ABSTRACT. In this paper the authors prove the following theorem:

Let $\Omega$ be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $V: \Omega \to \mathbb{R}$ a nonnegative convex function. Suppose $\lambda_1$ and $\lambda_2$ are the first and second nonzero eigenvalues of the equation

$$-\Delta f + Vf = \lambda f, \quad f|_{\partial\Omega} \equiv 0.$$ 

Then $\lambda_2 - \lambda_1 \geq \pi^2/d^2$, where $d$ is the diameter of $\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a smooth strictly convex bounded domain and $W: \Omega \to \mathbb{R}$ a nonnegative convex smooth function. The eigenvalues of the equation

(1) $$-\Delta f + Wf = \lambda f, \quad f = 0, \quad \text{on} \partial\Omega$$

can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots.$$ 


(2) $$\lambda_2 - \lambda_1 \geq \pi^2/4d^2,$$

where $d$ is the diameter of $\Omega$. In this paper the authors will use the method of [3 and 4] to prove the following theorem:

**THEOREM.** Let $\Omega$ be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $W: \Omega \to \mathbb{R}$ a nonnegative convex function. Suppose $\lambda_1$ and $\lambda_2$ are the first and second nonzero eigenvalues of (1). Then

(3) $$\lambda_2 - \lambda_1 \geq \pi^2/d^2,$$

where $d$ is the diameter of $\Omega$.

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In this paper the assumptions of all the lemmas are the same as those of the Theorem. We will not state them again.

Let $f_1$ and $f_2$ be the first and second eigenfunctions of (1); then $f_1(x) > 0, x \in \Omega$ [2], and $u = f_2/f_1$ is smooth to the boundary of $\Omega$ [3]. Suppose that

$$A = \max_{x \in \Omega} u(x); \quad -k = \min_{x \in \Omega} u(x).$$

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We may assume that $A \geq k$, otherwise, we can use $-f_2$ instead of $f_2$.

Since $\int_{\Omega} f_1 f_2 = 0$ and $f_1 > 0$, $k > 0$. Setting $\hat{u} = u/A$, we have $1 \geq \hat{u} \geq -k/A = -\hat{k}$ and $1 \geq \hat{k} > 0$;

$$v = \left( \hat{u} - \frac{1 - \hat{k}}{2} \right) \left( \frac{1 + \hat{k}}{2} \right)$$

and

$$a = \frac{1 - \hat{k}}{1 + \hat{k}}; \quad 1 > a \geq 0.$$

Then $v$ is a smooth function on $\bar{\Omega}$. By computing, we have

$$\Delta v = -\lambda(v + a) - 2(\nabla v \cdot \nabla \log f_1),$$

where $\lambda$ is $\lambda_2 - \lambda_1 > 0$.

**Lemma 1.** Let $z(v)$ be a smooth function defined on $\bar{\Omega}$ and $m > 0$ a constant. Suppose that

$$G(x) = m|\nabla v|^2 = z(v),$$

$P \in \partial\Omega$ and $G(P) = \max_{x \in \bar{\Omega}} G(x)$. Then $\nabla v(P) = 0$.

**Proof.** We can choose an orthonormal frame $l_1, l_2, \ldots, l_n$ around $P$ such that $l_1$ is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_i$ to denote the restriction of $l_i$ on $\partial\Omega$. Since $G(P)$ is the maximum of $G(x)$,

$$0 \leq \frac{\partial G}{\partial x_1} (P) = 2m \sum_{i=1}^{n} v_i v_{11} + z' v_1.$$

Furthermore, in $\Delta v = -\lambda(v + a) - 2(\nabla v \cdot \nabla \log f_1)$, $v$ and $\Delta v$ are all smooth on $\bar{\Omega}$; hence

$$\nabla v \cdot \nabla \log f_1 = \frac{1}{f_1} \left[ v_1 (f_1)_1 \sum_{i=2}^{n} v_i (f_1)_i \right]$$

achieves finite value on $\partial\Omega$. But $f_1|_{\partial\Omega} \equiv 0$, thus

$$\left. \left[ v_1 (f_1)_1 \sum_{i=2}^{n} v_i (f_1)_i \right] \right|_{\partial\Omega} \equiv 0.$$

Since $f_1 \equiv 0$ and $\partial\Omega$ and $l_i$, $2 \leq i \leq n$, are the tangent vectors of $\partial\Omega$, $(f_1)_i|_{\partial\Omega} \equiv 0$, $2 \leq i \leq n$. Hence,

$$v_1 (f_1)_1 \equiv 0 \quad \text{on} \quad \partial\Omega.$$

By Hopf’s lemma, $\partial f_1/\partial x_1 \neq 0$. Therefore,

$$v_1 \equiv 0 \quad \text{on} \quad \partial\Omega.$$

Putting (9) into (8), we have

$$0 \leq m \sum_{i=2}^{n} v_i v_{11} (P) = \frac{\partial G}{\partial x_1} (P).$$
From the definition of the second fundamental form in $R^n$, we have (note $v_1 = 0$)

$$v_{1i} = -\sum_{j=2}^{n} h_{ij} v_j,$$

where $(h_{ij})$ is the second fundamental form. Putting (11) into (10), we obtain

$$0 \leq -\sum_{i,j=2}^{n} mh_{ij} v_i v_j (P).$$

Since $\Omega$ is strictly convex, $(h_{ij})$ is positive definite; thus

$$0 \leq -m \sum_{i,j=2}^{n} h_{ij} v_i v_j (P) \leq 0.$$

Hence, $v_i (P) = 0$, $2 \leq i \leq n$, i.e., $\nabla v (P) = 0$.

**Lemma 2.** For any given $b > 1$,

$$\frac{\lvert \nabla v \rvert^2}{b^2 - v^2} \leq \lambda (1 + a).$$

**Proof.** For $\epsilon > 0$, consider the function defined on $\overline{\Omega}$

$$G(x) = \lvert \nabla v \rvert^2 + \lambda (1 + \epsilon + a) v^2.$$

Suppose $G (P) = \max_{x \in \overline{\Omega}} G (x)$. If $P \in \partial \Omega$, by Lemma 1, we have $\nabla v (P) = 0$ and

$$0 \leq 2b \Delta G (P) = 2b \lvert \nabla v \rvert^2 - 2 \nabla v \cdot \nabla (\nabla v \cdot \nabla \log f_1) - \lambda^2 (1 + \epsilon + a) v (v + a)$$

If $\nabla v (P) = 0$, then (12) is valid. If $\nabla v (P) \neq 0$, we can choose an orthonormal frame such that $v_i (P) = 0$, $2 \leq i \leq n$, and $v_1 (P) \neq 0$. (13) gives

$$v_{i1} = -\lambda (1 + \epsilon + a) v, \quad v_{1i} = 0, \quad 2 \leq i \leq n.$$
Putting (15) into (14), we obtain
\[0 \geq v_1^2 + (\epsilon + a)\lambda|\nabla v|^2 - 2v_1(\nabla v \cdot \nabla \log f_1) - \lambda^2(1 + \epsilon + a)v^2\]
\[ - \lambda^2(1 + \epsilon + a)av - 2\lambda(1 + \epsilon + a)v(\nabla v \cdot \nabla \log f_1)\]
\[= \lambda^2(1 + \epsilon + a)^2v^2 + (\epsilon + a)\lambda|\nabla v|^2 - \lambda^2(1 + \epsilon + a)av\]
\[-2\sum_{i=1}^{n} v_i v_{i1}(\log f_1)i - 2\sum_{i=1}^{n} v_i v_{i1}(\log f_1)i_1\]
\[-2(1 + \epsilon + a)v(\log f_1)i\]
\[= \lambda(\epsilon + a)G(P) - \lambda^2(1 + \epsilon + a)av + 2v_1(1 + \epsilon + a)v(\log f_1)\]
\[-2v^2(\log f_1)_{11} - 2(1 + \epsilon + a)v_1(\log f_1)\]
\[= \lambda(\epsilon + a)G(P) - \lambda^2(1 + \epsilon + a)av + 2v^2(\log f_1)_{11}.
\]
Hence
\[(16) \quad 0 \geq \lambda(\epsilon + a)G(P) - \lambda^2(1 + \epsilon + a)av - 2v^2(\log f_1)_{11}.
\]
Since $W$ and $\Omega$ are all convex, $\log f_1$ is concave [1], in particular, $-(\log f_1)_{11}(P) \geq 0$. Noting that $v \leq 1$, we have
\[(17) \quad \lambda(1 + \epsilon + a)G(x) \leq \lambda(1 + \epsilon + 1)\frac{a}{\epsilon + a} \leq \lambda(1 + \epsilon + a).
\]
From (12) and (17) we can obtain that $G(x) \leq (1 + \epsilon + a)\lambda$, $x \in \Omega$. This is
\[|\nabla v|^2 \leq \lambda(1 + \epsilon + a)(1 - v^2) \leq \lambda(1 + \epsilon + a)(b^2 - v^2).
\]
Letting $\epsilon \to 0$, we complete the proof of the lemma. Q.E.D.

**Lemma 3.** $\lambda \geq 1/(1 + a) \cdot \pi^2/d^2$.

**Proof.** By Lemma 2, we have
\[(18) \quad \frac{|\nabla (v/b)|}{\sqrt{1 - (v/d)^2}} \leq \lambda^{1/2}(1 + a)^{1/2}.
\]
Suppose that $q_1$ and $q_2 \in \Omega$ such that $v(q_1) = 1$, $v(q_2) = -1$, and let $L$ be the line segment between $q_1$ and $q_2$. $L$ lies on $\Omega$ completely, because it is convex. We integrate both sides of (18) along $L$ from $q_2$ to $q_1$ and obtain
\[\lambda^{1/2}(1 + a)^{1/2}d \geq \lambda^{1/2}(1 + a)^{1/2}length of L \geq \arcsin \frac{1}{b} - \arcsin \frac{-1}{b}.
\]
For any $b > 1$, (19) is valid and, letting $b \to 1$, the lemma is completely proved. Q.E.D.

If $a = 0$, then the Theorem is proved. Now suppose $a > 0$. From Lemma 2
\[\frac{|\nabla (v/b)|^2}{1 - (v/b)^2} \leq \lambda(1 + a), \quad b > 1.
\]
Set $\theta: \Omega \to R$, $\theta = \arcsin(v/b)$, $\arcsin(-1/b) \leq \theta \leq \arcsin(1/b)$. Then
\[\frac{|\nabla (v/b)|^2}{1 - (v/b)^2} = |\nabla \theta|^2 \leq \lambda(1 + a).
\]
Obviously, $\nabla \theta = 0$ if $v = 0$. Define $F: [\arcsin -1/b, \arcsin 1/b] \to R$ as
\begin{equation}
F(\theta_0) = \max_{x \in \Omega \atop \theta(x) = \theta_0} \frac{|\nabla(v/b)|^2}{1 - (v/b)^2}.
\end{equation}

$F(\theta_0(x))$ is continuous on $\bar{\Omega}$ and
\begin{equation}
F(\theta_0) \leq \lambda(1 + a), \quad \theta_0 \in [\arcsin -1/b, \arcsin 1/b].
\end{equation}

For any $\theta_0 \in [\arcsin -1/b, \arcsin 1/b]$ there must be an $x_0 \in \bar{\Omega}$ such that
\begin{equation}
\theta(x_0) = \theta_0 \quad \text{and} \quad F(\theta_0) = \frac{|\nabla(v/b)|^2}{1 - (v/b)^2}(x_0)
\end{equation}
are valid. Since $a > 0$, we can define a continuous function $\varphi$ on $\bar{\Omega}$ which satisfies
\begin{equation}
F(\theta) \equiv \lambda \left(1 + \frac{a}{b} \varphi(\theta)\right), \quad \varphi(\theta) \leq b.
\end{equation}

**LEMMA 4.** The $C^\infty$ function $y: [\arcsin -1/b, \arcsin 1/b] \to R$ satisfies
(i) $y(\theta) \geq \varphi(\theta)$, $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
(ii) there is an $x_0 \in \bar{\Omega}$ such that $\theta(x_0) = \theta_0$ and $y(\theta_0) = \varphi(\theta_0)$;
(iii) $y(\theta) \geq -1$ for any $\theta \in [\arcsin -1/b, \arcsin 1/b]$;
(iv) $y'(\theta_0) \geq 0$.

Then the following inequality is valid:
\begin{equation}
\varphi(\theta_0) \leq \sin \theta_0 - y'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} y''(\theta_0) \cos^2 \theta_0.
\end{equation}

**PROOF.** Consider the function $\Phi(x): \bar{\Omega} \to R$,
\begin{equation}
\Phi(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda(1 + cy) \right\} \cos^2 \theta,
\end{equation}
where $b > 1$ and $c = a/b$. Obviously, $\Phi(x) \leq 0$ for $x \in \bar{\Omega}$ and $\Phi(x_0) = 0$, i.e., $\Phi(x)$ attains its maximum at $x_0$, since
\begin{equation}
\Phi(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda \left(1 - \frac{v^2}{b^2}\right)(1 + cy).
\end{equation}

If $\nabla v(x_0) = 0$, then
\begin{equation}
0 = \Phi(x_0) = -\lambda(1 - v^2/b^2)(1 + cy)|_{x_0}
\end{equation}
and
\begin{equation}
y(x_0) = -1/c = -a/b < -1.
\end{equation}

This contradicts the assumption (iii). Thus $\nabla v(x_0) \neq 0$. By Lemma 1, $x_0 \notin \partial\Omega$, i.e., $x_0 \in \Omega$. According to the maximum principle, we have
\begin{equation}
\nabla \Phi(x_0) = 0,
\end{equation}
\begin{equation}
\Delta \Phi(x_0) \leq 0.
\end{equation}

For convenience we write $\Phi(x)$ as
\begin{equation}
\Phi(x) = \frac{1}{b^2} |\nabla v|^2 - \cos^2 \theta(1 + cy).
\end{equation}
Then

\[ \Phi_j = \frac{1}{b^2} \sum_i v_i v_{ij} - \lambda (1 + cy)(-2 \cos \theta \sin \theta) \theta_j - c \lambda \cos^2 \theta \theta'_j. \]

(22) gives that at \( x_0 \)

\[
(22') \quad \frac{2}{b^2} \sum_i v_i v_{ij} = \lambda [cy' \cos^2 \theta - 2(1 + cy) \cos \theta \sin \theta] \theta_j, \quad 1 \leq j \leq n.
\]

And also

\[
\Delta \Phi = \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla (\Delta v) - \lambda c \cos^2 \theta \Delta y
\]

\[ - \lambda (1 + cy) \Delta \cos^2 \theta - 2 \lambda c \nabla \cos^2 \theta \cdot \nabla y
\]

\[
= \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla (\Delta v) - \lambda c \cos^2 \theta (y''|\nabla \theta|^2 + y' \Delta \theta)
\]

\[ + 4 \lambda cy' \cos \theta \sin \theta |\nabla \theta|^2 - \lambda (1 + cy) \Delta \cos^2 \theta.
\]

From (23) we have that at \( x_0 \)

\[
0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \nabla v \cdot \nabla (\Delta v) - \lambda c \cos^2 \theta (y''|\nabla \theta|^2 + y' \Delta \theta)
\]

\[ + 4 \lambda cy' \cos \theta \sin \theta |\nabla \theta|^2 - \lambda (1 + cy) \Delta \cos^2 \theta.
\]

Since \( \nabla v(x_0) \neq 0 \), we can choose an orthonormal frame such that \( v_1(x_0) \neq 0 \) and \( v_i(x_0) = 0, \ 2 \leq i \leq n \). Then by \( (22') \) (note \( \sin \theta = v/b, \ \theta_j = v_j/b \cos \theta \))

\[
v_i = 0, \quad 2 \leq i \leq n,
\]

\[
v_{11} = (b/2) \lambda [cy' \cos \theta - 2(1 + cy) \sin \theta].
\]

Now we compute the terms in \( (23') \) at the particular frame

\[
\nabla v \cdot \nabla (\Delta v)_{x_0} = \sum_{i,j} v_i v_{jj} = \sum_j v_1 (v_{jj})_1
\]

\[
= v_1 \left[ -\lambda (v + a) - 2 \sum_i v_i (\log f_1)_i \right]
\]

\[
= -\lambda v_1^2 - 2v_1 \sum_i v_{1i} (\log f_1)_i - 2v_1 \sum_i v_i (\log f_1)_{1i}
\]

\[
= -\lambda v_1^2 + 2v_1 v_{11} (\log f_1)_1 = 2v_1^2 (\log f_1)_{11} \quad (\because \ (24)).
\]

From \( \Delta v/b = \Delta \sin \theta = \cos \theta \Delta \theta - \sin \theta |\nabla \theta|^2 \), we have

\[
\Delta \theta = \frac{1}{\cos \theta} \left[ \frac{\Delta v}{b} + \sin \theta |\nabla \theta|^2 \right].
\]

And

\[
\Delta \cos \theta = \Delta \left( 1 - \frac{v^2}{b^2} \right) = -\frac{1}{b^2} \Delta v^2 = -\frac{2}{b^2} (v \Delta v + |\nabla v|^2)
\]

\[
= -\frac{2}{b^2} v \Delta v - 2 \cos^2 \theta |\nabla \theta|^2.
\]
From $\Phi(x_0) = 0$, we have at $x_0$

\[ |\nabla \theta|^2 = \frac{1}{b^2 \cos^2 \theta} |\nabla v|^2 = \frac{1}{b^2 (1 - (v/b)^2)} |\nabla v|^2 = \frac{|\nabla v|^2}{b^2 - v^2} = \lambda(1 + cy). \]

Putting (25)–(28) into (23'), we obtain that at $x_0$

\[ 0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \left[ \frac{\Delta v}{b \cos \theta} + \frac{\sin \theta}{\cos \theta} \right] \lambda(1 + cy) \cos^2 \Theta \]

\[ - \lambda c \left[ \lambda y''(1 + cy) + y' \left( \frac{\Delta v}{b \cos \theta} + \frac{\sin \theta}{\cos \theta} \right) \lambda(1 + cy) \cos^2 \theta \right. \]

\[ + 4(1 + cy)\lambda^2 c y' \cos \theta \sin \theta - \lambda(1 + cy) \left[ -\frac{2}{b^2} v \Delta v - 2\lambda \cos^2 \theta (1 + cy) \right]. \]

Since $|\nabla v|^2 = \nu_1^2 = b^2 \cos^2 \theta |\nabla \theta|^2 = \lambda^2 \cos^2 \theta (1 + cy)$ (: (28)) and $\Delta v = -\lambda(v + a) - 2v \nabla \cdot \nabla \log f_1$, (29) can be written as

\[ 0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 - 2\lambda^2 (1 + cy) \cos^2 \theta - \lambda^2 c y''(1 + cy) \cos^2 \theta \]

\[ + 2\lambda^2 c y' \cos \theta (\sin \theta + c) + 3\lambda c y(1 + cy) \sin \theta \cos \theta \]

\[ + 2\lambda(1 + cy)^2 \cos^2 \theta - 2\lambda^2 (1 + cy) \sin \theta (\sin \theta + c) \]

\[ - \frac{4}{b^2} v_1^2 (\log f_1)_{11} - \frac{4}{b^2} v_1 v_{11} (\log f_1)_{11} \]

\[ + \frac{2}{b} \lambda [c y' \cos \theta - 2(1 + cy) \sin \theta] v_1 (\log f_1). \]

Putting the second formula of (24) into the above inequality and noting that $(\log f_1)_{11} \leq 0$, we have

\[ 0 \geq \frac{1}{2} \lambda^2 c^2 (y')^2 \cos^2 \theta + 2\lambda^2 (1 + cy)^2 - 2\lambda^2 (1 + cy) \cos^2 \theta \]

\[ + \lambda^2 c y'[(1 + cy) \sin \theta \cos \theta + \cos \theta (\sin \theta + c)] \]

\[ - \lambda^2 c (1 + cy) y'' \cos^2 \theta - 2\lambda^2 (1 + cy) \sin \theta (\sin \theta + c). \]

Dividing both sides of the above inequality by $\lambda^2 (1 + cy) > 0$, we have

\[ 0 \geq y' \left( \sin \theta \cos \theta + \cos \frac{\sin \theta + c}{1 + cy} \right) - y'' \cos^2 \theta + \frac{2}{c} (1 + cy) - \frac{2}{c} - 2 \sin \theta, \]

\[ 2y - 2 \sin \theta \leq y'' \cos^2 \theta - y' \left( \sin \theta \cos \theta + \cos \frac{\sin \theta + c}{1 + cy} \right). \]

Since $-1 \leq y(\theta_0) = \varphi(\theta_0) \leq b$, thus,

\[ |y(\theta_0)| \leq b, \quad y(\theta_0) \sin \theta_0 \leq |y(\theta_0)| \sin \theta_0 | \sin \theta_0 | \leq bv(\theta_0)/b \leq 1 \]

and

\[ c \geq cy \sin \theta, \quad c + \sin \theta \geq (1 + cy) \sin \theta, \quad \frac{c + \sin \theta}{1 + cy} \geq \sin \theta. \]

Since $y'(\theta_0) \geq 0$, we have

\[ \varphi(\theta_0) = y(\theta_0) \leq \sin \theta_0 - y'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} y''(\theta_0) \cos^2 \theta_0. \]
LEMMA 5. Define a function $\psi: [-\pi/2, \pi/2] \to R$ as
\[
\psi(\theta) = \frac{(4/\pi)(\theta + \cos \theta \sin \theta) - 2 \sin \theta}{\cos^2 \theta}, \quad \theta \in (-\pi/2, \pi/2),
\]
\[
\psi(-\pi/2) = -1, \quad \psi(\pi/2) = 1.
\]
Then $\psi$ is a $C^\infty$ function in $(-\pi/2, \pi/2)$ and is continuous on $[-\pi/2, \pi/2]$ and also $y = \psi(\theta)$ satisfies the following equation:
\[
y - \sin \theta + y' \sin \theta \cos \theta - \frac{1}{2} y'' \cos^2 \theta = 0,
\]
and $y'(\theta) \geq 0, \theta \in (-\pi/2, \pi/2)$.

PROOF. See reference [4].

LEMMA 6. Let $\varphi(\theta)$ be the function defined by (21). Then
\[
\varphi(\theta) \leq \psi(\theta), \quad \theta \in \left[\arcsin \frac{-1}{b}, \arcsin \frac{1}{b}\right],
\]
where $\psi(\theta)$ is defined by (30).

PROOF. We will use the reduction to absurdity. If
\[
\sigma = \varphi(\theta_0) - \psi(\theta_0) = \max_\theta \{\varphi(\theta) - \psi(\theta)\} > 0
\]
we could choose $\psi(\theta) + \sigma = \tilde{y}$ as $y$ in Lemma 4. Therefore,
\[
\varphi(\theta_0) = \tilde{y}(\theta_0) = \psi(\theta_0) + \sigma \leq \sin \theta_0 - \psi'(\theta_0) \sin \theta_0 \cos \theta_0 + \frac{1}{2} \psi''(\theta_0) \cos^2 \theta_0 = \psi(\theta_0).
\]
This contradicts (32). Q.E.D.

PROOF OF THE THEOREM. By Lemma 6, we have
\[
|\nabla \theta|^2 = \frac{|\nabla \psi|^2}{b^2 - \psi^2} \leq \lambda \left(1 + \frac{a}{b} \psi(\theta)\right),
\]
where $\psi(\theta)$ is the function defined by (30). Hence,
\[
\lambda^{1/2} \geq \frac{|\nabla \theta|}{\sqrt{1 + (a/b)\psi(\theta)}}.
\]
Obviously,
\[
\psi(0) = 0, \quad \psi(-\theta) = -\psi(\theta).
\]
Integrating both sides of (33) as in Lemma 3, we obtain
\[
\lambda^{1/2} d \geq \int_{\arcsin -1/b}^{\arcsin 1/b} \frac{d\theta}{\sqrt{1 + (a/b)\psi(\theta)}} = \int_{0}^{\arcsin 1/b} \left(\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}}\right) d\theta.
\]
Since $|\pm (a/b)\psi(\theta)| \leq 1$,
\[
\frac{1}{\sqrt{1 + (a/b)\psi(\theta)}} + \frac{1}{\sqrt{1 - (a/b)\psi(\theta)}} = 2 \left[1 + \sum_{p=1}^{\infty} \frac{1 \cdot 3 \cdots (4p-1)}{2 \cdot 4 \cdots (4p)} \left(\frac{a}{b}\right)^{2p} \psi^{2p}\right] \geq 2.
\]
Thus
\[
\lambda^{1/2} d \geq 2 \arcsin \frac{1}{b}, \quad \lambda \geq \frac{4}{d^2} \left(\arcsin \frac{1}{b}\right)^2.
\]
Letting $b \to 1$, we obtain $\lambda \geq \pi^2/d^2$. Q.E.D.
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