polynomials orthogonal on several intervals.

[1] This research was partially supported by NSF grant DMS 8503731 and was done while the author was on sabbatical leave from Arizona State University.

and the polynomials of the second kind $\{B_n^\lambda(x;k)\}$ are defined by

(1.4)
$$\begin{cases} B_0^\lambda(x;k) = 1, \qquad B_1^\lambda(x;k) = 2x, \\ 2xB_n^\lambda(x;k) = B_{n+1}^\lambda(x;k) + B_{n-1}^\lambda(x;k), \qquad k \nmid n+1, \\ 2(n+\lambda)xB_{nk-1}^\lambda(x;k) = nB_{nk}^\lambda(x;k) + (n+2\lambda)B_{nk-2}^\lambda(x;k), \qquad n > 0, \end{cases}$$

where $k = 2, 3, \ldots$ is a given integer. Al-Salam, Allaway and Askey formally derived orthogonality relations for $\{B_n^\lambda(x;k)\}$ and $\{c_n^\lambda(x;k)\}$ from the orthogonality relation of the continuous q-ultraspherical polynomials due to Askey and Ismail [**4**, **5**] and Askey and Wilson [**8**]. Both sets of sieved polynomials are orthogonal on $[-1,1]$. The sieved ultraspherical polynomials of the first (second) kind are orthogonal with respect to the weight function $w_1(x)$ ($w_2(x)$),

$$w_1(x) = |\sin(k\theta)|^{2\lambda}/\sin\theta, \qquad x = \cos\theta, \tag{1.5}$$

$$w_2(x) = \sin\theta|\sin(k\theta)|^{2\lambda}, \qquad x = \cos\theta. \tag{1.6}$$

Direct proofs of the orthogonality relations of $\{B_n^\lambda(x;k)\}$ and $\{c_n^\lambda(x;k)\}$ are in Askey and Shukla [**7**], Charris and Ismail [**11**], and Ismail [**14**]. Observe that both weight functions in (1.5) and (1.6) vanish at the $k-1$ points $x_1, x_2, \ldots, x_{k-1}$,

$$x_j := \cos(\pi j/k), \qquad j = 0, 1, \ldots, k. \tag{1.7}$$

So the $k+1$ points $x_0, \ldots, x_k$ break up the interval $[-1,1]$ to the union of k adjacent intervals. This suggests that the sieved ultraspherical polynomials are actually a special case of a more general set of polynomials orthogonal on the union of k disjoint intervals. The sought after more general polynomials should contain an additional continuous parameter c, say, and as c approaches a critical value c_0 the disjoint intervals become adjacent.

The purpose of this work is to study four sets of polynomials orthogonal on several disjoint intervals. Two of these sets are analogues of the Chebyshev polynomials of the first and second kinds and are probably the building blocks of a new theory that extends the Szegö theory from one interval to several intervals. It turns out that the continuous spectrum of all the four sets of polynomials is the closure $\overline{E}(c)$ of $E(c)$,

$$E(c) := \left\{x : \frac{-2\sqrt{c}}{1+c} < T_k(x) < \frac{2\sqrt{c}}{1+c}\right\}. \tag{1.8}$$

When $c < 1$ the discrete spectrum of our generalization of the Chebyshev polynomials consists of the zeros of $U_{k-1}(x)$. These zeros are located between the components of $E(c)$. The discrete spectrum is empty when $c \geqslant 1$. The other two polynomials are generalizations of the polynomials in Chapters 6 and 7 of [**6**].

In §2, we analyze the polynomials $\{J_n(x;k)\}$ generated by

$$\begin{cases} J_0(x;k) = 1, \qquad J_1(x;k) = 2x, \\ (1+c)xJ_{nk-1}(x;k) = J_{nk}(x;k) + cJ_{nk-2}(x;k), \qquad n > 0, \\ 2xJ_n(x;k) = J_{n+1}(x;k) + J_{n-1}(x;k), \qquad n > 0, \text{ if } k \nmid n+1, \end{cases} \tag{1.9}$$

where k is a prescribed integer greater than 1. The J_n's generalize the Chebyshev polynomials of the second kind $\{U_n(x)\}$. The corresponding generalization of the Chebyshev polynomials of the first kind are the polynomials $\{I_n(x; k)\}$,

$$(1.10)\quad \begin{cases} I_0(x;k) = 1, \qquad I_1(x;k) = x, \\ (1+c)xI_{nk}(x;k) = cI_{nk+1}(x;k) + I_{nk-1}(x;k), \qquad n > 0, \\ 2xI_n(x;k) = I_{n+1}(x;k) + I_{n-1}(x;k), \qquad n > 0, \text{ if } k \nmid n. \end{cases}$$

The I_n's will be studied in §4. Generating functions and explicit formulas for $\{I_n(x; k)\}$ and $\{J_n(x; k)\}$ will be included. The distribution functions for both sets of polynomials will be computed. In §3 we first state, very briefly, results on random walk polynomials that will be used in the subsequent sections. The results are due to Charris and Ismail [**11**]. We then show that the distribution function of a general set of sieved random walk polynomials of the second kind is $(1 - x^2)$ times the distribution function of the corresponding set of sieved random walk polynomials of the first kind.

We now state a result on generating functions that will be used in §2. Let $\{p_n(x)\}$ be the solution of

$$(1.11)\qquad 2xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \qquad n > 0,$$

where

$$(1.12)\qquad \alpha_n = \gamma_n = 1, \qquad \beta_n = 0 \quad \text{if } k \nmid n+1$$

and

$$(1.13)\qquad p_0(x) = 1, \qquad p_1(x) = 2x.$$

We proved [**15**]

$$(1.14)\qquad \sum_{n=0}^{\infty} p_{nk+l}(x)t^n = \left[U_l(x) + tU_{k-l-2}(x)\right]F(x,t), \qquad l = 0, 1, \ldots, k-1,$$

where $F(x, t)$ is a function of x, t and k but is independent of l.

In §5, we study a generalization of the random walk polynomials of Chapter 6 in [**6**]. These new polynomials are sieved analogues of random walk polynomials associated with a birth and death process with linear transition rates. The original example (not sieved) was first considered by Selliah [**25**] and the special case $a = 0$ was considered earlier by Carlitz [**10**] and, independently, by Karlin and McGregor [**18**]. The sieved example of §5 depends on two parameters a and b. It is orthogonal with respect to a measure that has an absolutely continuous component and a discrete part. The absolutely continuous component is supported on $\overline{E}(a)$ and when $a = 0$, the continuous spectrum becomes empty and the polynomials reduce to those in [**11**], but when $a = 1$ the discrete part disappears and the polynomials reduce to the sieved ultraspherical polynomials of the first kind. When $0 < a < 1$, $b > 0$ the discrete part is infinite. The discrete part has $2k$ points in certain cases. The corresponding sieved polynomials of the second kind are also introduced and the spectral results in §3 are used to compute the distribution function. In §6, we introduce basic (or q-) analogues of the results of §5. These q-analogues are sieved extensions of Chapter 7 in [6]. They include two sieved analogues of the q-ultraspherical polynomials. In particular, it contains a sieved generalization of the q-ultraspherical polynomials of Rogers. Rogers used his polynomials to prove the Rogers-Ramanujan identities. For a treatment of Rogers' methods, we refer the interested reader to Bressoud's interesting paper [9]. This paper ends with §7 where relevant remarks and some open problems are mentioned.

2. Sieved Chebyshev polynomials of the second kind. The numerators $\{J_n^*(x;k)\}$ of the continued fraction associated with $\{J_n(x;k)\}$ satisfy the recursion in (1.9) and the initial conditions

$$J_0^*(x;k)=0, \qquad J_1^*(x;k)=2. \tag{2.1}$$

Markov's theorem (Szegö [28, p. 57]) gives

$$\chi_k(x):=\lim_{n\to\infty}\left\{\frac{J_n^*(x;k)}{J_n(x;k)}\right\}=\int_{-\infty}^{\infty}\frac{d\sigma(t)}{x-t}, \qquad x\notin \operatorname{supp}(d\sigma), \tag{2.2}$$

where $d\sigma(t)$ is the positive measure the J_n's are orthogonal with respect to. Furthermore, $\chi_k(x)$ is the periodic continued fraction

$$\chi_k(x)=\frac{2|}{|2x}-\frac{d_1|}{|(1+d_1)x}-\cdots-\frac{d_n|}{|(1+d_n)x}-\cdots, \tag{2.3}$$

with

$$d_n=1, \quad k\nmid n+1, \qquad d_{nk-1}=c, \quad n=1,2,\ldots. \tag{2.4}$$

Therefore

$$\chi_k(x)=\frac{2|}{|2x}-\frac{1|}{|2x}-\cdots-\frac{c|}{|(1+c)x-\frac{1}{2}\chi_k(x)}. \tag{2.5}$$

For example

$$\chi_2(x)=\frac{2}{2x-c/[(1+c)x-\frac{1}{2}\chi_2(x)]},$$

which leads to

$$x\chi_2^2(x)-\chi_2(x)[2x^2(1+c)+1-c]+2(1+c)x=0.$$

For general k, $\chi_k(x)$ will also satisfy a quadratic equation which we shall now derive. Let us denote the right side of (2.5) by f_k/g_k and treat the $\chi_k(x)$ on the right side of (2.5) as a dummy variable $\chi(x)$ *independent* of k. Clearly we have

$$\frac{f_{k+1}}{g_{k+1}}=\frac{2}{2x-\frac{1}{2}f_k/g_k}=\frac{2g_k}{2xg_k-\frac{1}{2}f_k},$$

and we discover the recursions

$$f_{k+1}=2g_k, \qquad g_{k+1}=2xg_k-\tfrac{1}{2}f_k,$$

or equivalently,

$$f_{k+1}=2g_k, \quad 2xg_k=g_{k+1}+g_{k-1}, \qquad k=2,3,\ldots. \tag{2.6}$$

The initial conditions are

$$\frac{f_2}{g_2} = \frac{2(1+c)x - \chi(x)}{2(1+c)x^2 - x\chi(x) - c},$$

that is

$$g_2 = 2(1+c)x^2 - x\chi(x) - c, \qquad g_1 = (1+c)x - \tfrac{1}{2}\chi(x). \tag{2.7}$$

It is easy to see that the solution of the recurrence relation (2.6) with the side conditions (2.7) is given by

$$g_k = [(1+c)x - \tfrac{1}{2}\chi(x)]U_{k-1}(x) - cU_{k-2}(x). \tag{2.8}$$

This and $f_{k+1} = 2g_k$ establish

$$\chi_k(x) = \frac{[2(1+c)x - \chi_k(x)]U_{k-2}(x) - 2cU_{k-3}(x)}{[(1+c)x - \frac{1}{2}\chi_k(x)]U_{k-1}(x) - cU_{k-2}(x)}. \tag{2.9}$$

We used the fact that both $U_n(x)$ and $T_n(x)$ satisfy

$$2xp_n(x) = p_{n+1}(x) + p_{n-1}(x), \qquad p_n(x) = U_n(x) \text{ or } T_n(x), \tag{2.10}$$

and the identity

$$U_n(x) - U_{n-2}(x) = 2T_n(x). \tag{2.11}$$

Applying (2.10) and (2.11) we obtain

$$\begin{aligned} &\tfrac{1}{2}U_{k-1}(x)\chi_k^2(x) - [(1+c)T_k(x) + 2U_{k-2}(x)] \\ &\qquad + (1+c)U_{k-1}(x) + (1-c)U_{k-3}(x) = 0, \end{aligned} \tag{2.12}$$

from (2.9). This implies

$$\chi_k(x) = \left[(1+c)T_k(x) + 2U_{k-2}(x) - \sqrt{(1+c)^2T_k^2(x) - 4c}\right]/U_{k-1}(x), \tag{2.13}$$

since $\chi_k(x) \approx x^{-1}$ as $x \to \infty$.

Now assume the orthogonality relation is

$$\int_{-\infty}^{\infty} J_n(x:k)J_n(x;k)\,d\sigma(x) = \tau_n\sigma_{m,n}. \tag{2.14}$$

Since the coefficients in the recurrence relation (1.9) are bounded above and below by positive numbers, the support of $d\sigma$ will be bounded (Chihara [**12**, p. 109]) and (2.2) holds. The Perron-Stieltjes inversion formula

$$F(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t} \quad \text{iff} \quad \mu(t_2) - \mu(t_1) = \lim_{\varepsilon \to 0+} \int_{t_1}^{t_2} \frac{F(t-i\varepsilon) - F(t+i\varepsilon)}{2\pi i}\,dt \tag{2.15}$$

shows that $\sigma'(x)$, the absolutely continuous component of $d\sigma(x)$, is supported on $\overline{E}(c)$ and is given by

$$\sigma'(x) = \pi^{-1}\{4c - (1+c)^2T_k^2(x)\}^{1/2}/|U_{k-1}(x)|, \qquad x \in \overline{E}(c). \tag{2.16}$$

Note that the zeros of $U_{k-1}(x)$ lie outside $\bar{E}(c)$ when $c \neq 1$. The discrete part of $d\sigma$ coincides with the poles of the continued fraction $\chi_k(x)$. The poles of $\chi_k(x)$ are among $x_1, \dots, x_{k-1}$, see (1.7). A delicate analysis of (2.13) shows that $\chi_k(x)$ has no poles if $c > 1$, and has poles $x_1, \dots, x_{k-1}$ when $c < 1$. This analysis is based on the choice of the sign of the square root in (2.13). The key idea is to observe that

$$\left|(1+c)T_k(x) - \sqrt{(1+c)^2T_k^2(x) - 4c}\right| < \left|(1+c)T_k(x) + \sqrt{(1+c)^2T_k^2 - 4c}\right|$$

holds outside $\bar{E}(c)$. When $c < 1$, the mass at x_j, $j = 1, \dots, k-1$, is the residue of $\chi_k(x)$ at $x = x_j$ as can be seen from (2.15). The residue of $\chi_k(x)$ at $x = x_j$ is $(1-c)(1-x_j^2)/k$. Hence the orthogonality relation (2.14) becomes

$$\int_{E(c)} J_n(x;k)J_m(x;k)\sigma'(x)\,dx = \tau_n\sigma_{m,n} \quad \text{if } c > 1, \tag{2.17}$$

and

(2.18)

$$\int_{E(c)} J_n(x;k)J_m(x;k)\sigma'(x)\,dx + \sum_{j=1}^{k-1} \frac{2(1-c)(1-x_j^2)}{k} J_n(x_j;k)J_m(x_j;k) = \tau_n\delta_{m,n} \quad \text{if } c < 1,$$

where

$$\tau_{nk+l} = c^n, \quad l = 0, 1, \dots, k-2, \qquad \tau_{nk+k-1} = \frac{2c^{n+1}}{1+c}, \tag{2.19}$$

$E(c)$ is as in (1.8) and the x_j's are defined by (1.7).

We conclude this section by mentioning generating functions for the polynomials $\{J_n(x;k)\}$. Recall that

$$x(1+c)J_{nk-1}(x;k) = J_{nk}(x;k) + cJ_{nk-2}(x;k), \qquad n = 1, 2, \dots.$$

Multiplying the above recursion by t^n and adding the results we obtain

$$x(1+c)tG_{k-1}(x,t) = G_0(x,t) - 1 + ctG_{k-2}(x,t), \tag{2.20}$$

where

$$G_l(x;t) := \sum_{n=0}^{\infty} J_{nk+l}(x;k)t^n, \qquad l = 0, 1, \dots, k-1. \tag{2.21}$$

Now (1.14) and (2.20) yield

$$F(x,t) = \left\{1 - t(1+c)T_k(x) + ct^2\right\}^{-1},$$

and (1.14) gives

(2.22)

$$\sum_{n=0}^{\infty} J_{nk+l}(x;k)t^n = \left\{U_l(x) + tU_{k-l-2}(x)\right\}/\left\{1 - t(1+c)T_k(x) + ct^2\right\},$$

$l = 0, 1, \dots, k-1$. Consequences of the generating functions (2.22) are the curious evaluations

(2.23)

$$J_{nk+l}(x_j;k) = c^n(-1)^{jn}U_l(x_j), \qquad j = 1, 2, \dots, k-1,\ l = 0, 1, \dots, k-1,$$

and the full generating function

$$(2.24)\qquad \sum_{n=0}^{\infty} J_n(x;t)t^n = \frac{1-2t^kT_k(x)+t^{2k}}{1-2xt+t^2}\left\{1-t^k(1+c)T_k(x)+ct^{2k}\right\}^{-1}.$$

Another consequence is

$$(2.25)\qquad c^{-n/2}J_{nk+l}(x;k) = U_l(x)U_n(\cos\phi) + U_{k-l-2}(x)U_{n-1}(\cos\phi),$$

where

$$(2.26)\qquad \cos\phi = \frac{(1+c)}{2\sqrt{c}}T_k(x), \qquad 0 \leqslant \phi \leqslant k\pi.$$

Clearly ϕ is real when $x \in \bar{E}(c)$ and is purely imaginary when $x \notin \bar{E}(c)$. Furthermore, if $\bar{E}_1(c), \bar{E}_2(c), \ldots$ are the components of $\bar{E}(c)$ ordered such that $\bar{E}_j(c)$ is to the right of $\bar{E}_{j+1}(c)$, $j = 1,\ldots,$ then $0 < \phi < \pi$ on $\bar{E}_1(c)$, $\pi < \phi < 2\pi$ on $\bar{E}_2(c)$, and in general $(j-1)\pi < \phi < j\pi$ on $\bar{E}_j(c)$.

3. Random walk polynomials. A birth and death process is a stationary Markoff process whose states are the nonnegative integers. Its transition probabilities

$$p_{mn}(t) = \Pr.\{\chi(t) = n \mid X(0) = m\}, \qquad t > 0.$$

satisfy

$$p_{mn}(t) = \begin{cases} \beta_m t + o(t), & n = m+1, \\ \delta_m t + o(t), & n = m-1, \\ 1 - \beta_m t - \delta_m t + o(t), & n = m, \end{cases}$$

as $t \to 0+$. It is assumed that $\delta_0 \geqslant 0$, $\beta_n > 0$ and $\delta_{n+1} > 0$, $n = 0,1,\ldots.$ Associated with each birth and death process are: a random walk on the state space and a set of random walk polynomials $\{R_n(x)\}$. The polynomials $\{R_n(x)\}$ are defined recursively by

(3.1)
$$R_0(x) = 1, \quad B_0R_1(x) = x, \quad xR_n(x) = B_nR_{n+1}(x) + D_nR_{n-1}(x), \quad n > 0,$$

where

$$(3.2)\qquad B_n = \beta_n/(\beta_n+\delta_n), \qquad D_n = \delta_n/(\beta_n+\delta_n).$$

Charris and Ismail **[11]** showed how a sequence of random walk polynomials $\{R_n(x)\}$ satisfying (3.1) with

$$(3.3)\qquad B_n > 0, \qquad D_n > 0, \qquad B_n + D_n = 1, \quad n = 0,1,\ldots,$$

generates two sets of sieved orthogonal polynomials. Given an integer $k > 1$, the polynomials of the first kind $\{r_n(x)\}$ satisfy

$$(3.4)\quad r_0(x) = 1, \quad r_1(x) = x, \quad xr_n(x) = d_{n-1}r_{n+1}(x) + b_{n-1}r_{n-1}(x), \quad n > 0,$$

with

$$(3.5)\qquad b_n = d_n = \tfrac{1}{2} \ \text{ if } k \nmid n+1, \qquad b_{nk-1} = B_n, \qquad d_{nk-1} = D_{n-1}.$$

The polynomials $\{S_n(x)\}$ dual to $\{R_n(x)\}$ are generated by

(3.6) $S_0(x) = 1, \quad D_0 S_1(x) = x, \quad xS_n(x) = D_n S_{n+1}(x) + B_n S_{n-1}(x), \quad n > 0.$

Charris and Ismail [**11**] showed that the distribution function $\rho(x)$ of $\{r_n(x)\}$ satisfies

(3.7) $$\int_{-\infty}^{\infty} \frac{d\rho(t)}{x - t} = \lim_{n\to\infty} \frac{a_{nk} U_{k-1}(x) S_{n-1}(T_k(x))}{R_n(T_k(x)) - R_{n-2}(T_k(x))},$$

and the a_n's are as in (3.9) and (3.10). The support of $d\rho$ is $[-1,1]$ and ρ is normalized by

(3.8) $$\rho(-\infty) = 0, \qquad \rho(x) = \tfrac{1}{2}[\rho(x+0) + \rho(x-0)], \qquad \int_{-\infty}^{\infty} d\rho(x) = 1.$$

Let

(3.9) $a_0 = 1, \quad a_1 = 2, \quad a_n = d_0 d_1 \cdots d_{n-2}/[b_0 b_1 \cdots b_{n-1}], \quad n > 1.$

Hence

(3.10) $$a_{nk} = \frac{D_0 D_1 \cdots D_{n-2}}{B_0 B_1 \cdots B_{n-1}}, \quad a_{nk+l} = \frac{2D_0 \cdots D_{n-1}}{B_0 \cdots B_{n-1}}, \qquad n > 0, 0 < l < k.$$

In [**11**] it was also proved that

(3.11) $$\sum_0^{\infty} t^n a_{nk} r_{nk}(x) = (1 - t^2) \sum_{n=0}^{\infty} t^n R_n(T_k(x)),$$

(3.12) $$a_{nk} r_{nk}(x) = R_n(T_k(x)) - R_{n-2}(T_k(x)), \qquad n > 0,$$

(3.13) $$a_{nk+l} r_{nk+l}(x) = 2T_l(x) R_n(T_k(x)) - 2T_{k-l}(x) R_{n-1}(T_k(x)), \qquad n \geqslant 0,$$

and

(3.14) $$\sum_{n=0}^{\infty} t^n a_{nk+l} r_{nk+l}(x) = 2[T_l(x) - tT_{k-l}(x)] \sum_{n=0}^{\infty} t^n R_n(T_k(x)),$$

for $l = 1, 2, \ldots, k-1$.

The generating functions (3.11) and (3.12) imply

(3.15) $$\sum_{n=0}^{\infty} t^n a_n r_n(x) = (1 - t^2) \frac{(1 - 2t^k T_k(x) + t^{2k})}{1 - 2xt + t^2} \sum_{n=0}^{\infty} t^{kn} R_n(T_k(x)).$$

The connection between birth and death processes and random walks on one hand and orthogonal polynomials on the other hand was discovered by Karlin and McGregor [**17**, **18**].

The sieved polynomials of the second kind $\{s_n(x)\}$ satisfy

(3.16) $s_0(x) = 1, \quad s_1(x) = 2x, \quad xs_n(x) = b_n s_{n+1}(x) + d_n s_{n-1}(x), \quad n > 0,$

with b_n and d_n as in (3.5). The polynomials $\{r_n(x)\}$ and $\{s_n(x)\}$ satisfy [**11**],

(3.17) $$a_n r_n(x) = s_n(x) - s_{n-2}(x), \qquad n \geqslant 0,$$

if $s_{-1}(x)$ and $s_{-2}(x)$ are interpreted as zero. We now prove

THEOREM 3.1. *If $\{s_n(x)\}$ are orthogonal with respect to a distribution function $\sigma(x)$, then $d\sigma(x) = c(1 - x^2)d\rho(x)$, $\rho(x)$ being the distribution functions of $\{r_n(x)\}$ and c is a constant.*

PROOF. By Christoffel's formula (Szegö [**28**, §2.5]) it suffices to show that there is a sequence of constants $\{c_n\}$ such that

$$(3.18)\qquad c_n(1-x^2)s_n(x)=\begin{vmatrix} r_n(x) & r_{n+1}(x) & r_{n+2}(x)\\ r_n(1) & r_{n+1}(1) & r_{n+2}(1)\\ r_n(-1) & r_{n+1}(-1) & r_{n+2}(-1)\end{vmatrix}.$$

Observe that

$$(3.19)\qquad r_n(1)=1,\qquad r_n(-x)=(-1)^n r_n(x)$$

follow by induction on n from (3.4). Therefore (3.18) is equivalent to

$$(3.20)\qquad c_n(1-x^2)s_n(x)=2(-1)^n\left[r_n(x)-r_{n+2}(x)\right],$$

that is, in view of (3.16) and (3.17)

$$\begin{aligned} &c_n s_n(x)\{1-d_nb_{n-1}-b_nd_{n+1}\}-c_nb_nb_{n+1}s_{n+2}(x)-c_nd_nd_{n-1}s_{n-2}(x)\\ &\quad=2(-1)^n s_n(x)\left\{\frac{1}{a_n}+\frac{1}{a_{n+2}}\right\}-\frac{2(-1)^n}{a_n}s_{n-2}(x)-\frac{2(-1)^n}{a_{n+2}}s_{n+2}(x).\end{aligned}$$

The choice

$$(3.21)\qquad c_0=4,\qquad c_n=2(-1)^n b_0b_1\cdots b_{n-1}/[d_0d_1\cdots d_n],$$

satisfies the above identity, hence (3.20) holds and we have established (3.18). This completes the proof of Theorem 3.1.

In the process of proving Theorem 3.1 we proved

COROLLARY 3.2. *The relationship* (3.17) *has the inverse* (3.20) *where* c_n *is as in* (3.21).

Let the orthogonality relations of $\{r_n(x)\}$ and $\{s_n(x)\}$ be

$$(3.22)\qquad \int_{-1}^{1} r_n(x)r_m(x)\,d\rho(x)=\rho_n\delta_{m,n},$$

$$(3.23)\qquad \int_{-1}^{1} s_n(x)s_m(x)\,d\sigma(x)=\sigma_n\delta_{m,n}.$$

If both ρ and σ are normalized as in (3.8) then

$$(3.24)\qquad \rho_0=\sigma_0=1,\quad \rho_n=\frac{b_0\cdots b_{n-1}}{d_0\cdots d_{n-1}}d_{n-1},\quad \sigma_n=\frac{d_1\cdots d_n}{b_0\cdots b_{n-1}},\quad n>0.$$

In order to compute the constant c in Theorem 3.1 we note that

$$\begin{aligned} 1&=\int_{-1}^{1} d\sigma(x)=c\int_{-1}^{1}(1-x^2)\,d\rho(x)\\ &=c-c\int_{-1}^{1} r_1^2(x)\,d\rho(x)=c(1-\rho_1)=c(1-\tfrac{1}{2}),\end{aligned}$$

hence $c=2$. Therefore

$$(3.25)\qquad d\sigma(x)=2(1-x^2)\,d\rho(x).$$

4. Sieved Chebyshev polynomials of the first kind. We choose

$$D_n = c/(c+1), \qquad B_n = 1/(c+1), \tag{4.1}$$

so the R_n's satisfy

$$x(c+1)R_n(x) = R_{n+1}(x) + cR_{n-1}(x), \qquad n > 0, \tag{4.2}$$

while the S_n's satisfy

$$x(c+1)S_n(x) = cS_{n+1}(x) + S_{n-1}(x), \qquad n > 0. \tag{4.3}$$

The initial conditions are

$$R_0(x) = 1, \quad R_1(x) = x(c+1), \qquad S_0(x) = 1, \quad S_1(x) = x(c+1)/c. \tag{4.4}$$

It is easy to see that

$$R_n(x) = c^{n/2}U_n\left(\frac{x(c+1)}{2\sqrt{c}}\right), \qquad S_n(x) = c^{-n/2}U_n\left(\frac{x(c+1)}{2\sqrt{c}}\right). \tag{4.5}$$

In the present case, (3.10) becomes

$$a_{nk} = (c+1)c^{n-1}, \quad n > 1, \qquad a_{nk+l} = 2c^n, \quad n > 0, 0 < l < k. \tag{4.6}$$

Define ϕ (as in (2.26)) by

$$\cos\phi = \frac{1+c}{2\sqrt{c}}T_k(x), \tag{4.7}$$

and require $|\exp(-i\phi)| \leqslant |\exp(i\phi)|$. With this convention, we have for $x \notin \overline{E}(c)$

$$R_n(T_k(x)) \approx \frac{c^{n/2}\exp[i(n+1)\phi]}{2i\sin\phi}$$

and

$$S_n(T_k(x)) \approx \frac{c^{-n/2}\exp[i(n+1)\phi]}{2i\sin\phi}.$$

Now (3.7) gives

$$\int_{-1}^{1}\frac{d\rho(t)}{x-t} = \frac{U_{k-1}(x)\sqrt{c}\,(c+1)e^{i\phi}}{[ce^{2i\phi}-1]},$$

which reduces to

$$\int_{-1}^{1}\frac{d\rho(t)}{x-t} = \frac{2cU_{k-1}(x)}{\left[(c-1)T_k(x) + \sqrt{(1+c)^2T_k^2(x) - 4c}\right]}, \qquad x \notin [-1,1]. \tag{4.8}$$

We used (4.7) in obtaining (4.8). One can show that the zeros of the denominator of the right side of (4.8) are double zeros and are among the zeros of $U_{k-1}(x)$. In order to analyze the right side of (4.8) it is easier to first express it in the form

$$\left[(c-1)T_k(x) - \left\{(1+c)^2T_k^2(x) - 4c\right\}^{1/2}\right] / \left\{2U_{k-1}(x)(1-x^2)\right\}. \tag{4.9}$$

The inversion formula (2.15) gives

$$\frac{d\rho(x)}{dx} = \frac{\{4c - (1+c)^2 T_k^2(x)\}^{1/2}}{2\pi(1-x)^2 |U_{k-1}(x)|}, \qquad x \in E(c). \tag{4.10}$$

It can be shown that the expression (4.9) has no poles if $c \geqslant 1$ and has simple poles when $c < 1$. These poles occur at the zeros of $U_{k-1}(x)$ with residues $(1-c)/k$. This proves the orthogonality relations

$$\int_{-1}^{1} I_n(x;k) I_m(x;k) \rho'(x)\,dx = \sigma_n \delta_{m,n}, \quad \text{if } c \geqslant 1, \tag{4.11}$$

$$\int_{-1}^{1} I_n(x;k) I_m(x;k) \rho'(x)\,dx + \frac{(1-c)}{k} \sum_{j=1}^{k} I_m(x_j;k) I_n(x_j;k) = \sigma_n \delta_{m,n} \quad \text{if } c < 1, \tag{4.12}$$

where $\rho'(x)$ is as in (4.10) and

$$\sigma_0 = 1, \qquad \sigma_{nk} = c^{1-n}/(c+1), \quad n > 0, \qquad \sigma_{nk+l} = \tfrac{1}{2} c^{-n}, \quad l > 0. \tag{4.13}$$

In the case under consideration (4.5) implies

$$\sum_{n=0}^{\infty} R_n(x) t^n = \sum_{n=0}^{\infty} (t\sqrt{c})^n U_n\left(\frac{x(c+1)}{2\sqrt{c}} \right) = \frac{1}{1 - x(c+1)t + ct^2},$$

which, when combined with (3.11) and (4.6), yields

$$\sum_{n=0}^{\infty} t^n I_{nk}(x;k) = \frac{c - tT_k(x)}{c - t(c+1)T_k(x) + t^2}. \tag{4.14}$$

Furthermore, (3.14) and (3.15) imply two additional generating functions. Finally, (3.12), (4.5) and (3.13) give the explicit representations

$$(c+1) c^{n/2} I_{nk}(x;k) = cU_n\left(\frac{c+1}{2\sqrt{c}} T_k(x) \right) - U_{n-2}\left(\frac{c+1}{2\sqrt{c}} T_k(x) \right), \qquad n > 0, \tag{4.15}$$

$$c^{(n+1)/2} I_{nk+l}(x;k) = \sqrt{c}\, T_l(x) U_n\left(\frac{c+1}{2\sqrt{c}} T_k(x) \right) - T_{k-l}(x) U_{n-1}\left(\frac{c+1}{2\sqrt{c}} T_k(x) \right), \qquad n \geqslant 0, 0 < l < k. \tag{4.16}$$

The continued fraction of the present section and of §2 can be deduced from each other. One can also use Theorem 3.1, Corollary 3.2 and (3.24) to compute one of the distribution functions from the other one but we prefer giving two different proofs. The proof in this section is simple and illustrates the advantages of applying the results of §3. The proof in §2, although it is not systematic, is nevertheless worth noting.

5. Some sieved symmetric polynomials. We now analyse the sieved polynomials corresponding to

$$B_n = (n+1)/[b+(a+1)(n+1)], \qquad D_n = [b+a(n+1)]/[b+(a+1)(n+1)], \tag{5.1}$$

so the R_n's are generated by

$$(n+1)R_{n+1}(x) + [b+a(n+1)]R_{n-1}(x) = x[b+(a+1)(n+1)]R_n(x), \tag{5.2}$$

and $R_0(x) = 1$, $R_1(x) = (b+a+1)x$. Using the generating function

$$R(x,t) = \sum_0^\infty R_n(x)t^n, \tag{5.3}$$

we transform (5.2) and the initial conditions to the initial value problem

$$\frac{\partial R(x,t)}{\partial t} = \frac{(b+a+1)x-(2a+b)t}{1-(a+1)xt+at^2}R(x,t) \quad \text{and} \quad R(x,0)=1. \tag{5.4}$$

We shall follow the notation in Chapter 6 of Askey and Ismail's Memoir [**6**], so

$$\alpha \text{ and } \beta \text{ are roots of } 1+at^2-(a+1)xt=0, \qquad |\beta| \leqslant |\alpha|, \tag{5.5}$$

$$\frac{b(x-t)}{1+at^2-tx(a+1)} = \frac{A}{t-\alpha} + \frac{B}{t-\beta}, \tag{5.6}$$

$$A = \frac{b(x-\alpha)}{a(\alpha-\beta)}, \qquad B = \frac{b(x-\beta)}{a(\beta-\alpha)}. \tag{5.7}$$

The relationships (5.5)–(5.7) show that the solution to (5.4) is

$$\sum_{n=0}^\infty t^n R_n(x) = R(x,t) = (1-t/\alpha)^{A-1}(1-t/\beta)^{B-1}, \tag{5.8}$$

hence Darboux's method (Olver [**22**, §8.9]) gives

$$R_n(x) \approx \beta^{-n}\frac{(1-B)_n}{n!}\left(1-\frac{\beta}{\alpha}\right)^{A-1} \approx \frac{\beta^{-n}}{\Gamma(1-B)}\left(1-\frac{\beta}{\alpha}\right)^{A-1} n^{-B}, \tag{5.9}$$

where we used the well-known relationships

$$(\sigma)_n = \Gamma(\sigma+n)/\Gamma(\sigma) \quad \text{and} \quad \Gamma(a+n)/\Gamma(b+n) \approx n^{a-b}, \quad \text{as } n \to \infty. \tag{5.10}$$

We now derive a generating function for the dual polynomials $\{S_n(x)\}$ and then apply Darboux's method to determine the asymptotic behavior of $\{S_n(x)\}$. The S_n's are recursively defined as follows. $S_0(x) = 1$, $S_1(x) = (a+b+1)x/(a+b)$ and

$$[a(n+1)+b]S_{n+1}(x) + (n+1)S_{n-1}(x) = [b+(a+1)(n+1)]xS_n(x). \tag{5.11}$$

It will be more convenient to use

$$p_n(x) = \frac{a^n(1+b/a)_n}{(n+1)!}S_n(x). \tag{5.12}$$

In terms of the p's the recursion (5.11) becomes

$$(5.13)\quad (n+2)p_{n+1}(x)+(b+an)p_{n-1}(x)=x[b+(a+1)(n+1)]\,p_n(x).$$

The generating function

$$(5.14)\qquad p(x,t)=\sum_0^\infty p_n(x)t^n$$

transforms the difference equation (5.13) to

$$t[1+at^2-xt(1+a)]\frac{\partial p(x,t)}{\partial t}+[1+(b+a)t^2-xt(a+b+1)]\,p(x,t)=1$$

whose solution, subject to $p(x,0)=1$, is

$$(5.15)\quad p(x,t)=t^{-1}(1-t/\alpha)^A(1-t/\beta)^B\int_0^t(1-u/\alpha)^{-A-1}(1-u/\beta)^{-B-1}\,du,$$

with A and B as in (5.7). Darboux's method gives

$$p_n(x)\approx\frac{(1-\beta/\alpha)^A(-B)_n\beta^{-n-1}}{n!}\int_0^{\beta\rceil}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1}du.$$

That is, in view of (5.10)

$$(5.16)\quad p_n(x)\approx\frac{(1-\beta/\alpha)^A}{\Gamma(-B)}\beta^{-n-1}n^{-B-1}\int_0^{\beta\rceil}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1}du.$$

The integral in the above formulas is a Hadamard integral [**6**, Chapter V]. The asymptotic relationship

$$S_n(x)\approx\frac{a^{-n}(n+1)!(1-\beta/\alpha)^A}{(1+b/a)_n\Gamma(-B)}\beta^{-n-1}n^{-B-1}\int_0^{\beta\rceil}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1}du$$

follows from (5.12) and the asymptotic formula for $p_n(x)$ derived above. In this case, the a_{nk}'s of (3.10) are

$$a_{nk}=\left[\frac{b+(a+1)n}{b+an}\right]\frac{a^n(1+b/a)_n}{n!}.$$

This yields

(5.17)

$$a_{nk}S_{n-1}(x)\approx\beta^{-n}\left(1-\frac{\beta}{\alpha}\right)^A(a+1)\frac{n^{-B}}{\Gamma(-B)}\int_0^{\beta\rceil}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1}du.$$

Now (5.9) and (5.17) establish

(5.18)

$$\frac{a_{nk}S_{n-1}(x)}{R_n(x)-R_{n-2}(x)}\approx\frac{(-B)(a+1)(1-\beta/\alpha)}{(1-\beta^2)}\int_0^{\beta\rceil}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1}du.$$

Recall that β is a root of the quadratic equation in (5.5), hence

$$x=(1+a\beta^2)\beta^{-1}/(a+1).$$

Substituting this value of x in B of (5.7) we obtain

$$a(\alpha-\beta)B = b\left[\beta - \frac{1+a\beta^2}{\beta(a+1)}\right] = \frac{b(\beta^2-1)}{\beta(a+1)},$$

which implies

$$-\frac{B(a+1)}{1-\beta^2} = \frac{b}{a\alpha\beta(1-\beta/\alpha)} = \frac{b}{1-\beta/\alpha},$$

since $\alpha\beta = 1/a$, (5.5). The above analysis and (5.18) give

$$\text{(5.19)} \qquad \frac{a_{nk}S_{n-1}(x)}{R_n(x)-R_{n-2}(x)} \approx b\int_0^{\overline{\beta}|}\left(1-\frac{u}{\alpha}\right)^{-A-1}\left(1-\frac{u}{\beta}\right)^{-B-1} du,$$

which, when combined with (3.7) establish

$$\text{(5.20)} \quad X_k(x) = \int_{-1}^{1}\frac{d\psi(t)}{x-t} = bU_{k-1}(x)\int_0^{\overline{\beta_k}|}\left(1-\frac{u}{\alpha_k}\right)^{-A_k-1}\left(1-\frac{u}{\beta_k}\right)^{-B_k-1} du,$$

where α_k, β_k, A_k and B_k are as in (5.5)–(5.7) with x replaced by $T_k(x)$. The inversion formula (2.15) implies

(5.21)

$$\psi'(x) = \frac{b}{2\pi i}U_{k-1}(x)\int_{\underline{|\beta_k}}^{\overline{\alpha_k}|}\left(1-\frac{u}{\alpha_k}\right)^{-A_k-1}\left(1-\frac{u}{\beta_k}\right)^{-B_k-1} du, \qquad x \in E(a).$$

Note that by (5.7), $-\operatorname{Re} A = -\operatorname{Re} B = b/2a$. When $ab > 0$ the integral in (5.21) reduces to an ordinary beta integral. If ab is no longer positive, the Hadamard integral in (5.21) reduces to a beta integral as long as neither A nor b is a nonnegative integer. This leads to

$$\text{(5.22)} \quad \frac{d\psi}{d\phi} = \frac{b2^{-1+b/a}}{k\pi(a+1)\Gamma(b/a)}|\sin\phi|^{b/a}\exp\left\{\frac{b(a-1)}{a(a+1)}\left(\phi-\frac{\pi}{2}\right)\cot\phi\right\}$$
$$\cdot\left|\Gamma\left(\frac{b}{2a}+\frac{ib(1-a)}{2a(a+1)}\cot\phi\right)\right|^2,$$

and ϕ is as in (2.26) with c replaced by a, that is

$$\text{(5.23)} \qquad T_k(x) = 2\left[\sqrt{a}/(1+a)\right]\cos\phi, \qquad 0 \leqslant \phi \leqslant k\pi.$$

The nature of the discrete spectrum is similar to the discrete spectrum of the polynomials in Chapter 6 of [**6**]. Let $\{x_{n,j}\}$ be defined by

$$\text{(5.24)} \qquad 1 \geqslant x_{n,1} > x_{n,2} > \cdots > x_{n,k} > 0,$$
$$T_k(x_{n,j}) = \pm\sqrt{\frac{(b+2an)}{[b+(a+1)n][b+a(a+1)n]}}.$$

The polynomials under consideration are orthogonal with respect to a positive measure if and only if $a \geqslant 0$, $b+a > 0$. Table 1 identifies the discrete spectrum in different regions of the parameter space. These results can be proved along the same lines as the corresponding results of Chapter 6 in [**6**]. The only difference is the k

fold multiplicity of $\{x_{n,j}\}$ for a given n. The set $E(a)$ is free of discrete masses. The masses are located at the poles of the continued fraction $X_k(x)$ of (5.20). When $x = x_{n,j}$ supports a mass, the jump of $d\psi$ is the residue of $X_k(x)$ at $x = x_{n,j}$. Since the poles of $X_k(x)$ are all simple, we get

$$\begin{aligned}
&\lim_{x\to x_{n,j}} (x - x_{n,j}) X_k(x) \\
&= \lim_{x\to x_{n,j}} \frac{(x - x_{n,j})}{T_k(x) - T_k(x_{n,j})} \lim_{x\to x_{n,j}} \left[T_k(x) - T_k(x_{n,j})\right] X_k(x) \\
&= \frac{1}{kU_{k-1}(x)} \lim_{x\to x_{n,j}} \left[T_k(x) - T_k(x_{n,j})\right] X_k(x) \\
&= \frac{1}{k} \lim_{T_k(x)\to T_k(x_{n,j})} \left[T_k(x) - T_k(x_{n,j})\right] X_1(T_k(x)) \\
&= \frac{1}{k} \operatorname{Res}(X_1(y); y_n),
\end{aligned}$$

that is

$$\operatorname{Res}\left(X_k(x); x_{n,j}\right) = \frac{1}{k}\operatorname{Res}(X_1(y); y_n), \tag{5.25}$$

with

$$y_n = \pm(b + 2an)^{1/2}[b + (a+1)n]^{-1/2}[b + a(a+1)n]^{-1/2}. \tag{5.26}$$

The residues $\operatorname{Res}(X_1(y); y_n)$ have been computed in Chapter 6 of [**6**].

TABLE 1

Region	Discrete Spectrum
I. $0 \leqslant a < 1, b > 0$	$\{\pm x_{n,j} \mid n = 0, 1, \ldots, \; j = 1, 2, \ldots, k\}$
II. $1 \leqslant a, b > 0$	Empty
III. $a < 1, 0 < a + b < a$	$\{\pm x_{0,j} \mid j = 1, 2, \ldots, k\}$
IV. $a > 1, 0 < a + b < a$	$\{\pm x_{n,j} \mid n = 1, 2, \ldots, \; j = 1, 2, \ldots, k\}$

We now derive explicit formulas for $R_n(x)$ and then use them together with (3.12) and (3.13) to derive explicit formulas for the sieved polynomials. Applying the binomial theorem to (5.8) we obtain

$$R_n(x) = \sum_{j=0}^{n} \frac{(1-A)_j (1-B)_{n-j}}{j!(n-j)!} \beta^{n-j} \alpha^{-j},$$

which, in view of

$$(\sigma)_{n-j} = (\sigma)_n (-1)^j (1 - \sigma - n)_j, \tag{5.27}$$

implies

$$R_n(x) = \frac{(1-B)_n}{n!} \beta^{-n} {}_2F_1\left(\begin{matrix} -n, 1 - A \\ B - n \end{matrix} \middle| \frac{\beta}{\alpha} \right). \tag{5.28}$$

The Pfaff-Kummer transformation (Rainville [**24**, p. 60])

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) = (1-z)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix} \middle| \frac{z}{z-1} \right), \qquad |z| < 1, \left| \frac{z}{1-z} \right| < 1, $$

enables us to obtain the alternate representation

$$ (5.29) \quad R_n(x) = \frac{(1-B)_n}{n!} \beta^{-n} \left(1 - \frac{\beta}{\alpha}\right)^n {}_2F_1\left(\begin{matrix} -n, -n-1-b/a \\ B-n \end{matrix} \middle| \frac{\beta}{\beta - \alpha} \right) $$

from (5.28). We also used $A + B = -b/a$. We now reverse the sum in the ${}_2F_1$ of (5.29), i.e., replace the summation index j by $n - j$ and apply (5.27). This establishes

$$ (5.30) \qquad R_n(x) = \frac{(2+b/a)_n}{n!} \alpha^{-n} {}_2F_1\left(\begin{matrix} -n, 1-B \\ 2+b/a \end{matrix} \middle| 1 - \frac{\alpha}{\beta} \right). $$

Similarly

$$ (5.31) \qquad R_n(x) = \frac{(2+b/a)_n}{n!} \beta^{-n} {}_2F_1\left(\begin{matrix} -n, 1-A \\ 2+b/a \end{matrix} \middle| 1 - \frac{\beta}{\alpha} \right). $$

The explicit formula (5.31) can also be used to give an alternate proof of the asymptotic formula (5.9). If we use the notation (5.23), α, β, A, B become α_k, β_k, A_k, B_k, as before. Then

$$ (5.32) \qquad \begin{aligned} \alpha_k &= \frac{e^{i\phi}}{\sqrt{a}}, & \beta_k &= \frac{e^{-i\phi}}{\sqrt{a}}, \\ A_k &= -\frac{b}{2a} + i\frac{b(1-a)}{a(a+1)} \cot\phi, & B_k &= \frac{-b}{2a} - i\frac{b(1-a)}{a(a+1)} \cot\phi, \end{aligned} $$

and (5.30) and (5.31) yield

$$ (5.33) \qquad \begin{aligned} R_n(T_k(x)) &= \frac{(2+b/a)_n}{n!} e^{-in\phi} {}_2F_1\left(\begin{matrix} -n, 1-B_k \\ 2+b/a \end{matrix} \middle| 1 - e^{2i\phi} \right) \\ &= \frac{(2+b/a)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix} -n, 1-A_k \\ 2+b/a \end{matrix} \middle| 1 - e^{-2i\phi} \right). \end{aligned} $$

In the case of the polynomials under consideration, (3.10) is

$$ (5.34) \quad a_{nk} = \left[\frac{b + (a+1)n}{b + an} \right] \frac{a^n (1+b/a)_n}{n!}, \qquad a_{nk+l} = 2\frac{a^n(1+b/a)_n}{n!}, $$

$n \geqslant 0$, $0 < l < k$. Finally, we substitute for $R_n(T_k(x))$, a_{nk}, a_{nk+l} from (5.33) and (5.34) in (3.11), (3.15), and obtain generating functions and explicit representations for our polynomials. Note that when $a = 1$ our polynomials reduce to the Al-Salam, Allaway, Askey sieved ultraspherical polynomials of the first kind.

The sieved polynomials of the second kind can be treated similarly. §3 contains all the ingredients necessary to obtain their generating functions and explicit representations. The R_n's are as in (5.33) and the distribution function of the sieved polynomials of the second kind follows from (5.25), (5.22) and (3.25). The discrete spectrum in Region III will be different. The points $x_{0,1} = \pm 1$ will no longer support masses because the mass at $x_{n,j}$ is

$$2\left(1 - x_{n,j}^2\right) \operatorname{Res}\left(X_k(x); x_{n,j}\right)$$

so the masses at ± 1 get annihilated.

6. A basic analogue of §5. We now investigate a q-analogue of the polynomials in §5. This is a sieved analogue of Chapter 7 in Askey and Ismail [**6**]. We let

$$(6.1)\quad B_n = (1 - q^{n+1})/\left[b(1-q) + (a+1)(1-q^{n+1})\right], \qquad D_n = 1 - B_n.$$

As in Chapter 7 of [**6**], it will be more convenient to use the parameters a and c,

$$(6.2)\qquad c = a + b(1-q).$$

The R's start as $R_0(x) = 1$, $R_1(x) = a + 1 + (c-a)/(1-q)$ and satisfy the recursion formula

(6.3)

$$\left[c + 1 - (a+1)q^{n+1}\right] x R_n(x) = (1 - q^{n+1}) R_{n+1}(x) + (c - aq^{n+1}) R_{n-1}(x).$$

Multiplying both sides of (6.3) by t^{n+1} and adding the resulting recurrence relations we obtain

$$(6.4)\qquad R(x,t) = \frac{1 - q(a+1)xt + aq^2t^2}{1 - (c+1)xt + ct^2} R(x, qt),$$

where

$$(6.5)\qquad R(x,t) := \sum_{n=0}^{\infty} R_n(x) t^n.$$

The initial conditions were also used in deriving (6.4). Iterating (6.4), we obtain

$$R(x,t) = R(x, q^n t) \prod_{j=0}^{n-1} \frac{\left[1 - q^{j+1}(a+1)xt + aq^{2j+2}t^2\right]}{\left[1 - q^j(c+1)xt + q^{2j}ct^2\right]}.$$

As $n \to \infty$, $R(x, q^n t) \to R(x, 0) = R_0(x) = 1$. Therefore,

$$(6.6)\qquad R(x,t) = (qt/\alpha; q)_\infty (qt/\beta; q)_\infty / \left[(t/\mu; q)_\infty (t/\nu; q)_\infty\right],$$

where α and β are as in (5.5),

$$(6.7)\qquad (\sigma; q)_0 = 1, \qquad (\sigma; q)_n = \prod_{j=1}^{n} (1 - \sigma q^{j-1}), \quad n = \infty, 1, 2, 3, \ldots,$$

and

$$(6.8)\qquad \mu,\ \nu \text{ are roots of } ct^2 - (c+1)xt + 1 = 0, \qquad |\nu| \leqslant |\mu|.$$

Applying Darboux's method to (6.6), we obtain

$$(6.9)\qquad R_n(x) \approx \nu^{-n} (q\nu/\alpha; q)_\infty (q\nu/\beta; q)_\infty / \left[(q; q)_\infty (\nu/\mu; q)_\infty\right].$$

We now come to the dual polynomials $\{S_n(x)\}$. They satisfy the initial conditions $S_0(x) = 1$, $S_1(x) = x[1 + (1 - aq)/(c - aq)]$ and the difference equation

(6.10)

$$x\left[c + 1 - (a+1)q^{n+1}\right] S_n(x) = (c - aq^{n+1}) S_{n+1}(x) + (1 - q^{n+1}) S_{n-1}(x).$$

Again, we renormalize the polynomials by

$$S_n(x) = \frac{(q;q)_{n+1}/c^n}{(aq/c;q)_n} p_n(x). \tag{6.11}$$

The recursion (6.10) implies the following recursion for $\{p_n(x)\}$:

$$x[c + 1 - (a+1)q^{n+1}]\, p_n(x) = (1 - q^{n+2})p_{n+1}(x) + (c - aq^n)p_{n-1}(x).$$

Multiplying the above recurrence relation by t^{n+1} and adding for $n = 0, 1, \ldots,$ keeping in mind that $p_{-1}(x)$ is acutally zero and $p_0(x) = 1/(1-q)$, we get

$$p(x,t) = q\frac{[1 - (a+1)xt + at^2]}{[1-(c+1)xt + ct^2]} p(x, qt) + \frac{1}{1 + ct^2 - xt(c+1)}, \tag{6.12}$$

$p(x,t)$ being the generating function $p(x,t) = \sum_0^\infty t^n p_n(x)$. By iterating (6.12), we obtain the solution

$$p(x,t) = \left[\left(1 - \frac{t}{\mu}\right)\left(1 - \frac{t}{\nu}\right)\right]^{-1} {}_3\phi_2\left(\begin{matrix} t/\alpha,\ t/\beta,\ q;\ q,\ q \\ qt/\mu,\ qt/\nu \end{matrix}\right), \tag{6.13}$$

where the basic hypergeometric function is (Slater [**26**])

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \ldots, a_{r+1};\ q,\ x \\ b_1, \ldots, b_r \end{matrix}\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_{r+1};q)_n}{(b_1;q)_n \cdots (b_r;q)_n} \frac{x^n}{(q;q)_n}.$$

We again apply Darboux's method to (6.13) and conclude that

$$p_n(x) \approx \frac{\nu^{-n}}{1 - \nu/\mu}\, {}_2\phi_1\left(\begin{matrix} \nu/\alpha,\ \nu/\beta;\ q,\ q \\ q\nu/\mu \end{matrix}\right). \tag{6.14}$$

In the present case

$$a_{nk+k} = c^n(qa/c;q)_n[c + 1 - (a+1)q^{n+1}]/(q;q)_{n+1};$$

hence, by (6.11) and (6.14)

$$a_{nk+k}S_n(x) \approx (c+1)p_n(x) \approx \frac{(c+1)\nu^{-n}}{1 - \nu/\mu}\, {}_2\phi_1\left(\begin{matrix} \nu/\mu,\ \nu/\beta;\ q,\ q \\ q\nu/\mu \end{matrix}\right).$$

Combining the above asymptotic formula with (6.9) we see that

$$\begin{aligned} &\frac{a_{nk}S_{n-1}(x)}{R_n(x) - R_{n-2}(x)} \\ &\quad \approx \frac{(c+1)\mu\nu(q;q)_\infty(\nu/\mu;q)_\infty}{(\mu - \nu)(q\nu/\alpha;q)_\infty(q\nu/\beta;q)_\infty(1-\nu^2)}\, {}_2\phi_1\left(\begin{matrix} \nu/\alpha,\ \nu/\beta;\ q,\ q \\ q\nu/\mu \end{matrix}\right). \end{aligned} \tag{6.15}$$

Observe that (6.8) yields $\mu\nu = 1/c$, and (6.8) and (5.5) imply

$$\begin{aligned} (\alpha - \nu)(\beta - \nu) &= \alpha\beta - \nu(\alpha + \beta) + \nu^2 = \frac{1}{a} - \frac{\nu(a+1)x}{a} + \nu^2 \\ &= \frac{1}{a} - \frac{(a+1)}{a(c+1)}(1 + c\nu^2) + \nu^2 = \frac{(c-a)(1-\nu^2)}{a(c+1)}. \end{aligned}$$

Thus, we have $(1 - \nu/\alpha)(1 - \nu/\beta) = (c - a)(1 - \nu^2)/(1 + c)$ and, by (6.15)

$$\frac{a_{nk}S_{n-1}(x)}{R_n(x) - R_{n-2}(x)} \approx \frac{(c-a)(q;q)_\infty(\nu/\mu;q)_\infty}{c(\mu-\nu)(\nu/\alpha;q)_\infty(\nu/\beta;q)_\infty}\, {}_2\Phi_1\left(\begin{matrix}\nu/\alpha,\ \nu/\beta;\ q,\ q\\ q\nu/\mu\end{matrix}\right).$$

As in §5 we denote α, β, μ, ν by α_k, β_k, μ_k, ν_k when x is replaced by $T_k(x)$ in (5.5) and (6.8). If ψ is the distribution function, then (3.7) and the above asymptotic formula give

$$(6.16)\quad X_k(x) = \int_{-1}^{1}\frac{d\psi(t)}{x-t} = U_{k-1}(x)\frac{(c-a)(q;q)_\infty(\nu_k/\mu_k;q)_\infty}{c(\mu_k-\nu_k)(\nu_k/\alpha_k;q)_\infty(\nu_k/\beta_k;q)_\infty} \times {}_2\Phi_1\left(\begin{matrix}\nu_k/\alpha_k,\nu_k/\beta_k;q,q\\ q\nu_k/\mu_k\end{matrix}\right);$$

hence

$$(6.17)\qquad X_k(x) = U_{k-1}(x)X_1(T_k(x))$$

when $c = 0$ so $-a = b(1-q)$, the continuous spectrum degenerates and becomes empty and the discrete spectrum consists of the points $\{\pm x_{n,j}\}$,

(6.18)

$$1 \geqslant x_{n,1} > x_{n,2} > \cdots > x_{n,k} > 0, \qquad T_k(x_{n,j}) = \pm q^n\left\{\frac{-a}{1-(a+1)q^n}\right\}^{1/2}.$$

Formula (5.25) still holds, i.e.,

$$(6.19)\qquad \mathrm{Res}(X_k(x);x_{n,j}) = \frac{1}{k}\,\mathrm{Res}(X_1(y);y_n)$$

is valid, but now

$$(6.20)\quad y_n = \pm q^n\left\{\frac{-a}{1-(a+1)q^n}\right\}^{1/2} = \pm q^n\left\{\frac{b(1-q)}{1-q^n+b(1-q)q^n}\right\}^{1/2}.$$

The case $a = \beta$, $c = 1/\beta$ generates a sieved analogue of the continuous q-ultraspherical polynomials of L. J. Rogers. The sieved continuous ultraspherical polynomials of the first kind $\{c_n(x;\beta,k\,|\,q)\}$ satisfy

$$(6.21)\quad \begin{cases} c_j(x;\beta,k\,|\,q) = T_j(x), \qquad j = 0,1,\\ 2xc_n(x;\beta,k\,|\,q) = c_{n+1}(x;\beta,k\,|\,q) + c_{n-1}(x;\beta,k\,|\,q), \qquad k \nmid n,\\ x(1+\beta)(1-\beta q^m)c_{mk}(x;\beta,k\,|\,q) = \beta(1-\beta^2q^m)C_{mk+1}(x;\beta,k\,|\,q)\\ \qquad\qquad +(1-q^m)C_{mk-1}(x;\beta,k\,|\,q), \qquad m > 0.\end{cases}$$

The generating function (6.6) becomes

$$(6.22)\qquad \sum_{n=0}^{\infty}\beta^{n/2}t^nR_n(T_k(x)) = \frac{(\beta qte^{i\Phi},;q)_\infty(\beta qte^{i\Phi};q)_\infty}{(te^{i\Phi};q)_\infty(te^{-i\Phi};q)_\infty}$$

with

$$(6.23)\qquad T_k(x) = \frac{2\sqrt{\beta}}{1+\beta}\cos\Phi.$$

One can apply (3.12)–(3.15) and (6.22) and obtain generating and explicit relationships for the polynomials $\{c_n(x;\beta,k\,|\,q)\}$. The sieved continuous q-ultraspherical polynomials $\{B_n(x;\beta,k\,|\,q)\}$ are recursively generated by

(6.24)
$$\begin{cases} B_j(x;\beta,k\,|\,q) = U_j(x), \qquad j=0,1, \\ 2xB_n(x;\beta,k\,|\,q) = B_{n+1}(x;\beta,k\,|\,\beta) + B_{n-1}(x;\beta,k\,|\,\beta), \qquad k \nmid n+1, \\ x(\beta+1)(1-\beta q^m)B_{mk-1}(x;\beta,k\,|\,\beta) = \beta(1-q^m)B_{mk}(x;\beta,k\,|\,q) \\ \qquad\qquad +(1-\beta^2q^m)B_{mk-2}(x;\beta,k\,|\,q), \qquad m>0. \end{cases}$$

The case $c=1$ gives rise to another sieved q-analogue of the ultraspherical polynomials. The interested reader can derive their properties from the results of the present section and when $a=q^{2\lambda}$ and $q\to 1$ they reduce to the sets of sieved ultraspherical polynomials.

7. Concluding remarks. The continued fraction of §2 is periodic of period k and seems to be new. The theory of periodic continued fractions is very old and has extensive literature, Geronimus [**13**, §§III, VI]. Although a periodic continued fraction is always a solution of a quadratic equation, it is usually difficult to compute the coefficients of the quadratic equation for a periodic continued fraction of an arbitrary period. On the other hand, if the period is specified, finding the quadratic equation becomes a routine, but sometimes tedious, exercise in algebraic manipulations. So, it is quite surprising that one can compute the continued fraction in §2 (or §4) for arbitrary k. When a set of orthogonal polynomials satisfies a periodic three term recurrence relation of period k, the corresponding distribution function will have continuous spectrum consisting of k disjoint intervals. The discrete spectrum will be finite with at most one mass point per gap in the continuous spectrum. This is a discrete analogue of the Floquet theory for periodic differential equations. The example in §2 (and in §4) is extremal in the sense that it has the possible maximum of one discrete mass in every gap of the continuous spectrum. This is the case when $c<1$. When $c\geqslant 1$, the other extreme case occurs, and the discrete spectrum becomes empty. The case $k=2$ of the example in §2 is in Chihara's book [**12**, p. 91]. Another feature of the polynomials $\{I_n(x;c)\}$ and $\{J_n(x;c)\}$ is that their generating functions are rational functions of t (for all x). Certain ultraspherical, q-ultraspherical polynomials and their sieved analogues also have rational generating functions. This raises the question of characterizing all orthogonal polynomial sets whose generating functions are rational functions of t for all x. This characterization problem was raised by Ed Saff in a private communication. This class of polynomials also contains all orthogonal polynomials associated with periodic continued fractions. Al-Salam and Chihara [**2**] characterized a certain subclass of the class of polynomials described by Saff. A solution of Saff's problem will be most interesting.

In the past, several variations on the Chebyshev polynomials have been considered. One such variation is to study polynomials that satisfy

$$p_0(x)=1, \qquad p_1(x)=A_0x+B_0, \tag{7.1}$$

and, for $n > 0$,

(7.2) $$xp_n(x) = p_{n+1}(x) + p_{n-1}(x).$$

Another problem is to define $p_0(x)$ and $p_1(x)$ as in (7.1), then define $p_2(x), \ldots, p_j(x)$ by a recurrence relation

(7.3) $$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \qquad n = 1, 2, \ldots, j-1,$$

with $A_n A_{n-1} C_n > 0$, $n = 1, 2, \ldots, j$, in order to ensure orthogonality with respect to a positive measure. Then define $\{p_n(x): n > j\}$ by (7.2). In effect, one defines $p_j(x)$ and $p_{j-1}(x)$ in a way to be able to go forward via (7.2) and backward to get $p_{j-2}(x), p_{j-3}(x), \ldots, p_0(x) = 1$. One can ask the analogue of the first question for the I_n's and J_n's, so we use the same recurrence relation of $\{I_n(x; c)\}$ or $\{J_n(x; c)\}$ but change the initial conditions to (7.1). This problem is very easy and can be solved as follows. Use the recursion for $\{I_n(x; c)\}$; then

$$p_n(x) = I_n(x; c) + [(A_0 - 1)x + B_0]I_n^*(x; c), \qquad p_n^*(x) = A_0 I_n^*(x; c),$$

$I_n^*(x; c)$ being the numerator polynomials. The continued fraction associated with $\{p_n(x)\}$ is

(7.4) $$\frac{A_0 \chi_k(x)}{[(A_0 - 1)x + B_0]\chi_k(x) + 1},$$

where $\chi_k(x)$ is the continued fraction associated with $\{I_n(x; c)\}$. Inverting the Stieltjes transform in (7.4), we obtain the distribution function. It is clear from (7.4) that the continuous spectrum will remain as $\bar{E}(c)$ but the discrete spectrum will change. The analogue of the second question for $\{I_n(x; c)\}$ (or $\{J_n(x; c)\}$) seems to be much harder. This is the question of defining the polynomials $\{p_n(x)\}_0^j$ by (7.1) and (7.3), then define $\{p_n(x)\}_{j+1}^{\infty}$ by either the recurrence relation of $\{I_n(x; c)\}$ or $\{J_n(x; c)\}$ when $n \geqslant j$. Now that the polynomials are defined, one would ask what their distribution function is and what the associated continued fraction is? For a treatment of these problems, in the case of Chebyshev recursions, we refer the interested reader to Geronimus' addenda to Chapters III and VI in the Russian translation of Szegö's book [**13**, §§III, VI].

In all the examples of sieved random walk polynomials of the first kind which have been studied in [**11**] or in the present work, the associated continued fraction

(7.5) $$\chi_k(x) = \lim_{n \to \infty} \frac{a_{nk} S_{n-1}(T_k(x)) U_{k-1}(x)}{R_n(T_k(x)) - R_{n-2}(T_k(x))}$$

satisfies

(7.6) $$\chi_k(x) = U_{k-1}(x)\chi_1(T_k(x)).$$

In other words

(7.7) $$\lim_{n \to \infty} \frac{a_{nk} S_{n-1}(x)}{R_n(x) - R_{n-2}(x)} = \lim_{n \to \infty} \frac{R_n^*(x)}{R_n(x)},$$

where $\{R_n^*(x)\}$ are the numerators of $\{R_n(x)\}$, i.e., $\{R_n\}$ and $\{R_n^*\}$ satisfy the same recursion but $R_0^*(x) = 0$, $R_1^*(x) = 1/B_0$. Recall that $R_0(x) = 1$ and $R_1(x) = x/B_0$. One can easily show that

$$a_{nk}S_{n-1}(x) = R_n^*(x) - \Phi_{n-3}(x), \tag{7.8}$$

where $\Phi_n(x)$ is a polynomial of exact degree n. As a matter of fact all the examples I have seen so far indicate that

$$\Phi_n(x) \approx R_{n+1}^*(x) \quad \text{as } n \to \infty. \tag{7.9}$$

If one can prove (7.9), then (7.6) will follow because the left side of (7.7) will be equal to

$$\lim_{n\to\infty} \frac{R_n^*(x)\{1 - R_{n-2}^*(x)/R_n^*(x)\}}{R_n(x)\{1 - R_{n-2}(x)/R_n(x)\}},$$

which equals the right side of (7.7) since $R_n^*(x)/R_n(x)$ converges to a nonconstant function off the spectrum of $\{R_n(x)\}$. It will be very interesting to prove (7.9). Some restrictions on the asymptotic behavior of B_n and D_n will have to be imposed.

In the early sixties, Akhiezer wrote a series of papers on polynomials orthogonal on several intervals, see e.g. [**1**]. For references, conjectures and the current state of the art of Padé approximation on the complex plane cut along several slits, we refer the reader to Nuttal's interesting article [**21**]. In particular, note the connection between the Riemann problem and polynomials orthogonal on $[a, b] \cup [c, d]$, $a < b < c < d$, with respect to a weight function

$$\{-(a - x)(b - x)(c - x)(d - x)\}^{1/2}/p(x), \tag{7.10}$$

where $p(x)$ is a polynomial positive on $[a, b] \cup [c, d]$. The classical theory of S. N. Bernstein and G. Szegö considered the single interval case, say $[-1, 1]$ and weight functions

$$(1 - x^2)^{\pm 1/2}/p(x), \qquad ((1 - x)/(1 - x))^{1/2}/p(x), \tag{7.11}$$

Szegö [**28**, §2.6]. The weight functions in §§2 and 4 generalize (7.10) and (7.11). Our weight functions are of the type

$$\{\pi(x)\}^{1/2}/|p(x)| \quad \text{and} \quad \{\pi(x)\}^{1/2}/\left[(x - a_1)(a_{2l} - x)|p(x)|\right], \tag{7.12}$$

where $\pi(x) = -\prod_{j=1}^{2l}(a_j - x)$, $a_1 < a_2 < \cdots < a_{2l}$ and $p(x)$ is a polynomial. It is further assumed that $p(x)$ does not vanish on $E = \bigcup_{j=1}^{l}[a_{2j-1}, a_{2j}]$ and the polynomials are orthogonal on E. In the example of §6, $p(x)$ is not a polynomial. Here, we assume $c < a$, $c < -1$. The examples of §5 are of a completely different nature. They are symmetric sieved analogues of the polynomials of Pollaczek [**23**] and Szegö [**27**]. These are different from the symmetric sieved Pollaczek polynomials of [**14**]. The difference is that [**14**] sieves polynomials orthogonal on $[-1, 1]$ while in §5 we sieve polynomials orthogonal on $[-2\sqrt{c}/(1 + c), 2\sqrt{c}/(1 + c)]$. The cases $k = 1$ can be deduced from one another by rescaling, but that is not the case when $k > 1$. The polynomials in [**14**] are orthogonal on several adjacent intervals.

REFERENCES

1. N. I. Akhiezer, *Orthogonal polynomials on several intervals*, Soviet Math. **1** (1960), 989–992.

2. W. Al-Salam and T. Chihara, *On Reimer recurrences*, Portugal. Math. **38** (1979), 45–58.

3. W. Al-Salam, W. Allaway and R. Askey, *Sieved ultraspherical polynomials*, Trans. Amer. Math. Soc. **284** (1984), 39–55.

4. R. Askey and M. E. H. Ismail, *The Rogers q-ultraspherical polynomials*, Approximation Theory. III (E. W. Cheney, ed.), Academic Press, New York, 1980, pp. 175–182.

5. ______, *A generalization of ultraspherical polynomials*, Studies in Pure Math. (P. Erdös, ed.), Birkhäuser, Basel, 1983, pp. 55–78.

6. ______, *Recurrence relations, continued fractions and orthogonal polynomials*, Mem. Amer. Math. Soc. No. 300 (1984).

7. R. Askey and D. Shukla, *Sieved Jacobi polynomials* (to appear).

8. R. Askey and J. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. No. 319 (1985).

9. D. Bressoud, *On partitions, orthogonal polynomials and the expansion of certain infinite products*, Proc. London Math. Soc. (3) **42** (1981), 478–500.

10. L. Carlitz, *On some polynomials of Tricomi*, Boll. Un. Mat. Ital. (3) **13** (1958), 58–64.

11. J. Charris and M. E. H. Ismail, *On sieved orthogonal polynomials*. II: *Random walk polynomials*, Canad. J. Math. **38** (1986) (to appear).

12. T. Chihara, *An introduction to orthogonal polynomials*, Gordon & Breach, New York, 1978.

13. Ya. L. Geronimus, *Orthogonal polynomials*, Fitzmatgiz, Moscow, 1961, Appendix, pp. 414–494; English transl., Amer. Math. Soc. Transl. Series **2** (1977), 37–130.

14. M. E. H. Ismail, *On sieved orthogonal polynomials*. I: *Symmetric Pollaczek analogues*, SIAM J. Math. Anal. **16** (1985), 1093–1113.

15. ______, *On sieved orthogonal polynomials*. IV, J. Approx. Theory (to appear).

16. M. E. H. Ismail and J. Wilson, *Asymptotic and generating relations for the q-Jacobi and* $_4\phi_3$ *polynomials*, J. Approx. Theory **36** (1982), 43–54.

17. S. Karlin and J. McGregor, *The differential equations of birth and death processes, and the Stieltjes moment problem*, Trans. Amer. Math. Soc. **85** (1957), 489–546.

18. ______, *Many server queueing processes with Poisson input and exponential service time*, Pacific J. Math. **8** (1958), 87–118.

19. ______, *Random walks*, Illinois J. Math. **3** (1959), 66–81.

20. P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. No. 213 (1979).

21. J. Nuttal, *Asymptotics of diagonal Hermite Padé polynomials*, J. Approx. Theory **42** (1984), 299–386.

22. F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.

23. F. Pollaczek, *Sur une genralisation des polynomes de Legendre*, C. R. Acad. Sci. Paris **228** (1949), 1363–1365.

24. E. D. Rainville, *Special functions*, Chelsea, New York, 1971.

25. J. B. Selliah, *A system of orthogonal polynomials*, Tech. Report #13, Appl. Math. and Stat. Lab, Stanford University, Stanford, Calif. 1962.

26. L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, 1966.

27. G. Szegö, *On certain special sets of orthogonal polynomials*, Proc. Amer. Math. Soc. **1** (1950), 731–736; reprinted in *Collected papers*, Vol. 3, Birkhäuser, Boston, Mass., 1982, pp. 225–231.

28. ______, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1975.

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