ON LINKING DOUBLE LINES
BY
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ABSTRACT. A double line is a nonreduced locally Cohen-Macaulay scheme of degree two supported on a line in projective three-space. The heart of this work is to compute the associated Hartshorne-Rao module for such a curve. We can then say exactly when two such curves are in the same liaison class and in fact when they are directly linked. In particular, we find that C is only self-linked in characteristic two.

Introduction. Let \( k \) be an algebraically closed field and \( S = k[X_0, X_1, X_2, X_3] \). A double line \( C \subset \mathbb{P}^3_k \) is a nonreduced locally Cohen-Macaulay scheme of degree two supported on a line. The main purpose of this paper is to determine when two such curves can be linked. This is accomplished by a careful study of the Hartshorne-Rao module \( M(C) = \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, I_C(n)) \) (cf. [R1]) and has a somewhat surprising answer. In addition, we check when \( C \) can be self-linked.

In order to state the results, we first recall briefly the description of double lines due to Harris (cf. [H, pp. 32–33] for more detail). He gives a geometric description of a double line \( C \) in \( \mathbb{P}^3 \), produces the homogeneous ideal \( I(C) \), then verifies that the degree is in fact two and calculates the arithmetic genus.

To specify a double line \( C \) in \( \mathbb{P}^3 \) one has to choose the underlying line \( \lambda \) and then specify for each point \( P \in \lambda \) a normal direction to \( \lambda \). This is done by specifying for each \( P \in \lambda \) a plane \( H_P \) containing \( \lambda \). Equivalently, we have a map \( \psi: \lambda \to \mathbb{P}^3/\lambda \) (where \( \mathbb{P}^3/\lambda \cong \mathbb{P}^1 \) represents the pencil of planes through \( \lambda \)).

Without loss of generality, let \( \lambda \) be the line given by \( X_2 = X_3 = 0 \). Then we can write
\[
\psi([a_0, a_1, 0, 0]) = F(a_0, a_1)X_2 + G(a_0, a_1)X_3,
\]
where \( F \) and \( G \) are homogeneous polynomials of degree equal to \( \deg \psi = d \) (say), with no common zero. We can also think of this as a map of \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \):
\[
\psi([X_0, X_1]) = [F(X_0, X_1), G(X_0, X_1)].
\]
This information is lumped together in the ideal of \( C \), which is
\[
I(C) = (X_2^2, X_2X_3, X_3^2, X_2F(X_0, X_1) - X_3G(X_0, X_1)).
\]
Note that \( X_2F - X_3G \) corresponds to a surface \( S \) of degree \( d + 1 \) which is nonsingular in a neighborhood of \( \lambda \) (since \( F \) and \( G \) have no common zero). So we can equivalently write
\[
I(C) = (I(\lambda)^2, I(S)).
\]
Finally, Harris gives an elementary argument to show that the Hilbert polynomial for \( C \) is \( 2x + d + 1 \). Hence the degree of \( C \) is 2 and the arithmetic genus \( p_a(C) = -d \).
We can also compute the Hilbert function for $C$. For $n \leq 1$ this is trivial. For $n \geq 2$ observe that any element of $I(C)$ of degree $n$ can be uniquely expressed in the form

$$X_2^2 P(X_0, X_1, X_2) + X_2 X_3 Q(X_0, X_1, X_2, X_3) + X_3^2 R(X_0, X_1, X_3) \quad (n \leq d)$$

or

$$X_2^2 P(X_0, X_1, X_2) + X_2 X_3 Q(X_0, X_1, X_2, X_3) + X_3^2 R(X_0, X_1, X_3) + [X_2 F(X_0, X_1) - X_3 G(X_0, X_1)] S(X_0, X_1) \quad (n > d + 1),$$

where $P$, $Q$ and $R$ are homogeneous of degree $n - 2$ and $S$ is homogeneous of degree $n - d - 1$. Therefore

$$\dim \left( \frac{S}{I(C)} \right)_n = \begin{cases} \binom{n+3}{3} - \binom{n}{2} - \binom{n+1}{3} - \binom{n}{2} & \text{if } n \leq d, \\ \binom{n+3}{3} - \binom{n}{2} - \binom{n+1}{3} - \binom{n-1}{2} - (n - d) & \text{if } n \geq d. \end{cases}$$

Turning to liaison, it turns out that the cases $d = 0$ and $d = 1$ are trivial. The main result of the paper is the following:

**Theorem.** Let $C$ and $C'$ be double lines and assume $p_a(C) \leq -2$. Then $C$ is linked to $C'$ if and only if

(a) they have the same line $\lambda$ as support,

(b) they have the same arithmetic genus $-d$,

(c) the corresponding maps $\psi, \psi': \lambda \to \mathbb{P}^3/\lambda$ differ by an automorphism of the target $\mathbb{P}^3/\lambda$.

To achieve this, §1 gives some preliminary results and §2 describes a special type of complete intersection which is used to simplify the computations. The heart of the paper is in §3 where we describe the global sections of the sheaves $\mathcal{O}_C(n)$, compute the Hartshorne-Rao module $M(C)$, and prove the main theorem. §4 gives an interesting interpretation which is purely geometric, from which we establish the fact that double lines of arithmetic genus $\leq -2$ are only self-linked in characteristic two. (More generally, we see when $C$ and $C'$ can be directly linked.)

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1. **First results.** A key strategy for this paper will be to study the sheaves $\mathcal{O}_C(n)$ for all $n$, which will tell us what we need to known about $M(C)$. In this section we give a first description of $\mathcal{O}_C$, the main point being the exact sequence (3). As a consequence we compute the dimensions of the components of $M(C)$.

To begin, let $S$ be the surface corresponding to $X_2 F - X_3 G$ as in the Introduction. Then since $I(S) \subset I(C) \subset I(\lambda)$ it follows that $\lambda \subset C \subset S$.

**Lemma 1.1.** On $S$, $\lambda \cdot \lambda = 1 - d$.

**Proof.** This is an immediate application of the adjunction formula, recalling that $\omega_S = \mathcal{O}_S(d - 3)$ (since $\deg S = d + 1$) so $K_S = (d - 3) H$ (where $K_S$ is the canonical divisor class and $H$ is the class of a hyperplane section), and the geometric genus $\pi(\lambda) = 0$:

$$0 = \pi(\lambda) = \frac{\lambda \cdot \lambda + K_S \cdot \lambda}{2} + 1 = \frac{\lambda \cdot \lambda + (d - 3)}{2} + 1.$$
Now, we have an exact sequence of sheaves

\[ 0 \to \mathcal{I}_{C,S} \to \mathcal{I}_{\lambda,S} \to \mathcal{I}_{\lambda,C} \to 0 \]

from which we deduce

(1) \[ \mathcal{I}_{\lambda,S}/\mathcal{I}_{C,S} \cong \mathcal{I}_{\lambda,C}. \]

Since \( \lambda \hookrightarrow C \) we have a natural map \( \mathcal{O}_C \to \mathcal{O}_\lambda \), which yields the exact sequence

\[ 0 \to \mathcal{I}_{\lambda,C} \to \mathcal{O}_C \to \mathcal{O}_\lambda \to 0 \]

or, substituting the isomorphism (1) (thinking of this as a sequence of sheaves on \( S \))

(2) \[ 0 \to \mathcal{I}_{\lambda,S}/\mathcal{I}_{C,S} \to \mathcal{O}_C \to \mathcal{O}_\lambda \to 0. \]

But since \( I(C) = (I(\lambda)^2,I(S)) \) it follows that \( \mathcal{I}_{C,S} = \mathcal{I}_{\lambda,S}^2 \). Now, \( \mathcal{I}_{\lambda,S}/\mathcal{I}_{\lambda,C} \) is the conormal bundle of \( \lambda \) on \( S \) (recall \( S \) is smooth around \( \lambda \)), which is a line bundle of degree \(- (\lambda \cdot \lambda) = d - 1 \) on \( \lambda \cong \mathbb{P}^1 \) (cf. [Ha, p. 361]). Therefore we can write (2) in the form

(3) \[ 0 \to \mathcal{O}_\lambda(d-1) \xrightarrow{\alpha} \mathcal{O}_C \xrightarrow{r} \mathcal{O}_\lambda \to 0. \]

The maps \( \alpha \) and \( r \) will be described following Proposition 3.1.

From this we can quickly compute the dimensions of the components of the Hartshorne-Rao module \( M(C) \):

**LEMMA 1.2.**

\[ \dim M_n(C) = \begin{cases} 
0, & n \leq -d, \\
d + n, & -d \leq n \leq 0, \\
d - n, & 0 \leq n \leq d, \\
0, & n \geq d. 
\end{cases} \]

**PROOF.** From the exact sequence of sheaves

\[ 0 \to \mathcal{I}_C(n) \to \mathcal{O}_{P^3}(n) \to \mathcal{O}_C(n) \to 0 \]

we get the associated long exact sequence in cohomology:

(4) \[ 0 \to H^0(P^3, \mathcal{I}_C(n)) \to H^0(P^3, \mathcal{O}(n)) \to H^0(C, \mathcal{O}_C(n)) \to M_n(C) \to 0. \]

Having (3) and twisting, we can compute \( h^0(\mathcal{O}_C(n)) \) for any \( n \). (Note that \( h^1(\mathcal{O}_\lambda(n + d - 1)) = 0 \) for \( n \geq -d \) and \( h^0(\mathcal{O}_\lambda(n)) = 0 \) for \( n \leq -1 \).)

Then recalling the Hilbert function calculation of the Introduction, the result is easily obtained using (4). \( \square \)

In other words, \( \{ \dim M_n(C) | n \in \mathbb{Z} \} \) is the sequence

\[ \ldots, 0, 0, 1, 2, \ldots, d - 1, d, d - 1, \ldots, 2, 1, 0, 0, \ldots \]

where the “peak” occurs for \( n = 0 \). The “diameter” of \( M(C) \) is thus \( 2d - 1 \), and
this gives a necessary condition for liaison:

**COROLLARY 1.3.** If two double lines $C$ and $C'$ are linked, then the corresponding maps $\psi: \lambda \to \mathbb{P}^3/\lambda$ and $\psi': \lambda' \to \mathbb{P}^3/\lambda'$ have the same degree $d$ (and, equivalently, $C$ and $C'$ have the same arithmetic genus $-d$).

**REMARK 1.4.** It follows that a double line can have a Hartshorne-Rao module of arbitrarily large “diameter”, that it can have arbitrarily large components, and that it can have arbitrarily many nonzero components in negative degrees. Furthermore, for $d > 0$, $C$ is “extremal” in its liaison class in the following sense: Since $\dim M_0(C) > \dim M_1(C)$, it follows from Proposition 2.8 of [M] that no curve in the liaison class can have a Hartshorne-Rao module which is leftward shift of $M(C)$. □

**REMARK 1.5.** A double line is also extremal in the sense that (for $d > 0$) there is obviously no curve in the liaison class of strictly smaller degree. This paper describes which other degree two curves are in the liaison class.

For $d = 0$, $C$ is a plane curve and hence a complete intersection. $M(C) = 0$ and $C$ is arithmetically Cohen-Macaulay. $C$ is thus linked to any line and any (plane) conic in $\mathbb{P}^3$, including all other double lines with $d = 0$.

For $d = 1$, $M(C) \cong k$ so $C$ is arithmetically Buchsbaum (cf. [GMV2]). $C$ is linked to any pair of skew lines (cf. [R1]), as well as to any other double line with $d = 1$.

For $d > 1$, the components of $M(C)$ are as described above, but now there may be some nontrivial module structure. In fact, it was shown in [GMV2] that such a $C$ is never arithmetically Buchsbaum (i.e. there does exist nontrivial module structure). Because of this added complexity, it is reasonable to expect that not all double lines with the same $d$ will be linked. □

2. Basic links. We are assuming that $C$ is a double line with ideal $(X_2^2, X_2X_3, X_3^2, X_2F(X_0, X_1) - X_3G(X_0, X_1))$, where $\deg F = \deg G = d$. Observe that if $T$ is a reducible quadratic polynomial of the form $(\alpha X_2 + \beta X_3)(\gamma X_2 + \delta X_3)$ (i.e. the corresponding surface is the union of two planes, both containing the line $\lambda: (X_2 = X_3 = 0)$), then $T \in I(C)$.

If $T_1$ and $T_2$ are both of this form, with no common factor, then the corresponding complete intersection curve $X$ links $C$ to a curve $C'$ which is again locally Cohen-Macaulay. Since $\deg X = 4$ we have $\deg C' = 2$, and clearly $C'$ is supported on $\lambda$. Then by Harris’ description,

$I(C') = (X_2^2, X_2X_3, X_3^2, X_2F'(X_0, X_1) - X_3G'(X_0, X_1)),$

where $F'$ and $G'$ have no common root. Furthermore, by Corollary 1.3 we have $\deg F' = \deg G' = \deg F = \deg G = d$.

For our purposes it will actually be enough to look at complete intersections of the form $(T_1, T_2)$, where $T_1 = (X_2)(X_2 + \tau X_3)$ and $T_2 = (X_3)(\mu X_2 + X_3)$. Observe that $T_1$ and $T_2$ have no common factor (and hence give a link) if and only if the determinant $|\begin{smallmatrix} \tau & -1 \\ 1 & \mu \end{smallmatrix}| \neq 0$. We shall call such a link a basic link.

Given $I(C)$ and the basic link $(T_1, T_2)$, we can compute the ideal of the residual double line $C'$. Recall

$$I(C') = (T_1, T_2): I(C).$$
Since we know \( I(C') = (X_2^3, X_2X_3, X_3^2, X_2F'(X_0, X_1) - X_3G'(X_0, X_1)) \) with \( \deg F' = \deg G' = d \), we have only to identify \( F' \) and \( G' \) in terms of \( F, G, \tau \) and \( \mu \).

By (5),
\[
[X_2F'(X_0, X_1) - X_3G'(X_0, X_1)]\times [X_2F(X_0, X_1) - X_3G(X_0, X_1)] = P(X_0, X_1)(X_2 + \tau X_3) + Q(X_0, X_1)(X_3)(\mu X_2 + X_3)
\]
for some \( P, Q \). Then \( FF' = P \) and \( GG' = Q \), so one deduces that \( F'(-\tau F - G) = G'(-\mu G + F) \). But \( F' \) and \( G' \) have no common root, nor do \( (-\tau F - G) \) and \( (\mu G + F) \). Therefore, up to scalar multiplication we have
\[
F' = \mu G + F, \quad G' = -\tau F - G.
\]

Equivalently,
\[
\begin{bmatrix} F'(X_0, X_1) \\ G'(X_0, X_1) \end{bmatrix} = \alpha \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} F(X_0, X_1) \\ G(X_0, X_1) \end{bmatrix}
\]
for some scalar \( \alpha \neq 0 \). We conclude

**PROPOSITION 2.1.** Let \( C \) and \( C' \) have ideals as above. Then \( C \) is (directly) linked to \( C' \) by a basic link if and only if \( [I(C')] = A[I(C)] \), where \( \det A \neq 0 \) and \( \text{trace} A = 0 \) (but the main diagonal is not zero).

**REMARK 2.2.** We know that direct linking is a symmetric relation. Hence if we link \( C \) to \( C' \) as above and then apply the same surfaces to \( C' \), we must get \( C \) back. This is neatly reflected in the fact that
\[
\begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} = \begin{bmatrix} 1 - \tau \mu & 0 \\ 0 & 1 - \tau \mu \end{bmatrix} = (1 - \tau \mu) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

**REMARK 2.3.** Since
\[
\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

it follows that \( C \) is linked to itself by a sequence of three basic links. (Note that by Hartshorne's theorem (cf. [R1]) this implies that \( M(C) \) is self-dual.)

**REMARK 2.4.** As a corollary to Proposition 2.1, note that \( C \) is self-linked (cf. [R2]) by a basic link if and only if \( \text{char} k = 2 \) (and we take \( \tau = \mu = 0 \)). We shall say more about self-linkage in §4.

**3. Main results.** As indicated earlier, we are interested in studying the sheaf \( \mathcal{O}_C(n) \). The key point is the following:

**PROPOSITION 3.1.** The global sections of \( \mathcal{O}_C(n) \) are of the form
\[
H^0(C, \mathcal{O}_C(n)) = \left\{ A(X_0, X_1) + B(X_0, X_1) \cdot \frac{X_2}{G(X_0, X_1)} \quad A \text{ and } B \text{ are polynomials of degree } n \text{ and } d - 1 + n \text{ respectively,} \right\}.
\]

**PROOF.** We shall prove the inclusion \( \supseteq \), and then the result will follow from a check of vector space dimensions, using sequence (3).
Recall that \(I(C) = (X_2^2, X_2X_3, X_3^2, X_2F(X_0, X_1) - X_3G(X_0, X_1))\). Let \(U_0\) be the open set \(\{X_0 \neq 0, G(X_0, X_1) \neq 0\}\) on \(\mathbb{P}^3\), and consider the exact sequence
\[
0 \to \mathcal{O}_C(n)(U_0) \to \mathcal{O}_{\mathbb{P}^3}(n)(U_0) \to \mathcal{O}_C(n)(U_0) \to \cdots.
\]
For any open set \(U\), \(\mathcal{O}_{\mathbb{P}^3}(n)(U) = \{P/Q \mid \deg P = \deg Q + n, Q \text{ never zero on } U\}\). For \(I_C(n)(U) \subset \mathcal{O}_{\mathbb{P}^3}(n)(U)\) we have the further condition \(P \in I(C \cap U)\).

Hence it follows that (up to isomorphism) at least the following inclusions must hold:
\[
\mathcal{O}_C(n)(U_0) \supseteq \frac{\mathcal{O}_{\mathbb{P}^3}(n)(U_0)}{I_C(n)(U_0)} \supseteq \left\{ \frac{A(X_0, X_1)}{X_0^p} + \frac{B(X_0, X_1) \cdot X_2}{X_0^q G(X_0, X_1)} \middle| \begin{array}{c} \deg A = p + n \\ \deg B = q + d - 1 + n \end{array} \right\}.
\]
Similarly, on \(U_1 = \{X_1 \neq 0, G(X_0, X_1) \neq 0\}\) we have
\[
\mathcal{O}_C(n)(U_1) \supseteq \frac{\mathcal{O}_{\mathbb{P}^3}(n)(U_1)}{I_C(n)(U_1)} \supseteq \left\{ \frac{D(X_0, X_1)}{X_1^s} + \frac{E(X_0, X_1) \cdot X_2}{X_1^t G(X_0, X_1)} \middle| \begin{array}{c} \deg D = s + n \\ \deg E = t + d - 1 + n \end{array} \right\}.
\]
Two such sections patch together to give a section over \(U = U_0 \cup U_1\) if and only if \(p = q = s = t = 0\), as in the statement of the proposition. But such a section over \(U\) in fact extends to a global section of \(\mathcal{O}_C(n)\) (i.e. it extends to \(\{G(X_0, X_1) = 0\}\)) because on \(C\) we have \(X_2/G = X_3/F\) and \(F\) and \(G\) have no common root.  □

**Remark 3.2.** As a result of this proposition, we may view the map \(\alpha\) in the exact sequence (3) as multiplication by \(X_2/G(X_0, X_1)\) and \(r\) as the restriction to the first term.  □

From Proposition 3.1 we can immediately describe \(M(C)\):

**Corollary 3.3.** \(M(C) \cong [k[X_0, X_1]/(F, G)](d - 1)\) as an \(S\)-module.

**Proof.** We have the usual exact sequence (4):
\[
0 \to I(C)_n \to S_n \to H^0(C, \mathcal{O}_C(n)) \to M_n(C) \to 0.
\]
Let \(R = S/I(C)\), so \(M(C) \cong H^0_s(C, \mathcal{O}_C(n))/R\) (where \(H^0_s\) represents the direct sum over all \(n\)).

Now given an element \(A + B \cdot X_2/G \in H^0(C, \mathcal{O}_C(n))\) as in Proposition 3.1, we see that \(A\) becomes zero in \(M(C)\), and it is a simple exercise to check that \(B\) also becomes zero whenever it is a linear combination \(PF + QG\). Also, multiplication by \(X_2\) and \(X_3\) are zero in \(M(C)\).  □

We can now prove our main result. Note that the cases \(d = 0\) and \(d = 1\) were taken care of in Remark 1.5.

**Theorem 3.4.** Let \(C\) and \(C'\) be double lines and assume \(p_a(C) \leq -2\). Then \(C\) is linked to \(C'\) if and only if
(a) they have the same line \(\lambda\) as support,
(b) they have the same arithmetic genus \(-d\),
(c) the corresponding maps \(\psi, \psi': \lambda \to \mathbb{P}^3/\lambda\) differ by an automorphism of the target \(\mathbb{P}^3/\lambda\).

**Proof.** Again, we assume without loss of generality that \(C\) is supported on the line \(\lambda: (X_2 = X_3 = 0)\). Now, assume \(C\) is linked to \(C'\). By Remark 2.3 we may
assume that $C$ is evenly linked to $C'$, so $M(C) \cong M(C')$ by Hartshorne's theorem (cf. [R1]) and Lemma 1.2. (The latter assures that there is no shift.)

Consider the homomorphism $\phi_{1-d,1}: S_1 \rightarrow \text{Hom}(M_{1-d}(C), M_{2-d}(C))$. By Corollary 3.3, $\ker \phi_{1-d,1} = \langle X_2, X_3 \rangle$, so we have (a). We know (b) from Corollary 1.3. Given these, (c) says that $[G] = A[F']$ for some invertible $2 \times 2$ matrix $A$. Equivalently, $F$ and $G$ span the same subspace of $k[X_0, X_1]$ that $F'$ and $G'$ do.

Consider the homomorphism $\phi_{1-d,d}: S_d \rightarrow \text{Hom}(M_{1-d}(C), M_1(C))$ and let $K$ be its kernel. Let $I(\lambda)$ be the ideal $(X_2, X_3) \subset S$. From Corollary 3.3 we have that $K$ is generated by $I(\lambda)_d$, $F(X_0, X_1)$ and $G(X_0, X_1)$. We thus have

$$I(\lambda)_d \subset K \hookrightarrow S_d \phi_{1-d,d} \rightarrow \text{Hom}(M_{1-d}(C), M_1(C)).$$

Modding out by $I(\lambda)_d$ we then have

$$\langle F(X_0, X_1), G(X_0, X_1) \rangle = K/I(\lambda)_d \hookrightarrow S_d/I(\lambda)_d \cong k[X_0, X_1].$$

Since both $K$ and $I(\lambda)$ are isomorphism invariants, we get that the subspace $\langle F, G \rangle$ of $k[X_0, X_1]$ is also, and so we are done.

Conversely, it suffices to check the following: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2)$. Then

1. if $b = c = 0$, certainly $a \neq 0$ and $d \neq 0$ and we have

$$A = a \begin{bmatrix} d & 0 \\ a + 1 & -1 \end{bmatrix} \begin{bmatrix} d & -1 \\ a + 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

2. if at least one of $b, c$ is nonzero then there exist scalars $\alpha \neq 0, \tau, \mu, \tau', \mu'$ such that

$$A = \alpha \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} 1 & \mu' \\ -\tau' & -1 \end{bmatrix}.$$

(The latter is tedious but straightforward.) Then Proposition 2.1 gives the result.

4. Projective geometry and self-linkage. In this section we give an interesting geometric interpretation of the preceding sections which was pointed out by A. Landman. It will then be easy to say when a double line is self-linked, generalizing a result of [R2]. For this section, we assume char $k \neq 2$.

As always, we assume that the double line $C$ is supported on the line $\lambda$: $(X_2 = X_3 = 0)$, and that $d > 1$. The quadric hypersurfaces containing $C$ are then given by homogeneous polynomials of the form $aX_2^2 + bX_2X_3 + cX_3^2$ (which are always reducible). These hypersurfaces thus form a projective space $P^2$. The ideal generated by two such polynomials is then represented by a line in this $P^2$, or by a point in the dual space $(P^2)^*$. (Henceforth, unless stated otherwise, an ideal shall be understood to be of this special form.)

For the purposes of liaison we need a complete intersection ideal, that is one whose two generators have no common factor. We shall call these allowable ideals. We may thus ask which lines in $P^2$ (or which points in $(P^2)^*$) correspond to allowable ideals. We now find the complement of this set:

**Lemma 4.1.** (a) The "unallowable" ideals form a smooth conic in $(P^2)^*$.

(b) This conic is dual to the "conic of squares" in $P^2$. 

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PROOF. (a) Let \( I = (aX_2^2 + bX_2X_3 + cX_3^2, dX_2^2 + eX_2X_3 + fX_3^2) \). One shows using the resultant that these polynomials have a common root if and only if
\[
(a - cd)^2 - (ae - bd)(bf - ce) = 0.
\]
Now, the line in \( \mathbb{P}^2 \) through the points \([a, b, c]\) and \([d, e, f]\) has equation
\[
\begin{vmatrix}
Y_0 & Y_1 & Y_2 \\
a & b & c \\
d & e & f \\
\end{vmatrix}
\]
(where \( Y_0, Y_1, Y_2 \) are the coordinates for this \( \mathbb{P}^2 \)). Hence the corresponding point in \( (\mathbb{P}^2)^* \) is \([ae - bd, -(af - cd), bf - ce]\), and the result follows from (6).

(b) A line in \( (\mathbb{P}^2)^* \) corresponds to a pencil of lines in \( \mathbb{P}^2 \), i.e. the collection of those ideals containing a given polynomial \( L_1L_2 \). This line in \( (\mathbb{P}^2)^* \) will meet the “unallowable” conic twice in general, corresponding to the ideals \((L_1L_2, L_1L_3)\) and \((L_1L_2, L_2L_3)\) \( (L_3 \text{ arbitrary})\). This line will be tangent to the “unallowable” conic precisely when the corresponding point in \( \mathbb{P}^2 \) represents a square \( L_2^2 \), and so we are done. (Note that the squares really do form a conic in \( \mathbb{P}^2 \), namely \( Y_1^2 - 4Y_0Y_2 = 0 \).

REMARK 4.2. We get several amusing facts about these ideals for free from this description. For example,
(a) A general ideal \((L_1L_2, L_3L_4)\) contains exactly two squares (up to scalar multiplication).
(b) Two distinct ideals of this form have exactly one polynomial in common (up to scalar multiplication).
(c) Given a general polynomial \( L_1L_2 \), there are exactly two ideals containing it which are not allowable.

We now consider our basic links of \( \S 2 \). The set of all polynomials of the form \( X_2(X_2 + \tau X_3) \) corresponds to a line in \( \mathbb{P}^2 \), as does the set of all polynomials of the form \( X_3(\mu X_2 + X_3) \). These lines meet at \( X_2X_3 \) (which is at infinity) and each is tangent to the conic of squares (by duality, since each represents an unallowable ideal). See Figure 1.
The following is then immediate:

**Proposition 4.3.** Given an allowable ideal $(L_1L_2, L_3L_4)$, we can find generators which give a basic link if and only if the ideal does not contain the polynomial $X_2X_3$.

**Proof.** An allowable ideal corresponds to a line $P^2$ not tangent to the conic of squares. It then must meet the two distinguished lines either in two distinct points or at the point corresponding to $X_2X_3$ (see Figure 1).

We now turn to the question of when a double line in self-linked, i.e. when it can be linked to itself in one step. In the case $d = 0$, $C$ is clearly self-linked since it is a plane curve (and so a complete intersection). For $d = 1$, on the other hand, it is shown in [R2] that $C$ is self-linked if and only if char $k = 2$. This was generalized in [GMV1], where it is shown that if $C_n$ is the curve defined by the ideal $(I(\lambda)^n, X_0X_3 - X_1X_2)$, then $C$ is never self-linked for $n > 2$. ($n = 1$ is Rao’s result.) We now give a different generalization:

**Theorem 4.4.** A double line $C$ of arithmetic genus $< -1$ (i.e. $d \geq 2$) is self-linked if and only if char $k = 2$.

**Proof.** In view of Remark 2.4, the only thing left to prove is that for char $k \neq 2$, $C$ cannot be self-linked by a nonbasic link.

By Proposition 4.3, “almost all” links are basic. Hence we have only to consider allowable ideals of the form $(X_2X_3, L_1L_2)$. Clearly we may write this as $(X_2X_3, aX_2^2 + bX_3^2)$, where $a \neq 0$ and $b \neq 0$. Then writing $I(C)$ and the linked ideal $I(C')$ as in §2, a similar calculation shows

$$\begin{bmatrix} F' \\ G' \end{bmatrix} = \alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}.$$ 

Therefore $C$ is not self-linked. □

**Remark 4.5.** We can now generalize Proposition 2.1 to say that $C$ is directly linked to $C'$ if and only if

$$\begin{bmatrix} F' \\ G' \end{bmatrix} = A \begin{bmatrix} F \\ G \end{bmatrix},$$

where $\det A \neq 0$ and trace $A = 0$. □

**References**


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