VECTOR BUNDLES AND PROJECTIVE MODULES

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ABSTRACT. Serre and Swan showed that the category of vector bundles over a compact space $X$ is equivalent to the category of finitely generated projective modules over the ring of continuous functions on $X$. In this paper, titled after the famous paper by Swan, this result is extended to an arbitrary topological space $X$. Also the well-known homotopy classification of the vector bundles over compact $X$ up to isomorphism is extended to arbitrary $X$. It is shown that the $K_0$-functor and the Witt group of the ring of continuous functions on $X$ coincide, and they are homotopy-type invariants of $X$.

1. Statement of results. Let $F$ be either the real numbers $R$, the complex numbers $C$, or the quaternions $H$. For a topological space $X$, let $F^X$ denote the ring of all continuous functions $X \rightarrow F$.

When $X$ is compact Hausdorff, it is well known (Swan [5]) that the category $\mathcal{P}(F^X)$ of finitely generated projective $F^X$-modules is equivalent to the category of $F$-vector bundles over $X$. Later Goodearl [2] observed that the equivalence holds in the more general case of paracompact Hausdorff $X$ if we restrict ourselves to the bundles of finite type. This restriction excludes vector bundles of unbounded dimension which cannot come from $\mathcal{P}(F^X)$.

The first goal of this paper is to extend this result (with an appropriate definition of finite type) to an arbitrary topological space $X$.

DEFINITION. A bundle over $X$ (see [5]) is of finite type if there is a finite partition $S$ of 1 on $X$ (i.e. a finite set $S$ of nonnegative continuous functions on $X$ whose sum is 1) such that the restriction of the bundle to the set $\{x \in X : f(x) \neq 0\}$ is trivial for each $f$ in $S$.

For example, when $X$ is compact Hausdorff, every bundle over $X$ is of finite type. For a normal space $X$ our definition of finite type is equivalent to the following definition [2]: there is a finite open covering $T$ of $X$ such that the restriction of the bundle to each $U \in T$ is trivial.

THEOREM 1. The category $\mathcal{P}(F^X)$ is equivalent to the category of $F$-vector bundles of finite type over $X$.

Our second theorem concerns the classification problem for the bundles of finite type over a topological space $X$, i.e. the problem of identification of the isomorphism classes in $\mathcal{P}(F^X)$ with homotopy classes. Let $G(F^n)$ be the set of all subspaces of the $n$-dimensional vector space $F^n$ over $F$. As usual, $G(F^n)$ is endowed with...
the topology of the disjoint union of the Grassmann manifolds \( G_m(F^n) \), where \( 0 \leq m \leq n \). The topology on \( G(F^n) \) can be given explicitly by identifying \( G(F^n) \) with the subset \( Y_1 = \{ p = p^2 = p^* \in M_n F \} \) or \( Y_2 = \{ a \in M_n F : a^2 = aa^* = 1_n \} \) of the matrix ring \( M_n F \) (which has the topology of the direct product of \( n^2 \) copies of \( F \)). The following 2 subsets of \( M_n F \) are homotopically equivalent to \( G(F^n) \), so they can replace \( G(F^n) \) in Theorem 2 below: \( Y_3 = \{ e = e^2 \in M_n F \} \), \( Y_4 = \{ a = a^* \in \text{GL}_n F \} \). The inclusion \( F^n \subset F^{n+1} \) induces the inclusion \( G(F^n) \subset G(F^{n+1}) \) which corresponds to the map \( b \mapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \) on the sets \( Y_1 \) and \( Y_3 \) and to the map \( b \mapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \) on the sets \( Y_2 \) and \( Y_4 \).

**THEOREM 2.** There is a bijection between any two of the following five sets:

1. the isomorphism classes of \( F \)-vector bundles of finite type over \( X \);
2. the isomorphism classes in \( \mathcal{P}(F^X) \);
3. the isomorphism classes in \( \mathcal{P}(F_0^X) \), where \( F_0^X \subset F^X \) is the subring of bounded functions on \( X \);
4. the isomorphism classes of finitely generated projective \( F^X \)-modules with positive definite hermitian forms;
5. the inductive limit of the homotopy classes of continuous maps from \( X \) to \( G(F^n) \) as \( n \to \infty \).

**COROLLARY.** The set of isomorphism classes in \( \mathcal{P}(F^X) \) is a homotopy-type invariant of \( X \). In particular, \( K_0 F^X \), the Grothendieck group of \( \mathcal{P}(F^X) \), is a homotopy-type invariant of \( X \).

**REMARK.** The rings \( F^X \) and \( F_0^X \) in Theorem 2 can be replaced by dense subrings (see [6]). For example, if \( X \) is a manifold and \( A \) is the ring of smooth functions on \( X \), then there is a bijection between the isomorphism classes in \( \mathcal{P}(F^X) \) and \( \mathcal{P}(A) \). In particular, \( K_0 F^X = K_0 A \) is a homotopy-type invariant of \( X \). Note that the algebraic \( K_0 \)-functor does not take into account any topology on rings.

Our third goal is to compute the Witt group of hermitian forms over \( F^X \). Let \( \mathcal{P}^+(F^X) \) denote the category of hermitian spaces \( (P, \Phi) \) over \( F^X \), where \( P \in \mathcal{P}(F^X) \) and \( \Phi \) is a nonsingular hermitian form on \( P \) (nonsingular here means that the corresponding map \( P \to P^* \) is an isomorphism). The morphisms in \( \mathcal{P}^+(F^X) \) preserve scalar products. In particular, they have trivial kernels.

**THEOREM 3.** The set of isomorphism classes in \( \mathcal{P}^+(F^X) \) is the direct square of the set of isomorphism classes of objects in \( \mathcal{P}(F^X) \).

**COROLLARY.** The Grothendieck group of the category \( \mathcal{P}^+(F^X) \), i.e. the Witt-Grothendieck group of \( F^X \), is isomorphic to \( K_0 F^X \times K_0 F^X \). The Witt group of \( F^X \) is isomorphic to \( K_0 F^X \).

Recall that the Witt group is the cokernel of the homomorphism of the Grothendieck groups induced by the hyperbolic functor \( \mathcal{P}(F^X) \to \mathcal{P}^+(F^X) \).

2. **Proof of Theorem 1.**

**LEMMA 4.** Every object \( P \) in \( \mathcal{P}(F^X) \) is isomorphic to the column space of a square hermitian idempotent matrix \( e = e^2 = e^* \) over \( F^X \).

**PROOF.** Since \( P \) is isomorphic to a direct summand of the free module \( (F^X)^n \) for some natural number \( n \), it is isomorphic to the column space (i.e. to the image
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\[ a(F^X)^n \] of an \( n \)-by-\( n \) idempotent matrix \( a = a^2 \) over \( F^X \). Since \( a = a^2 \), we have

\[ (a^*)^2 = a^* \]

and

\[ (1 + (1 - 2a)(1 - 2a^*))a^* = 2aa^* = a(1 + (1 - 2a)(1 - 2a^*)) \]

where 1 stands for the identity matrix \( 1_n \in M_n F^X \). Since \( (1 - 2a)(1 - 2a^*) \) is a hermitian semipositive matrix, there is a hermitian matrix \( g = g^* \) in \( GL_n F^X \) such that \( g^* = 1 + (1 - 2a)(1 - 2a^*) \). We set \( e = gag^{-1} \). Then \( e^2 = ga^2g^{-1} = gag^{-1} = e \) and \( e^* = g^{-1}a^*g = gag^{-1} = e \). So \( P \) is isomorphic to the column space of this \( e = e^2 = e^* \). The lemma is proved.

Every object \( P \) in \( \mathcal{P}(F^X) \) gives an \( F \)-vector bundle \( \Gamma^*(P) \) over \( X \) in the usual way. Namely, the \( F \)-vector space at a point \( x \) of \( X \) is \( P \) evaluated at \( x \); i.e. \( P \otimes_v F^X \), where \( v: F^X \to F \) is given by \( v(f) = f(x) \) for any continuous function \( f: X \to F \). When \( P \) is the column space of a matrix \( e = e^2 \in M_n F^X \), the fiber at \( x \) is just \( e(x)F^n \). The local triviality of \( \Gamma^*(P) \) follows from the following lemma.

**Lemma 5.** If \( e = e^2 \in M_n F^X, x, y \in X, and g = e(x)e(y) + (1 - e(x))(1 - e(y)) \in GL_n F \), then the matrices \( e(x) \) and \( e(y) \) over \( F \) are similar.

**Proof.** We have \( e(x)g = ge(y) \). When \( g \in GL_n F \), we conclude that \( g^{-1}e(x)g = e(y) \). The lemma is proved.

**Corollary 6.** For every object \( P \) in \( \mathcal{P}(F^X) \), the corresponding \( F \)-vector bundle \( \Gamma^*(P) \) over \( X \) is of finite type.

**Proof.** By Lemma 4, \( P \) is isomorphic to the column space of \( e = e^2 = e^* \in M_n F^X \) for some \( n \), \( e \). The set \( Y_1 = \{ p = p^2 = p^* \in M_n F \} \) is compact (note that \( |p| \leq 1 \) for each \( p \) in \( Y_1 \)). Therefore there is a finite partition \( S' \) of \( 1 \) on \( Y_1 \) such that \( |p - q| < 1/3 \) whenever \( f'(p)f'(q) \neq 0 \) for some \( f' \) in \( S' \). (Note that \( F^n \) has the usual metric \( ||(z_j)||^2 = \sum z_j \) which makes it a Banach space, so \( M_n F \) is a Banach algebra with respect to the operator norm.)

The matrix \( e \) above can be considered as a continuous map \( X \to Y_1 \). So the partition \( S' \) gives a finite partition \( S \) of \( 1 \) on \( X \). We have \( |e(x) - e(y)| < 1/3 \) whenever \( x \) and \( y \) are in the same part (i.e. \( f(x)f(y) \neq 0 \) for some \( f \) in \( S \)).

For any part \( U(f) = \{ x \in X: f(x) \neq 0 \} \), where \( f \in S \), and any \( x, y \) in \( U(f) \) we have, as in Lemma 5, \( e(x)g = ge(y) \), where

\[ g = e(x)e(y) + (1 - e(x))(1 - e(y)) = 1 - (e(y) - e(x))(1 - 2e(y)) \]

Since

\[ |g - 1| = |(e(y) - e(x))(1 - 2e(y))| \leq |e(y) - e(x)||1 - 2e(y)| < (1/3) \cdot 3 = 1, \]

we conclude that \( g \in GL_n F \). Fixing \( x \in U(f) \), we have \( e(y) = g^{-1}e(x)g \), where \( g \) depends continuously on \( y \in U(f) \). Since \( F \) is a division algebra, the column space of \( e(z) \) is a finite dimensional vector space over \( F \) for each \( z \) in \( X \). Therefore the column space of the restriction of \( e \) to \( U(f) \) is a finitely generated free \( F^{U(f)} \)-module. So the restriction of the bundle \( \Gamma^*(P) \) to \( U(f) \) is trivial. Thus, the bundle is of finite type. The corollary is proved.

**Lemma 7.** For every \( F \)-vector bundle \( \xi \) of finite type over \( X \), the \( F^X \)-module \( \Gamma(\xi) \) of its global sections is finitely generated and projective (i.e. \( \Gamma(\xi) \in \mathcal{P}(F^X) \)).

**Proof.** Let \( S \) be a finite partition of \( 1 \) on \( X \) such that \( \xi \) is trivial over each \( U(f) = \{ x \in X: f(x) \neq 0 \} \), \( f \in S \). For each \( f \) in \( S \) we pick a free basis.
\[ a_1(f), \ldots, a_n(f) \] for \( \Gamma(\xi)|_{U(f)} \). We would like to extend these \( a_j(f) \) to global sections of the bundle. However, in general, this can be done only after a modification of \( a_j(f) \).

We write \( a_j(f) = \sum c_{i,j}(f, g)a_i(g) \) on \( U(f) \cap U(g) \) for every \( g \in S \). It is clear that there is a continuous nonnegative function \( f' \) on \( X \) such that \( f' \) and \( f \) have the same zero set and \( c_{i,j}(f, g)f' \to 0 \) on \( U(f) \cap U(g) \) whenever \( f(x) \to 0 \), for all \( i, j \) and \( g \). Then

\[
b_j(f) = \begin{cases} a_j(f)f' & \text{on } U(f), \\ 0 & \text{elsewhere on } X \end{cases}
\]

is a continuous global section of the bundle. Moreover, \( \{b_j(f)|_{U(f)}: 1 \leq j \leq n(f)\} \) is a basis for \( \Gamma(\xi)|_{U(f)} \).

Let us show now that \( \{b_j(f): 1 \leq j \leq n(f), f \in S\} \) is a generating set for the \( \mathcal{F}^X \)-module \( \Gamma(\xi) \). Let \( s \in \Gamma(\xi) \). We write \( s|_{U(f)} = \sum c_j(f)b_j(f) \) with \( c_j(f) \in \mathcal{F}^U(f) \). There is a modification (depending on \( s \)) of the partition \( S \) such that the open covering \( \{U(f): f \in S\} \) of \( X \) stays the same, but for the new partition we have that \( c_i(f)f \to 0 \) on \( U(f) \) whenever \( f \to 0 \). Then \( d_j(f) \in \mathcal{F}^X \) for \( d_j(f) \) defined by

\[
d_j(f) = \begin{cases} c_j(f)f & \text{on } U(f), \\ 0 & \text{elsewhere on } X \end{cases}
\]

Moreover,

\[
s = \sum_{f \in S} \sum_{j=1}^{n(f)} d_j(f)b_j(f). \]

Therefore we have a surjective map from \( \mathcal{F}^X)^N \), where \( N = \sum_{f \in S} n(f) \) is the number of the generators, onto \( \Gamma(\xi) \). Using a positive definite hermitian form on \( \mathcal{F}^X)^N \) (say, the standard form \( (a, b) \mapsto \sum a_j^*b_j \) for \( a = (a_j), b = (b_j) \) in \( \mathcal{F}^X)^N \)), we obtain, as in [5], that the kernel of the homomorphism \( \mathcal{F}^X)^N \to \Gamma(\xi) \) is a direct summand of \( \mathcal{F}^X)^N \). So \( \Gamma(\xi) \) is a projective \( \mathcal{F}^X \)-module. Thus, Lemma 7 is proved.

Putting together Lemmas 4 and 7, we conclude, as in [5] (see also [3 and 4]) that the functors \( \Gamma \) and \( \Gamma^* \) give an equivalence of the category \( \mathcal{P}(\mathcal{F}^X) \) and the category of \( \mathcal{F} \)-vector bundles over \( X \) of finite type.

3. Proof of Theorem 2.

Isomorphism of the sets (1) and (2). This follows from Theorem 1.

Isomorphism of the sets (2) and (3). By Lemma 4 every object \( P \) of \( \mathcal{P}(\mathcal{F}^X) \) is isomorphic to an object extended from some object of \( \mathcal{P}(\mathcal{F}_0^X) \). Let now \( P \) and \( Q \) be objects of \( \mathcal{P}(\mathcal{F}_0^X) \) which are isomorphic over the bigger ring \( \mathcal{F}^X \). We want to prove that \( P \) and \( Q \) are isomorphic in \( \mathcal{P}(\mathcal{F}_0^X) \).

We represent \( P \) and \( Q \) as the column spaces of matrices \( p = p^2 = p^* \) and \( q = q^2 = q^* \) in \( M_n\mathcal{F}_0^X \). Using the stabilization operations \( e \mapsto (e_0) \) (which allowed us to take the same \( n \) for \( p \) and \( q \)), and an isomorphism of \( P \) and \( Q \) over \( \mathcal{F}^X \), we can assume that \( g^{-1}pg = q \) for some \( g \in \text{GL}_n \mathcal{F}^X \).

Let \( g = g'h \) be the polar decomposition of \( g \), i.e. \( g' = g'^* \) is hermitian, \( h = h'^{-1} \) is orthogonal (unitary), \( g'^2 = gg'^* \), and the centralizer of \( g' \) in \( M_n\mathcal{F}^X \) is the same as that of \( gg'^* \), so \( pg' = g'p \). Then \( h \in \text{GL}_n \mathcal{F}^X \) is bounded and \( h^{-1}ph = h^*pg = q \). So \( P \) and \( Q \) are isomorphic over \( \mathcal{F}_0^X \).
Isomorphism of the sets (2) and (4). Let us show that every object $P$ in $\mathcal{P}(F^X)$ carries a positive definite hermitian form. By Lemma 4, $P$ is isomorphic to the column space $e(F^X)^n$ of a matrix $e = e^2 = e^*$ in $M_nF^X$. We define a hermitian positive definite form

$$\Phi: e(F^X)^n \times e(F^X)^n \to F^X$$

by

$$\Phi((a_i), (b_i)) = \sum a_i^* b_i.$$

Note that $(1 - e)(F^X)^n$ is the orthogonal complement to $e(F^X)^n$.

Let us show now that any two positive definite hermitian forms $\Phi$ and $\Psi$ on $P$ are isomorphic. Following [4, Theorem 8.8], consider $h = \Phi^{-1} \Psi \in \text{Aut}(P)$ (where $\Phi$ and $\Psi$ are regarded as maps $P \to P^*$). Then $h$ is selfadjoint and positive with respect to $\Phi$ (namely,

$$\Phi(hu, v) = (\Phi hu)v = (\Psi u)v = \Psi(u, v) = \Psi(v, u)^* = \Phi(hv, u)^* = \Phi(u, hv)$$

and $\Phi(hv, v) = \Psi(v, v) > 0$ for $0 \neq v \in P$). Let $g$ be its selfadjoint positive square root. Then $\Psi = \Phi h = \Phi g^2 = g^* \Phi g$ (i.e. $\Psi(u, v) = \Phi(hu, v) = \Phi(g^2 u, v) = \Phi(gu, gv)$), so $\Phi$ and $\Psi$ are isomorphic.

Isomorphism of the sets (2) and (5). We identify $G(p_n)$ with $\{p = r = p^* \in M_nF\}$ and $\text{Hom}(X, G(F^n))$ (i.e. the continuous maps $X \to G(F^n)$) with $\{e = e^2 = e^* \in M_nF^X\}$.

By Lemma 4, every object $P$ in $\mathcal{P}(F^X)$ is isomorphic to the column space of such a matrix $e$. Let us show that the image of $e$ in $\lim \pi_1(X, G(F^n))$ does not depend on the choice of $e$ (here $\pi(X, G(F^n))$ denotes the homotopy classes in $\text{Hom}(X, G(F^n))$).

Let $a = a^2 = a^*$ in $M_mF^X$ and $b = b^2 = b^*$ in $M_nF^X$ have isomorphic column spaces. Using the stabilization maps $e \mapsto (e_{00})$ which do not change images in the inductive limit, we can assume that $m = n$ and the matrices $a$ and $b$ are similar. That is, $b = gag^{-1}$ with $g$ in $\text{GL}_n F^X$.

Then we have, in $M_{2n}F^X$, that

$$\begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

By the Whitehead lemma, the matrix $(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix})$ is the product of elementary matrices and, hence, homotopic to the identity matrix. So $(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix})$ are homotopic in the set $\{e = e^2 \in M_{2n}F^X\}$. This homotopy can be retracted to a homotopy in the subset $\{e = e^2 = e^* \in M_{2n}F^X\}$ (using, for example, the proof of Lemma 5 with $X$ replaced by $X \times \{t: 0 \leq t \leq 1\}$). Thus, $(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix})$ are homotopic in $\text{Hom}(X, G(F^{2n}))$.

So we have a well-defined map from the isomorphism classes of objects in $\mathcal{P}(F^X)$ to $\lim \pi(X, G(F^n))$. This map is onto, as one can see by assigning to each $e = e^2 = e^*$ in $M_nF^X$ its column space, which gives a map from $\lim \pi_1(\text{Hom}(X, G(F^n)))$ to $\mathcal{P}(F^X)$ (up to isomorphism in $\mathcal{P}(F^X)$, this map is onto by Lemma 4).

The following lemma completes our proof.
Lemma 8. Any two matrices in $\text{Hom}(X, G(F^n))$ which are homotopic are similar. In particular, their column spaces are isomorphic.

Proof. Let $a, b \in \text{Hom}(X, G(F^n))$ be homotopic. That is, there is a continuous map $c: X \times \{t: 0 \leq t \leq 1\} \to G(F^n)$ such that $c(\cdot, 0) = a$ and $c(\cdot, 1) = b$.

Let $\Pi G(F^n)$ denote the set of continuous maps $\{t: 0 \leq t \leq 1\} \to G(F^n)$. Then $c$ can be regarded as a continuous map $X \to \Pi G(F^n)$. Note that the metric on $G(F^n)$ (induced by the norm on $M_n F$) gives a metric on $\Pi G(F^n)$.

Since $\Pi G(F^n)$ is a metric space, it is paracompact. Let $S'$ be a locally finite partition of 1 on $\Pi G(F^n)$ such that $|p - q| < 1/9$ whenever $p$ and $q$ are in the same part. The map $c$ gives a locally finite partition $S$ of $X$ such that $|c(x, t) - c(y, t)| < 1/9$ for all $t$ whenever $x$ and $y$ are in the same part, i.e. $f(x)f(y) \neq 0$ for some $f \in S$.

For each $f$ in $S$ we pick a point $x = x(f)$ in $U(f) = \{z \in X: f(z) \neq 0\}$ and a positive number $\varepsilon(f)$ such that $|c(x, t) - c(x, s)| < 1/9$ whenever $|t - s| < \varepsilon(f)$. Then

$$|c(z, t) - c(z, s)| \leq |c(x, t) - c(x, s)| + |c(z, t) - c(x, t)| + |c(z, s) - c(x, s)|$$

$$< 1/9 + 1/9 + 1/9 = 1/3$$

for every $z$ in $U(f)$ whenever $|t - s| < \varepsilon(f)$.

We set $\delta(z) = \sum_{f \in S} \varepsilon(f)f(z)$ for every $z$ in $X$. Then $\delta$ is a continuous positive function on $X$ such that $|c(z, t) - c(z, s)| < 1/3$ whenever $|t - s| < \delta(z)$.

For every $z$ in $X$ and any integer $k \geq 0$ we set

$$t_k(z) = \min(1, k\delta(z)), \quad c_k(z) = c(z, t_k(z)).$$

Then $c_0 = a$, $c_k(z) = b(z)$ for $k \geq 1/\delta(z)$, and $|c_k(z) - c_{k+1}(z)| < 1/3$ for all $z$ and $k$.

We set

$$g_k = c_k c_{k+1} + (1 - c_k)(1 - c_{k+1}) \quad \text{for all } k.$$

Then (see Lemma 4 and its proof) $c_k g_k = g_k c_{k+1}, |1 - g_k| < 1$; hence $g_k \in GL_n(F_X)$ for all $k$. Moreover, $g_k(z) = 1$ for $k \geq 1/\delta(z)$.

Finally, we set $G(z) = g_0(z)g_1(z)\cdots$. Then $g \in GL_n F_X$ and $ag = gb$; hence $g^{-1}ag = b$. Lemma 8 is proved.

Theorem 2 is proved.

Remark. Using that $G(F^n)$ is homotopically equivalent to $\{h = h^* \in GL_n F_X\}$ we obtain that any two homotopic nonsingular hermitian forms over $F_X$ are isomorphic (when $X$ is compact, this is well known; see [4]). In [1] the converse is claimed (that every two isomorphic forms are homotopic) when $F = C$ and $X$ is a finite CW complex. However, this claim is false, as the following counterexample shows. Let

$$X = S^3 = \{(x_i) \in R^4: \sum x_k^2 = 1\}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$g = \begin{pmatrix} x_1 + x_2 i & x_4 i - x_3 \\ x_3 + x_4 i & x_1 - x_2 i \end{pmatrix} \in GL_2 F, \quad b = g^*ag.$$
GL_1 F^X (E_2 F^X); hence g ∈ GL_1 F^X (E_2 F^X). But homotopically g is the identity map S^3 → S^3 and hence cannot be passed through the embedding S^1 ⊂ S^3 (which corresponds to the embedding GL_1 C ⊂ GL_2 C). Of course, a and b are stably homotopic.

REMARK. Lemma 8 can be easily extended to an arbitrary Banach algebra F (rather than F = R, C, or H). Namely, we can assert the following result.

THEOREM 9. Let F be a Banach algebra with 1. Then for any topological space X and any n, every two homotopic maps X → {p = p^2 ∈ M_n F} give similar matrices in M_n F^X. Therefore there is a bijection between the finitely generated projective F^X-modules up to isomorphism and \( \lim \pi(X, \{p = p^2 ∈ M_n F\}) \). In particular, this set of isomorphism classes and, hence, K_0 F^X are homotopy-type invariants of X.

Moreover, checking the properties of F which were actually used in the proof of Theorem 2, we see that we can replace the condition that F is a Banach algebra with 1 in Theorem 9 by the following weaker condition: F is a metrizable topological R-algebra with 1 such that the map \( z ↦ z^{-1} \) is defined and continuous in a neighbourhood of 1.

4. Proof of Theorem 3. This theorem can be restated as follows. For any nonsingular hermitian form \( \Phi \) on any object \( P \) in \( P(F^X) \) there is an orthogonal decomposition \( P = P_1 ⊕ P_2 \) such that the restriction of \( \Phi \) on \( P_1 \) (resp. \( P_2 \)) is positive (resp. negative); moreover, such a decomposition is unique up to isomorphism.

For compact X this statement is Theorem 8.13 of [4]. This theorem is stated in [4] in terms of bundles, so the condition that X is compact is needed to obtain a positive definite hermitian form. This condition can be replaced by the condition that the bundle is of finite type. Therefore to prove our Theorem 3 we can just repeat the arguments of [4].

REMARK. The referee attracted my attention to the fact that some ideas and computations in this paper can be traced back to the book *Rings of operators* by I. Kaplansky (Benjamin, 1968). In particular, our Lemma 4 is contained in Theorem 26 of the book. Also the referee pointed out the paper *Partitions of unity in the theory of fibrations* by A. Dold (Ann. of Math. 78 (1963), 223–255) which is one of the first studies of fibrations using locally finite partitions of unity.

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