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Ω-STABLE LIMIT SET EXPLOSIONS

BY
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ABSTRACT. Certain diffeomorphisms of two-dimensional manifolds are considered. These diffeomorphisms have a finite hyperbolic limit set which contains a limit set cycle. The only nontransverse cycle connection in these cycles is a complete coincidence of one component of the unstable manifold of one periodic point with one component of the stable manifold of some other periodic point. A one-parameter family of diffeomorphisms containing the original diffeomorphism is described. It is shown that for parameter values arbitrarily near the parameter value corresponding to the original map these diffeomorphisms have a much enlarged limit set and are Ω-stable.

1. Introduction. It is well known that, for Axiom A diffeomorphisms, the presence of cycles prevents Ω-stability [**5**]. On the other hand, the absence of cycles guarantees Ω-stability [**8**]. We are concerned with the following question: Given f satisfying Axiom A with cycles, can f be C^r approximated by an Ω-stable g? This question has been answered affirmatively for manifolds of dimension two by Newhouse and Palis [**3**]. In this paper they describe a perturbation taking f to g with g satisfying Axiom A without cycles. Furthermore $\Omega(f) = \Omega(g)$. That this result does not generalize to higher dimensions was established by a pair of counterexamples. Pugh, Walker and Wilson [**6**] demonstrated a flow in three dimensions, and Danker [**1**] developed a diffeomorphism in three (and higher) dimensions, both of which are Axiom A systems with cycles and have the property that any attempt to purge the cycle creates an Ω-explosion (i.e., uncountably many new nonwandering points are created). Evidently the perturbation described by Newhouse and Palis will not, in general, serve to approximate f satisfying Axiom A with cycles by Ω-stable g since, in general, it is not always possible that $\Omega(f) = \Omega(g)$. If the systems represented by the above examples are to be approximated by Ω-stable systems, then a new perturbation must be found.

The new perturbation must give rise to an Ω-stable explosion. That is, the perturbation will necessarily explode the given nonwandering set into a larger one in such a way that the resulting system is Ω-stable.

In another paper, Newhouse and Palis [**4**] studied Ω-stable explosions for Axiom A diffeomorphisms with cycles that occur on the boundary of the Morse-Smale systems. A characteristic generic feature of these systems is a nondegenerate tangency of (second order contact between) the unstable manifold of some periodic point with the stable manifold of some periodic point. A characteristic feature of the examples above, however, is complete coincidence of (infinite order contact between) such manifolds (or of components of such). In this paper we study explosions from

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certain cycles in two dimensions with coincident cycle connections and show that Ω-stable explosions exist for such systems.

First we establish some background.

Let M be a compact C^∞ Riemannian manifold and let $\mathrm{Diff}^r(M)$, $r \geq 1$, be the space of C^r diffeomorphisms of M with the C^r topology. If $f \in \mathrm{Diff}^r(M)$ and $x \in M$, denote by $O(x) = O(x, f)$ the union $\bigcup_{n\in\mathbf{Z}} f^n(x)$. We call $O(x)$ the orbit of x (under f).

Recurrence.

P: PERIODIC POINT SET. We call $p \in M$ periodic of period n if $f^n(p) = p$ and n is the least such integer. Denote by $P = P(f)$ the set of all periodic points of f.

L: LIMIT SET. Define the limit set $L = L(f)$ by $L = \{x \in M | \forall \text{nbhd}\, U \text{ of } x, \exists y \in U \text{ and a monotone sequence of integers } \{m_n\}, |m_n| \to \infty \ni f^{m_n}(y) \in U\}$.

Ω: NONWANDERING SET. Define the nonwandering set $\Omega = \Omega(f)$ by $\Omega = \{x \in M | \forall \text{nbhd}\, U \text{ of } x, \exists y \in U \text{ and an integer } n \neq 0 \ni f^n(y) \in U\}$.

It is clear that $P \subset L \subset \Omega$ and each is an invariant subset of M. L and Ω are each closed.

Ω-STABILITY. We say $f \in \mathrm{Diff}^r(M)$ is Ω-stable ($f \in \Omega S$) if there exists a neighborhood U of f in $\mathrm{Diff}^r(M)$ such that $g \in U$ implies the existence of a homeomorphism $h\colon \Omega(f) \to \Omega(g)$ such that

$$\begin{array}{ccc} \Omega(f) & \xrightarrow{f} & \Omega(f) \\ h\downarrow & & \downarrow h \\ \Omega(g) & \to_g & \Omega(g) \end{array}$$

commutes; i.e., $f|_{\Omega(f)}$ and $g|_{\Omega(g)}$ are topologically conjugate. The importance of the Ω-stable diffeomorphisms is that they form the largest class of diffeomorphisms whose orbit structures are well understood.

HYPERBOLICITY. A compact f-invariant set Λ is hyperbolic if there are a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$ preserved by the derivative Tf of f, a Riemannian metric $|\cdot|$ on M, and a constant $0 < \lambda < 1$ such that $|Tf(v)| \leq \lambda|v|$ for $v \in E^s$ and $|Tf(v)| \geq \lambda^{-1}|v|$ for $v \in E^u$.

AXIOM A. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is said to satisfy Axiom A ($f \in$AxA) if (a) Ω is hyperbolic, and (b) $\overline{P} = \Omega$.

STABLE AND UNSTABLE MANIFOLDS. Let Λ be a hyperbolic set with $|\cdot|$ as in the definition of hyperbolicity, and let d be the topological metric induced by $|\cdot|$. For $x \in \Lambda$ define the stable manifold of x, $W^s(x)$, and the unstable manifold of x, $W^u(x)$, by

$$W^s(x) = \{y \in M | d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x) = \{y \in M | d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

Let $W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x)$. Define $W^u(\Lambda)$ in a similar fashion.

SPECTRAL DECOMPOSITON THEOREM (SMALE [**7**]). *If $f \in$ AxA, then there is a unique decomposition of Ω into disjoint indecomposable subsets (called basic sets) $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ such that each Ω_i is a compact, invariant, hyperbolic subset of M on which f is topologically transitive.*

A more general theorem has been proved by Newhouse [**2**].

THEOREM. *If L is hyperbolic, then $L = \Lambda_1 \cup \cdots \cup \Lambda_n$ (uniquely) where the Λ_i are pairwise disjoint closed invariant hyperbolic topologically transitive sets with periodic points dense. Further, each Λ_i has a local product structure (see definition below).*

AXIOM A CYCLES. If $f \in$AxA and $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_n$ is the unique decomposition of $\Omega(f)$, then a sequence $\{\Omega_{i_0}, \ldots, \Omega_{i_k}\}$ of disjoint basic sets is an Axiom A cycle provided

$$\hat{W}^u(\Omega_{i_j}) \cap \hat{W}^s(\Omega_{i_{j+1}}) \neq \varnothing, \qquad j = 0, \ldots, k-1,$$

and

$$\hat{W}^u(\Omega_{i_k}) \cap \hat{W}^s(\Omega_{i_0}) \neq \varnothing,$$

where

$$\hat{W}^\sigma(\Lambda) = W^\sigma(\Lambda) \setminus \Lambda \quad \text{for } \sigma = u \text{ or } s.$$

L-CYCLES. If $L(f)$ is hyperbolic, define L-cycles by analogy to Axiom A cycles where Ω_{i_k} is replaced by Λ_{i_k}.

It is important to note that, for both types of cycles, not all connections between stable and unstable manifolds can be transverse, as this would imply that each Λ_{i_k} is a subset of the same basic set, which contradicts the uniqueness of the spectral decomposition.

Ω-STABILITY THEOREM (SMALE [**8**]). *If $f \in$ AxA and has no Axiom A cycles, then $f \in \Omega S$.*

A more general version is due to Newhouse [**2**].

THEOREM. *If $L(f)$ is hyperbolic and has no L-cycles, then $f \in \Omega S$.*

LOCAL STABLE AND UNSTABLE MANIFOLDS. If Λ is a hyperbolic set and $x \in \Lambda$, define for $\varepsilon > 0$:

$$W^s_\varepsilon(x) = \{y \in M | d(f^n(x), f^n(y)) \leq \varepsilon \text{ for } n \geq 0\}$$

and

$$W^u_\varepsilon(x) = \{y \in M | d(f^n(x), f^n(y)) \leq \varepsilon \text{ for } n \leq 0\}.$$

These are called the local stable and unstable manifolds of x respectively.

LOCAL PRODUCT STRUCTURE. If there is an $\varepsilon > 0$ so that $W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \subset \Lambda$ for all $x, y \in \Lambda$, then Λ is said to have a local product structure.

ADAPTED NEIGHBORHOODS. If Λ is a hyperbolic set with a local product structure, then there exists a neighborhood U of Λ so that $\bigcap_{n \in \mathbf{Z}} f^n(U) = \Lambda$. Such a neighborhood is said to be adapted to Λ with respect to f. Furthermore if g is C^1 near f, then $\Lambda_g = \bigcap_{n \in \mathbf{Z}} g^n(U)$ is also hyperbolic with a local product structure and Λ_g is homeomorphic to Λ.

SECTOR FIELDS. If $F = E^1 \oplus E^2$ is a direct sum decomposition of a vector space with norm $|\cdot|$ and $\varepsilon > 0$ is a positive number, let $S_\varepsilon(E^1) = S_\varepsilon(E^1, E^2) = \{(v_1, v_2) | \, |v_2| \leq \varepsilon |v_1|$ and (v_1, v_2) is the decomposition of $v \in F$ under $F = E^1 \oplus E^2\}$. If $U \subset M$, $T_U M = E^1 \oplus E^2$ is a direct sum decomposition and $\varepsilon \colon U \to [\delta, 1/\delta] \subset \mathbf{R}$ for some small positive δ, then the mapping $x \in U \mapsto S_{\varepsilon(x)} E^2_x$ is a sector field on U.

ALMOST HYPERBOLIC STRUCTURE. If Λ is any compact f-invariant set and V is a compact neighborhood of Λ so that $\Lambda = \bigcap_{n\in\mathbf{Z}} f^n(V)$, we say V has an almost hyperbolic structure (E^1, E^2, ε) or $(E^1, E^2, S_\varepsilon E^2)$ if

1. $T_V M = E^1 \oplus E^2$ is a not necessarily continuous splitting.
2. $\varepsilon\colon V \to \mathbf{R}_+$ is a bounded (and bounded away from zero) function.
3. There is a constant $\lambda > 1$ such that

$$\begin{aligned}
&\text{(a)}\quad T_x f(S_{\varepsilon(x)}E^2_x) \subset S_{\varepsilon(f(x))}E^2_{f(x)},\\
&\qquad\; m(T_x f|S_{\varepsilon(x)}E^2_x) \geq \lambda, \qquad x \in V \cap f^{-1}(V),\\
&\text{(b)}\quad T_x f^{-1}(T_x M - S_{\varepsilon(x)}E^2_x) \subset T_{f^{-1}(x)}M - S_{\varepsilon(f^{-1}(x))}E^2_{f^{-1}(x)},\\
&\qquad\; m(T_x f^{-1}|T_x M - S_{\varepsilon(x)}E^2_x) \geq \lambda, \qquad x \in V \cap f(V),
\end{aligned}$$

where if $A\colon F \to F$ is a linear map on a vector space, $m(A) = \inf_{|v|=1} |Av|$ is the minimum norm of A.

A compact invariant set Λ is hyperbolic with a local product structure iff it has a neighborhood V endowed with an almost hyperbolic structure. In fact if Λ is hyperbolic with a local product structure, then the splitting and function $\varepsilon\colon V \to \mathbf{R}_+$ may be taken to be continuous.

We shall be concerned with L-cycles for the following reason. Every Axiom A cycle must have at least two nontransverse cycle connections. For if all connections are transverse but one, then the nontransverse connection is a nonwandering point which is not hyperbolic, contradicting Axiom A. Exploding an Axiom A cycle would therefore require making at least two simultaneous perturbations. While this can be done, it only complicates the picture. On the other hand, L-cyles may have only one nontransverse cycle connection, and these simpler cycles capture entirely the essence of the systems we wish to study. Furthermore, the Pugh, Walker and Wilson example and Dankner's example may be perturbed into systems that, although not Axiom A, do have hyperbolic limit sets and have L-cycles with only one nontransverse connection. Accordingly we make the following

CONJECTURE. If $f \in$ Axiom A and has a cycle, then there exists g C^r near f so that $L(g) = \Omega(f)$ and g has an L-cycle with only one nontransverse connection.

2. Statement of the theorem. We describe the system which will be the focus of this study. Let $f \in \mathrm{Diff}^r(M^2)$, $r \geq 2$, satisfy

(1) $L(f)$ is finite and hyperbolic, and $L(f) = S_1 \cup P \cup S_2$ where S_1 and S_2 consist of the sinks and sources of f respectively and $P = \{p_1, \ldots, p_l\}$ where each p_i is a fixed point of f. $\mathrm{Spec}(T_{p_i} f) = \{\mu_i, \lambda_i\}$ with $0 < \mu_i < 1 < \lambda_i$.

(2) P contains a cycle.

(3) One component of $W^u(p_l) \setminus \{p_l\}$ coincides with one component of $W^s(p_1) \setminus \{p_1\}$. We call this embedded open 1-disk Θ.

(4) All other cycle connections are transverse.

(5) For $i = 1, \ldots, l$, p_i has a C^2 linear neighborhood. That is, there exists a C^2 chart (ϕ_i, U_i) so that

$$\phi_i \circ f \circ \phi_i^{-1} = \begin{vmatrix} \mu_i & 0 \\ 0 & \lambda_i \end{vmatrix} \quad \text{and} \quad \phi_i(p_i) = \overline{0}.$$

THEOREM 2.1. *Let f be as above. Then there exists $g \in \mathrm{Diff}^r(M)$, arbitrarily C^r near f, so that g is structurally stable and $L(g)$ is infinite.*

REMARK 2.2. (a) We note here that, as stated, this is not a new theorem. Indeed θ may be broken as in Figure 2.3. This does not change L. Now a nondegenerate tangency may be created as in Figure 2.4. When the hook is pushed through $W^s(P_1)$ an L-explosion takes place. This is precisely the situation considered by Newhouse and Palis in [**4**], where it is shown that the hook may be pushed in such a way that the resulting system is structurally stable. We shall prove the theorem, however, using a new type of perturbation which may be represented by Figure 2.5. We consider this type of perturbation because the scheme represented in Figures 2.3 and 2.4 may not be carried out (without radically changing L) for certain systems in higher dimensions (the modified Dankner system for example). This perturbationn in two dimensions is thus motivated by the hope that it may be generalized to higher dimensions. We remark, further, that this perturbation creates structurally stable systems which are topologically distinct from those created by the Newhouse and Palis perturbation.

(b) While we are primarily concerned with Ω-stable explosions, the conclusion of Theorem 2.1 refers to the stronger condition of structural stability. This is merely

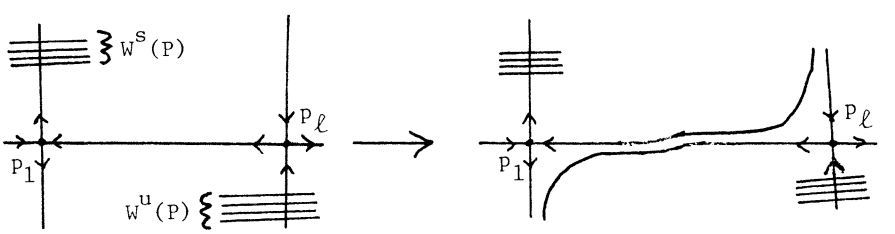

FIGURE 2.3

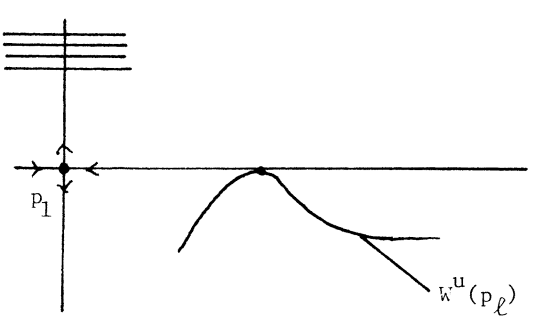

FIGURE 2.4

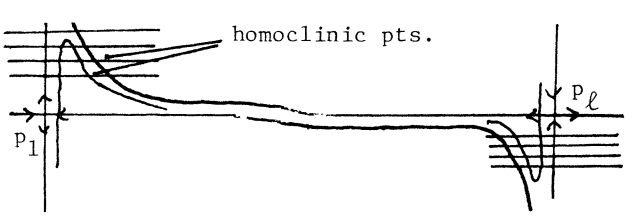

FIGURE 2.5

the result of the strong hypotheses (in particular $L(f)$ finite) which may not be present in more general systems where this perturbation may be applicable.

(c) Since $L(f) = S_1 \cup P \cup S_2$ is hyperbolic, there is a filtration $\emptyset = M_0 \subset M_1 \subset M_2 \subset M_3 = M$ for f so that

$$\bigcap_{n\in\mathbf{Z}} f^n(\overline{M_1 \setminus M_0}) = S_1, \quad \bigcap_{n\in\mathbf{Z}} f^n(\overline{M_2 \setminus M_2}) = W^u(P) \cap W^s(P),$$

and

$$\bigcap_{n\in\mathbf{Z}} f^n(\overline{M_3 \setminus M_2}) = S_2.$$

Let

$$U = \overline{M_2 \setminus M_1}, \quad \Lambda = \bigcap_{n\in\mathbf{Z}} f^n(U), \quad \text{and} \quad \Lambda_1 = W^s(P) \pitchfork W^u(P).$$

So we have that $\Lambda = \Lambda_1 \cap \Theta$ is an isolated invariant set for f with isolating neighborhood U. Furthermore, $L(f)$ is hyperbolic outside U, so all of the pathology of f is isolated in Λ in the sense that if Λ were hyperbolic, then f would be structurally stable. We note further that within Λ the nonhyperbolicity is all found in Θ since $\Lambda \setminus \Theta = \Lambda_1 = W^u(P) \pitchfork W^s(P)$ is a hyperbolic invariant set.

These are the diffeomorphism considered in Theorem 2.1. We will establish that theorem by showing that such an f may be C^r approximated by $g \in \mathrm{Diff}^r(M)$ via the type of perturbation considered in Remark 2.2(a) in such a way that

$$\Lambda(g) \equiv \bigcap_{n\in\mathbf{Z}} g^n(U)$$

is a hyperbolic basic set. This result has some interest in itself apart from Ω-stability considerations.

The idea is to exploit the hyperbolic structure of Λ_1 and extend this structure to all of $\Lambda(g)$ for carefully chosen g.

Now Λ_1 is a hyperbolic set, so Λ_1 has an adapted neighborhood V_1 complete with an almost hyperbolic structure $(E^1_{V_1}, E^2_{V_1}, \varepsilon)$. Furthermore there exists a compact set V_2 so that

(1) $\Theta \subset O(V_2)$,

(2) $\bigcap_{n\in\mathbf{Z}} f^n(V_2) = \emptyset$, and

(3) $\bigcap_{n\in\mathbf{Z}} f^n(V_1 \cup V_2) = \bigcap_{n\in\mathbf{Z}} f^n(U) = \Lambda$.

The set V_2 may be chosen to be a compact neighborhood of any fundamental domain of Θ and we may take $U = V_1 \cup V_2$.

The theorem will be proved by showing that appropriate g may be found so that $\Lambda_g = \bigcap_{n\in\mathbf{Z}} g^n(U)$ is an infinite, zero-dimensional, hyperbolic basic set of $L(g)$. To this end we will make use of the following lemma of Newhouse and Palis [**4**].

(2.6) LEMMA NP. *Let $g \in \mathrm{Diff}^r(M)$ with $\Lambda_1 \subset M$ compact and g-invariant. Let V_1 be an adapted neighborhood of Λ_1 with respect to g. Let $V_2 \subset M$ be compact and satisfy $g^2(V_2) \cap V_2 = \emptyset = g^{-2}(V_2) \cap V_2$.*

If there exists a compact subset $V_2' \subset V_2$ so that the following hold:

(A) $V_2 \cap [\bigcap_n g^n(V_1 \cup V_2)] \subset V_2'$.

(B) *The splitting on V_1 may be extended to a (not necessarily continuous) splitting $T_{V_1\cup V_2'}M = E^2_{V_1\cup V_2'} \oplus E^2_{V_1\cup V_2'}$ on $V_1 \cup V_2'$.*

(C) *There exists* $\lambda > 1$, $Q \in \mathbf{N}$ *and* $\varepsilon\colon V_2' \to \mathbf{R}_+$ *so that*

(1) *For each* $x \in V_2'$ *there exists* $k = k(x)$ *so that* $0 < k(x) < Q$ *and*

(a) $T_x g^k(S_{\varepsilon(x)}E_x^2) \subset S_{\varepsilon(g^k(x))}E^2_{g^k(x)}$,

(b) $m(T_x g^k | S_{\varepsilon(x)}E_x^2) \geq \lambda$, *and*

(c) $g^k(x) \in g^{-1}(V_1) \cap V_1 \cap g(V_1)$.

(2) *For each* $x \in V_2'$ *there exists* $j = j(x)$ *so that* $-Q < j(x) < 0$ *and*

(a) $T_x g^j(T_x M \setminus S_{\varepsilon(x)}E_x^2) \subset T_{g^j(x)}M \setminus S_{\varepsilon(g^j(x))}E^2_{g^j(x)}$,

(b) $m(T_x g^j | T_x M \setminus S_{\varepsilon(x)}E_x^2) \geq \lambda$, *and*

(c) $g^j(x) \in g^{-1}(V_1) \cap V_1 \cap g(V_1)$.

Then $\bigcap_{n\in\mathbf{Z}} g^n(V_1 \cup V_2)$ *is hyperbolic.*

We will produce g arbitrarily C^r near f; V_2'; $E^1_{V_2'}, E^2_{V_2'}$ and $S_\varepsilon E^2$ on V_2', so these satisfy the conditions of Lemma NP, and we conclude that Λ_g is hyperbolic. First we pause to establish some machinery.

3. Two key lemmas. In this section we define the coordinate neighborhoods W_i of the fixed points p_i and establish two key technical lemmas regarding their properties.

For $i \in \{1, \ldots, l\}$ let W_i be a compact neighborhood of p_i such that

$$\phi_i\colon W_i \overset{C^2}{\twoheadrightarrow} B_1 \times B_1, \quad \text{where } B_1 = [-1,1] \subset \mathbf{R},$$

and if $x \in W_i \cap f^{-1}(W_i)$, then

$$\phi_i^{-1} \circ \begin{bmatrix} \mu_i & 0 \\ 0 & \lambda_i \end{bmatrix} \circ \phi_i(x) = f(x); \quad \phi_i(W_i \cap W^s(p_i)) = (B_1 \times \{0\}),$$
$$\phi_i(W_i \cap W^u(p_i)) = (\{0\} \times B_1), \quad \phi_i(p_i) = (0,0).$$

Define a splitting $T_{W_i}M = \hat{E}^1_{W_i} \oplus \hat{E}^2_{W_i}$ on W_i to be the coordinate splitting

$$\hat{E}^1_x = T_x(\phi_i^{-1}(B_1 \times \{\pi_2 \circ \phi_i(x)\})), \qquad x \in W_i,$$
$$\hat{E}^2_x = T_x(\phi_i^{-1}(\{\pi_1 \circ \phi_i(x)\} \times B_1)), \qquad x \in W_i,$$

where $\pi_2\colon B_1 \times B_1 \to B_1$ by $\pi_2(a,b) = b$ and $\pi_1\colon B_1 \times B_1 \to B_1$ by $\pi_1(a,b) = a$. If $v_x \in T_xM$, define the u-slope of v_x as $\rho_u(v_x) = |\hat{v}_2|/|\hat{v}_1|$, where $v_x = \hat{v}_1 \oplus \hat{v}_2$ is the decomposition under $\hat{E}^1_x \oplus \hat{E}^2_x$. Define s-slope as $\rho_s(v_x) = |\hat{v}_1|/|\hat{v}_2|$. If Σ is an embedded 1-disk define $\rho_u(\Sigma) = \sup\{\rho_u(V) | V \in T_\Sigma M\}$. Define $\rho_s(\Sigma)$ by the obvious analogy.

LEMMA 3.0. *There exists* $c_1 \in \mathbf{R}$ *with* $\frac{1}{2} > c_1 > 0$ *and the* W_i*'s may be chosen so small that*

(1) $W_i \subset V_1$, $i = 1, \ldots, l$.

(2) $S_{c_1}\hat{E}^2_x \subset S_{\varepsilon(x)}E^2_x$ $\forall x \in W_i$, $i = 1, \ldots, l$.

(3) *If* Σ *is a component of* $W^s(P) \cap W_i$, *then* $\rho_u(\Sigma) < c_1/2$ *and* $\phi_i^{-1} \circ \pi_1 \circ \phi_1 : \Sigma \to W^s(p_i) \cap W_i$ *is an embedding.*

(4) *If* $x \in W_i \cap f^{-n}(W_j)$, $i \neq j$, $n \in \mathbf{N}$, *then*

(a) $T_x f^n(S_{c_1}\hat{E}^2_x) \subsetneq S_{c_1}\hat{E}^2_{f^n(x)}$ *and*

(b) $m(T_x f^n | S_{c_1}\hat{E}^2_x) > 1$.

(3′) *If* Σ *is a component of* $W^u(P) \cap W_i$, *then* $\rho_s(\Sigma) < c_1/2$ *and* $\phi_i^{-1} \circ \pi_2 \circ \phi_i : \Sigma \to W^u(p_i) \cap W_i$ *is an embedding.*

(4′) *If* $x \in W_i \cap f^n(W_j)$, $i \neq j$, $n \in \mathbf{N}$, *then*
(a) $T_x f^{-n}(S_{c_1}\hat{E}^1_x) \subsetneq S_{c_1}\hat{E}^1_{f^{-n}(x)}$ *and*
(b) $m(T_x f^{-n}|S_{c_1}\hat{E}^1_x) > 1$.

Hereafter we shall write π_s for $\phi_i^{-1} \circ \pi_1 \circ \phi_i$ and π_u for $\phi_i^{-1} \circ \pi_2 \circ \phi_i$ without reference to i.

PROOF OF LEMMA 3.0. (1) This is obvious and remains true under any further shrinkage of any W_i.

(2) Recall that $\Lambda_1 = p_1 \cup p_l \cup [\bigcup_{i=1}^{l-1} W^i(p_i) \cap \bigcup_{i=2}^{l} W^s(p_i)]$ is a hyperbolic set with local product structure and that V_1 is an adapted neighborhood of Λ_1. So $T_{V_1}M$ has a continuous splitting $T_xM = E^1_x \oplus E^2_x$, $x \in V_1$, and $\varepsilon\colon V_1 \to \mathbf{R}_+$ is continuous. For $i \in \{1, \dots, l\}$ with $W_i \subset V_1$,

$$\hat{E}^2_{p_i} = T_{p_i}W^u(p_i) \subset S_{\varepsilon(p_i)}E^2_{p_i}.$$

By continuity of the splitting and ε, $\exists t > 0$ so that

$$x \in B_t(p_i) \Rightarrow \hat{E}^2_x \subset S_{\varepsilon(x)}E^2_x.$$

($B_t(p_i)$ is the closed ball of radius t about p_i.) For each $x \in B_t(p_i)$ let

$$C_x = \sup\{c \in \mathbf{R} | S_c\hat{E}^2_x \subset S_{\varepsilon(x)}E^2_x\},$$
$$C_x > 0 \quad \forall x \in B_t(p_i).$$

Let $c_1^i = \inf_{x \in B_t(p_i)} C_x$. Then $c_1^i > 0$ and if $c \in \mathbf{R}$ is such that $c_1^i > c > 0$, then

$$S_c\hat{E}^2_x \subset S_{\varepsilon(x)}E^2_x \quad \forall x \in B_t(p_i).$$

Now redefine W_i so that $W_i \subset B_t(p_i)$. Once this is done for each $i \in \{1, \dots, l\}$ let

$$c_1 = \min\{\tfrac{1}{3}, c_1^1, c_1^2, \dots, c_1^l\}.$$

Now (2) is satisfied and will remain satisfied under any further shrinkage of any W_i.

(3) & (3′). This is a result of the λ-lemma and possible further shrinking of W_i's. Note that once (3) and (3′) are satisfied any further shrinking of any W_i will keep (3) and (3′) satisfied.

(4)(a) With W_i's chosen for (1)–(3′), define $W_i^k = \bigcap_{n=-k}^{k} f^n(W_i)$ and for convenience define $\phi_i^k\colon W_i^k \twoheadrightarrow B_1 \times B_1$ (onto) by

$$\phi_i^k = \begin{bmatrix} \mu_i^{-k} & 0 \\ 0 & \lambda_i^k \end{bmatrix} \circ \phi_i|_{W_i^k},$$

so ϕ_i^k maps W_i^k onto $B_1 \times B_1$ and

$$f|_{W_i^k \cap f^{-1}(W_i^k)} = (\phi_i^k)^{-1} \circ \begin{bmatrix} \mu_i & 0 \\ 0 & \lambda_i \end{bmatrix} \circ \phi_i^k.$$

Now for $k \in \mathbf{N}$, suppose $x \in W_i^k \cap f^{-n}(W_j^k)$ and $v_x \in S_{c_1}\hat{E}^2_x$. Since the W_i's satisfy parts (1) and (2) of the lemma and since $(E^1_{V_1}, E^2_{V_2}, \varepsilon)$ is an almost hyperbolic structure on V_1 (in particular $f(S_{(x)}E^2_x) \subset S_{\varepsilon(f(x))}E^2_{f(x)}$), $T_x f^{n-k} v_x \in S_{\varepsilon(y)}E^2_y$, where $y \in \bigcap_{\nu=-k}^{0} f^\nu(W_j)$, so $T_x f^{n-k} v_x$ will spend the next k iterates in $T_{W_j}M$.

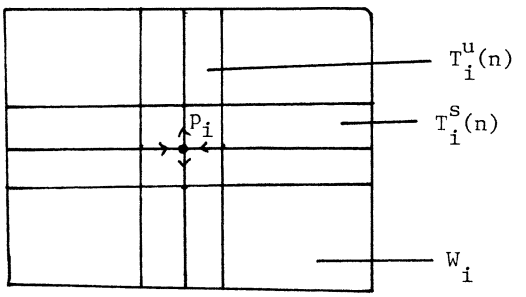

FIGURE 3.1

Each iterate will increase the u-slope of $T_x f^{n-k} v_x$ by a factor of λ_j/μ_j. Let $\mu = \inf\{\varepsilon(x) | x \in \bigcup_{i=1}^l W_i\}$. Since $\bigcup_{i=1}^l W_i$ is compact, $\mu > 0$. Thus

$$\rho_u(T_x f^n(v_x)) \geq (\lambda_i/\mu_i)^k \mu,$$

so if $\nu = \min_{i=1,\ldots,l}\{\lambda_i/\mu_i\}$, then

$$\rho_u(T_x f^n(v_x)) \geq \nu^k \mu \quad \text{or} \quad \rho_s(T_x f^n(v_x)) \leq 1/\nu^k \mu.$$

Now let $K \in \mathbf{N}$ be such that $1/\nu^K \mu < c_1$. So part (4)(a) of the lemma is satisfied for W_i^K's. Expansion (part (4)(b)) follows directly from the expansion property of the a.h.s. $(E_{V_1}^1, E_{V_1}^2, \varepsilon)$. Similar arguments establish (4′)(a) and (4′)(b). So by replacing W_i by W_i^K, Lemma 3.0 is proved.

Hereafter we assume that c_1 and the W_i's are chosen so as to satisfy Lemma 3.0. Furthermore we shall identify W_i with $\phi_i(W_i)$. Whenever coordinates are used to describe an object in W_i, it is assumed that the reader understands that the coordinates arise from ϕ_i. The careful reader may wish to insert ϕ_i's and ϕ_i^{-1}'s where appropriate. For $i \in \{1, \ldots, l\}$ and $n \in \mathbf{N}$ define

$$T_i^s(n) = \bigcap_{k=0}^{n} f^{-k}(W_i) = B_1 \times [-\lambda_i^{-n}, \lambda_i^{-n}]$$

and

$$T_i^u(n) = \bigcap_{k=0}^{n} f^k(w_i) = [-\mu_i^n, \mu_i^n] \times B_1.$$

Observe that if $x \in T_i^s(n)$, then $O(x)$ spends at least the next n iterates in W_i and if $x \in T_i^u(n)$, then in backward time $O(x)$ spends at least the next n interates in W_i. See Figure 3.1.

Define for each $i \in \{1, \ldots, l\}$

$$\hat{F}_i^\sigma = W_i \setminus T_i^\sigma(1), \qquad \sigma = u \text{ or } s.$$

Define F_1^u to be that component of $\hat{F}_1^u$ which meets $W^s(P)$. Define F_l^s to be that component of $\hat{F}_l^s$ which meets $W^i(P)$.

REMARK 3.2. Only one component of $\hat{F}_1^u$ can meet $W^s(P)$ if $L(f)$ is finite.

Let W_1^+ be the closure of that component of $W_1 \setminus W^s(p_1)$ which contains F_1^u, and let W_l^+ be the closure of that component of $W_l \setminus W^u(p_l)$ which contains F_l^s.

REMARK 3.3. If $x \in \operatorname{int}(W_1 \setminus W_1^+)$, then $O_+(x)$ is eventually contained in M_1, and if $x \in \operatorname{int}(W_l \setminus W_l^+)$, then $O(x)$ is eventually contained in $M_3 \setminus M_2$.

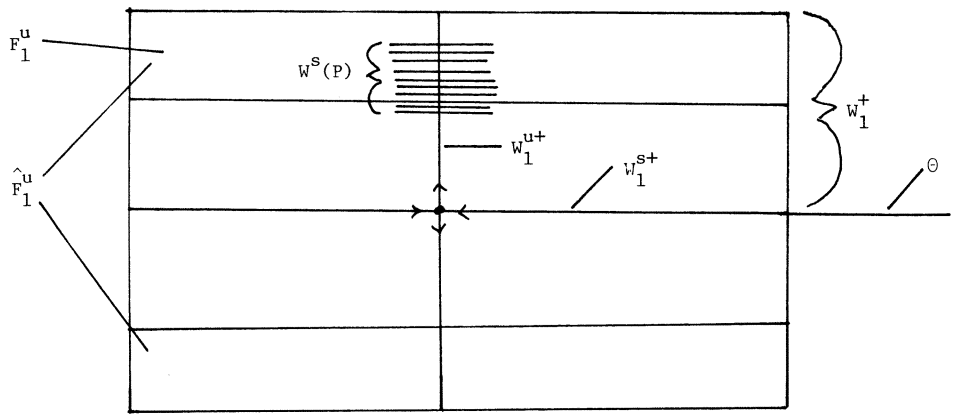

FIGURE 3.4

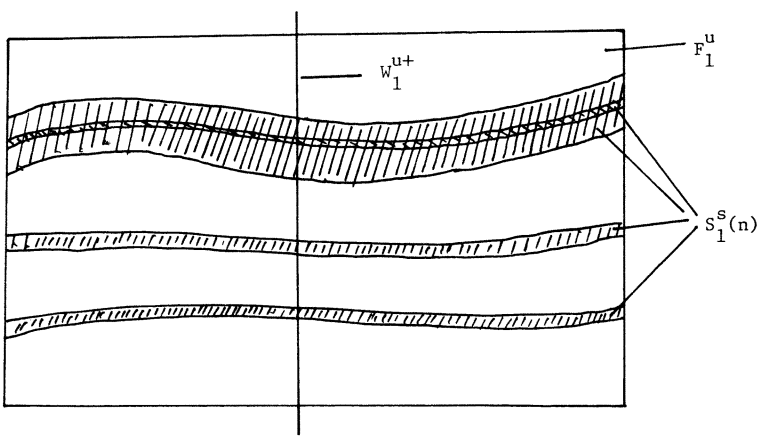

FIGURE 3.5

Let $W_i^\sigma = W_i \cap W^\sigma(p_i)$, $\sigma = u$ or s, and let $W_l^{s+} = W_l^s \cap W_l^+$, $W_l^{u+} = W_l^u \cap \Theta$, $W_1^{s+} = W_1^s \cap \Theta$ and $W_1^{u+} = W_1^u \cap W_1^+$. See Figure 3.4.

For $n \in \mathbf{N}$, define

$$S_1^s(n) = F_1^u \cap O\left(\bigcup_{k=2}^{l} T_k^s(n)\right)$$
$$= \left\{x \in F_1^u | \exists j \in \{2, \ldots, l\} \text{ and } n_0 \in \mathbf{N} \text{ so that } \bigcup_{k=n_0}^{n_0+n} f^k(x) \subset W_j\right\}.$$

Let $S_l^u(n) = F_l^s \cap O_+(\bigcup_{k=1}^{l-1} T_k^u(n))$. See Figure 3.5.

LEMMA 3.6. *There exist sets $\hat{S}_1^s(n)$, $\hat{S}^u(n)$, a polynomial η of degree at most $l-2$, and real numbers $K_1 > 0$ and $\lambda > 1$ so that for large n,*

(1) $S_1^s(n) \cap T_1^u(n) \subset \hat{S}_1^s(n)$.

(1′) $S^u(n) \cap T_l^s(n) \subset \hat{S}_l^u(n)$.

(2) $\hat{S}_1^s(n) = \bigcup_{k=1}^{m_1(n)} (\pi_u^{-1}\hat{I}_k) \cap T_1^u(n)$ *where* $m_1(n) < \eta(n)$ *and each* $\hat{I}_k$ *is a closed interval with* $l(\hat{I}_k) < K_1\lambda^{-n}$.

(2′) $\hat{S}_l^u(n) = \bigcup_{k=1}^{m_l(n)} (\pi_s^{-1}\hat{I}'_k) \cap T_l^s(n)$, *where* $m_1(n) < \eta(n)$ *and each* $\hat{I}'_k$ *is a closed interval with* $l(\hat{I}'_k) < K_1\lambda^{-n}$.

PROOF OF LEMMA 3.6. For $i = \{1, \ldots, l-1\}$, let $A_i = W_i^u \cap F_1^u$, so A_i is a fundamental domain of $W^u(p_i)$. Define $\mathrm{sl}(i)$ (for string length) to be the maximal length of a sequence $\{p_{j_1}, \ldots, p_{j_k}\} \subset P$ such that

(1) $p_{j_1} = p_i$ and $p_{j_k} = p_l$ and

(2) $\mathrm{card}(A_{j_k} \cap W^s(p_{j_{k+1}}))$ is finite and nonzero.

Define $\mathrm{sl}(l) = 0$. Now sl imposes a partial ordering on $\{1, \ldots, l\}$ by $i <_1 j$ iff $\mathrm{sl}(i) < \mathrm{sl}(j)$. If $W_i^u \pitchfork W^s(p_j) \neq \emptyset$, then $j <_1 i$. Let $B_i(n) = \bigcup_{j \leq_1 i} O_-(T_j^s(n))$. Define $\eta_i(n)$ to be the number of components of $A_i \cap B_i(n)$. Let $N_i = \eta_i(0)$. We have the following

CLAIM 3.7. $\eta_i(n)$ is a polynomial of degree $\mathrm{sl}(i) - 1$.

PROOF. We use induction on $\mathrm{sl}(i)$. Let j be such that $\mathrm{sl}(j) = 1$, so $\eta_j(n) = \mathrm{card}(A_j \cap W^s(p_l))$, which is finite and constant for all n. Thus η_j is a polynomial of degree $0 = \mathrm{sl}(j) - 1$. Now suppose for some k that $\eta_i(n)$ is a polynomial of degree $\mathrm{sl}(i) - 1$ for all i such that $\mathrm{sl}(i) < k$. Let j satisfy $\mathrm{sl}(j) = k$. We wish to show that $\eta_j(n)$ is a polynomial of degree $k - 1$. Consider $A_j \cap B_j(0)$. This set consists of N_j components. Let c be any such component. For some $m > 0$, $f^m(c)$ is a nearly vertical 1-disk in some W_i with $\mathrm{sl}(i) < \mathrm{sl}(j) = k$. Of course, $\mathrm{card}(f^m(c) \cap B_j(0)) = 1$. See Figure 3.8.

Now for arbitrary $n > 0$, $f^m(c)$ is still a nearly vertical 1-disk in W_i, but $\mathrm{card}(f^n(c) \cap B_j(n))$ has increased. We shall count the number of components of $f^m(c) \cap B_j(n)$. $f^m(c)$ still intersects $T_i^s(n)$ (instead of $T_i^s(0)$), but the shrinking of $T_i^s(0)$ to $T_i^s(n)$ has exposed n fundamental domains of W_i^u, each of which carry $\eta_i(n)$ nearly horizontal components of $B_i(n)$. Thus the number of components of $f^m(c) \cap B_j(n)$ is $n \cdot \eta_i(n) + 1$, which, by induction hypothesis, is a polynomial of degree $1 + \mathrm{sl}(i) - 1$. Since there are but finitely many such components c, $\eta_j(n)$ is a polynomial of degree $\max_{i <_1 j}\{\mathrm{sl}(i)\} = k - 1$. So the claim is established. We may conclude that the number of components of $S_1^s(n)$ is a polynomial of degree $\leq l - 2$. A similar argument shows that the number of components of $S_l^u(n)$ is also bounded by a polynomial of degree at most $l - 2$. Let $m_1(n)$ and $m_l(n)$ be the number of components of $S_1^s(n)$ and $S_l^u(n)$ respectively. Let η be a polynomial of

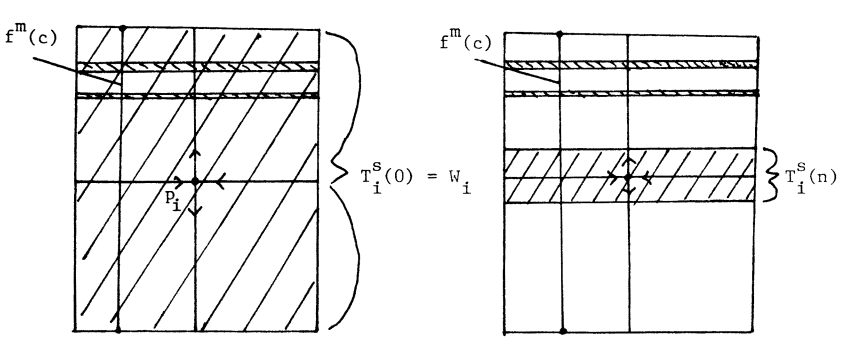

FIGURE 3.8

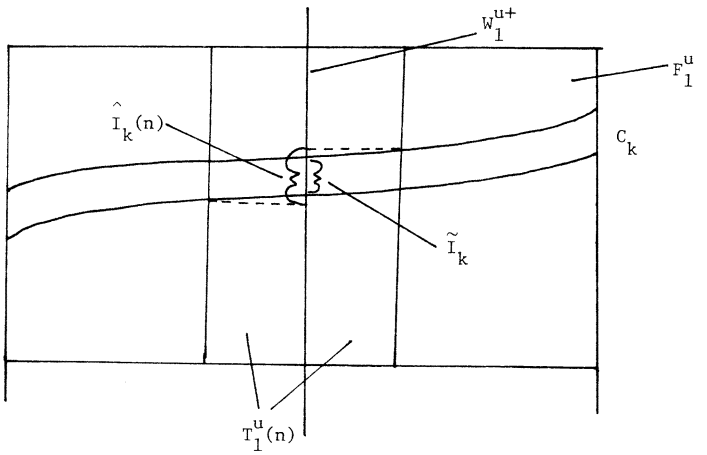

FIGURE 3.9

degree $l-2$, so that $m_1(n) < \eta(n)$ and $m_l(n) < \eta(n)$. For $n \in \mathbf{N}$ let $\{C_k\}_{k=1}^{m_1(n)}$ be the collection of components of $S_1^s(n)$. Let $\tilde{L}_k = C_k \cap W_1^u$ and $\hat{I}_k = \pi_u C_k$. So $\tilde{I}_k \subset \hat{I}_k \subset (W_1^u \cap F_1^u)$. (See Figure 3.9.) For each C_k there is some $e(k)$ so that $f^{e(k)}\tilde{I}_k$ is a nearly vertical 1-disk in some $T_i^s(n)$ and, by Lemma 3.0(4), $l(\tilde{I}_k) < l(f^{e(k)}C_k)$. But $l(f^{e(k)}C_k) < K(\lambda_k^{-1})^n$, so $l(\tilde{I}_k) < K(\lambda_i^{-1})^n$. Each C_k contains one component Σ_k of $W^s(p_i)$ ($C_k \subset F_1^u \cap f^{-e(k)}T_i^s(n)$). For large n, the horizontal boundaries of C_k approximate Σ_k so, by Lemma 3.0(3), the u-slope of these boundaries is less than c_1. We conclude that $l(\hat{I}_k) < 2c_1\mu_1^n + K\lambda_i^{-n}$. Let $\lambda = \min\{\mu_1^{-1}, \lambda_1, \ldots, \lambda_l\}$. Then for some K_1,

$$l(I_k) < K_1\lambda^{-n} \quad \text{for all } k \in \{1, \ldots, m(n)\}.$$

Define $\hat{S}_1^s(n) = [\bigcup_{k=1}^{m_1(n)} \pi_u^{-1}\hat{I}_k] \cap T_1^u(n)$. Since $\hat{S}_1^s(n) = \pi_u^{-1}(\pi_u(S_1^s(n) \cap T_1^u(n))) \cap T_1^u(n)$, $S_1^s(n) \cap T_1^u(n) \subset S_1^s(n)$. A similar construction produces $\{\hat{I}_k'\}$, $S_l^u(n)$ and the appropriate estimates, so the lemma is proved.

4. More lemmas. Recall that it is our goal to produce a map g arbitrarily C^r near f, a set $V_2' \subset V_2$, and a splitting and sector field on V_2' so that the conditions of Lemma NP are satisfied. We first describe a one-parameter family of maps g_b.

For $b > 0$ small let g_b be a map which agrees with f outside a small compact neighborhood N of a fundamental domain D of θ between W_l and W_1 and satisfies $g_b^n(z) = L_b \circ f^n(z)$ for $z \cap \overline{f}^n(V_2)$, where, in the coordinates of V_2, $L_b(x,y) = (x, y + b/x^p)$, where $\mu_1^p = \lambda_1^{-1}$. Thus in V_2 (and by iteration in W_1), $W^u(p_l)$ is a curve of the form $y = b/x^p$. Clearly $g_b \to f$ as $b \to 0$ in the C^r topology. The g which we seek will be g_b for some suitable, small b. See Figure 4.0. We note, for small b, that $g = g_b$, Λ_1, V_1 and V_2 are as in Lemma NP (i.e. Λ_1 is g_b-invariant, etc.). For positive integers n_1, n_2 and n_3 and real $b > 0$ let $A_1(n_1, b)$ be the first pass of $g_b^{n_1}(W_l)$ through W_1. Precisely, this is

$$A_1(n_1, b) = \Big\{ x \in g_b^{n_1}(W_l) \cap W_1 \,|\, \text{if } k \text{ is the least positive integer so that } g_b^{-k}(x) \notin W_1, \text{ then } \bigcup_{j=k}^{n_1} g_b^{-j}(x) \cap W_1 = \varnothing \Big\}.$$

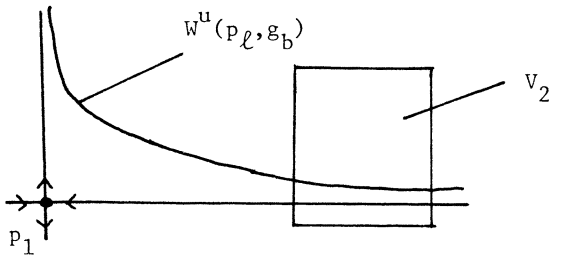

FIGURE 4.0

Let us denote the first pass of $g_b^{n_1}(W_l)$ by $[g_b^{n_1}(W_l)]^*$, so $A_1(n_1, b) = [g_b^{n_1}(W_l)]^*$.

Using the same notation, let

$$A_2(n_1, n_2, b) = \left[\bigcup_{j=0}^{n_1-1} g_b^j(S_l^u(n_2))\right]^*$$

and

$$\hat{A}_2(n_1, n_2, b) = \left[\bigcup_{j=0}^{n_1-1} g_b^j(\hat{S}_l^u(n_2))\right]^*.$$

If b is sufficiently small compared to n_2, then $A_2(n_1, n_2, b) \subset \hat{A}_2(n_1, n_2, b)$.

Let $B_1(n_3) = T_1^s(n_3)$. Let $B_2(n_3, n_2)$ be the first pass of $\bigcup_{j=0}^{n_3-1} g_b^{-j}(S_1^s(n_2))$ through W_1. Let $\hat{B}_2(n_3, n_2)$ be defined as B_2 but with $\hat{S}_j^s(n_2)$ replacing $S_j^s(n_2)$. Now define V_2' by

$$V_2'(n_1, n_2, n_3, b) = (A_1(n_1, b) \cup A_2(n_1, n_2, b)) \cap (B_1(n_1) \cup B_2(n_3, n_2)) \cap V_2.$$

Let

$$\hat{V}_2'(n_1, n_2, n_3, b) = (A_1(n_1, b) \cup \hat{A}_2(n_1, n_2, b)) \cap (B_1(n_1) \cup \hat{B}_2(n_3, n_2)) \cap V_2$$

(see Figure 4.1). We will see that for the appropriate choice of n_2 depending on b, $V_2' \subset \hat{V}_2'$.

It will be seen that for suitable b and n, $E^2_{V_1'}$ and $E^2_{V_2'}$ may be defined so that $T_{V_2'}M = E^1_{V_2'} \oplus E^2_{V_2'}$ and thereby condition (B) of Lemma NP will be satisfied. First we need some definitions.

If $z \in \mathbf{R}$ is such that $|z| \in (\mu_l, 1]$ let

$$\tilde{\mathcal{L}}_z = \pi_s^{-1}(z), \quad \text{where } \pi_s\colon W_l \to W_l^s.$$

If $|z| \in (0, \mu_l]$, then $z = \mu_l^k z_0$ for some $k \in \mathbf{N}$ and $|z_0| \in (\mu_l, 1]$. Let $\tilde{\mathcal{L}}_z = \tilde{\mathcal{L}}_z(b) = g_b^k \tilde{\mathcal{L}}_{z_0}$. Let $\tilde{\mathcal{L}}_0 = W^u(p_l)$. We may think of z as the point of attachment of the leaf $\tilde{\mathcal{L}}_z$ to W_l^s. Observe that g_b permutes the leaves of $\tilde{\mathcal{L}}$ insofar as $g_b\tilde{\mathcal{L}}_z = \tilde{\mathcal{L}}_{\mu_l z}$. For $|z| \le 1$, let $\mathcal{L}_z = (\tilde{\mathcal{L}}_z)^*$ be the first pass of $\tilde{\mathcal{L}}_z$ through W_1. Every suitably thin fundamental neighborhood V of $W^s \cap \Theta$ is foliated by $\mathcal{L} \cap V$, provided b is sufficiently small. We assume that V_2 is so thin and is thus foliated by $\mathcal{L} \cap V$.

E^1 and $E^2 \subset T_{V_2'}M$ may now be defined. For $x \in V_2'$, let $E_x^1 = \hat{E}_x^1$ (the horizontal coordinate splitting of W_1) and $E_x^2 = T_x\mathcal{L}$. Let V_3 be a thin fundamental neighborhood of W_1^{s+} of the form $V_3 = [\mu_1, 1] \times [-\delta, \delta]$. For small b, $\mathcal{L}_0 \cap V_3$ is a

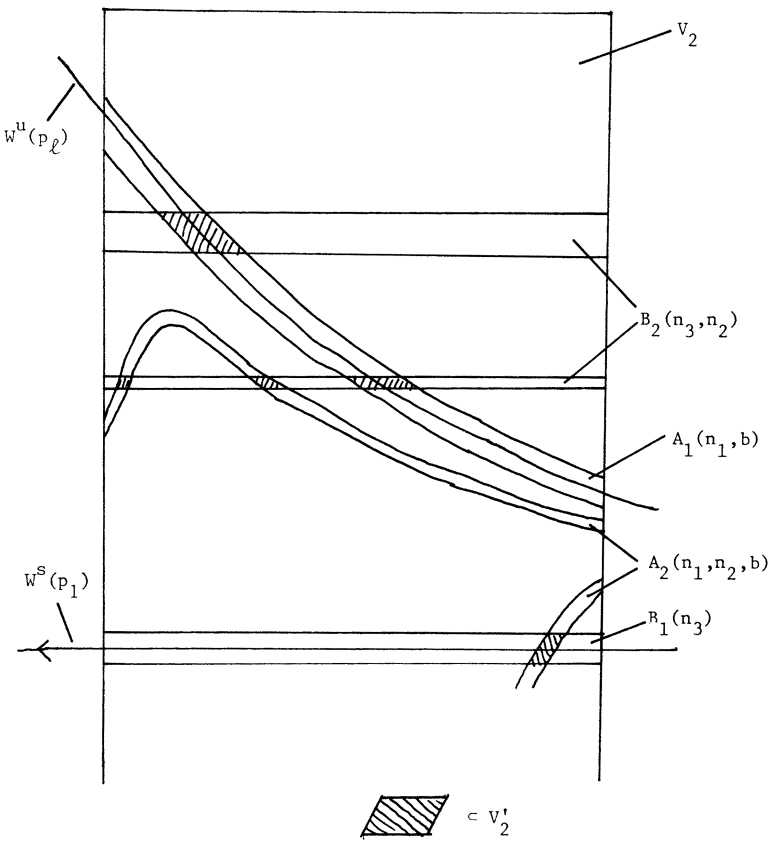

FIGURE 4.1

curve in V_3 given by graph$(\hat{F}(\cdot,0))$, where $\hat{F}(\cdot,0)\colon [\mu_1,1] \to \mathbf{R}$ by $\hat{F}(x,0) = b/x^p$, where $\mu_1^p = \lambda_1^{-1}$. For small $|z|$, $\mathcal{L}_z \cap V_3$ is a curve in V_3 given by graph$(\hat{F}(\cdot,z))$, where $\hat{F}(\cdot,z)\colon [\mu_1,1] \to \mathbf{R}$. It may be seen that $\hat{F}\colon [\mu_1,1] \times [-\varepsilon,\varepsilon] \to \mathbf{R}$ is at least C^2 in x and C^1 in z. We may write

$$\hat{F}(x,z) = \hat{F}(x,0) - a(x)\cdot z + r(x,z) = b/x^p - a(x)\cdot z + r(x,z)$$

where a is a C^2 function r is C^2 in x and

$$\lim_{z\to 0} \frac{r(x,z)}{z} = \lim_{z\to 0} \frac{\partial r(x,z)/\partial x}{z} = 0.$$

We take $a(x) > 0$ on $[\mu_1,1]$.

Now that the form of $\mathcal{L}$ is known on V_3, the same form may be extended by iteration under g to $O(V_3) \cap W_1$ so, in particular, we assume that $\mathcal{L}$ is generated by $\hat{F}\colon (0,1] \times [-\varepsilon,\varepsilon] \to \mathbf{R}$ and $\hat{F}(x,z) = b/x^p - a(x)\cdot z + r(x,z)$. Further, for the moment, we will assume that $\mathcal{L}$ is determined by

$$F(x,z) = b/x^p - a(x)\cdot z$$

and consider the more general case later. It will then be seen that asymptotically (that is, as $b \to 0$) these foliations have the same essential behavior.

We now have an alternative description of $A_1(n_1, b)$ and $\hat{A}_2(n_2, b)$.

For small $b > 0$ and large n_1 and n_2, $A_1(n_1, b) = \bigcup_{|z| \le \mu_l^{n_1}} \mathcal{L}_z$ and $\hat{A}_2(n_1, n_2, b)$ is a finite union of components each of which is an uncountable union of leaves of $\mathcal{L}$. For each $\mathcal{L}_z \subset \hat{A}_2$, $\mathcal{L}_z$ is given in the coordinates of W_1 by $y = b/x^p - a(x) \cdot z$. A point $q \in \mathcal{L}_z \subset \hat{A}_2$ is a critical point of $\mathcal{L}_z$ if $T_q\mathcal{L}_z$ is horizontal. Suppose $\mathcal{L}_z(b)$ is critical at q with $\pi_s(q) = t_0$. Then

$$0 = -pb/t_0^{p+1} - a'(t_0) \cdot z \quad \text{or} \quad b/z = (-a'(t_0) \cdot t_0^{p+1})/p.$$

Define $h\colon W_1^{s+} \to \mathbf{R}$ by

$$h(t) = [-a'(t)]t^{p+1}/p.$$

So $\mathcal{L}_z(b)$ is critical over t_0 if $h(t_0) = b/z$. The condition that q is a nondegenerate critical point of $\mathcal{L}_z(b)$ is

$$h(t_0) = b/z \quad \text{and} \quad h'(t_0) \neq 0, \qquad \text{for } \pi_s(q) = t_0.$$

By $\mathcal{C}(n_1, n_2, b)$, let us denote the collection of critical points of $\hat{A}_z(n_1, n_2, b)$ in W_1. It can be seen that for large n_1 (as compared to b), $A_1(n_1, b)$ contains no critical points. It will be shown that for certain n_1, n_2 and b, $\mathcal{C}(n_1, n_2, b)$ contains only nondegenerate critical points.

Define $G_l^s\colon W_l^{s+} \to \mathcal{T}_l^s \approx S^1$ by $G_l^s(x) = e^{-2\pi i x/\ln \mu_l}$. If $z \in W_l^{s+}$, then $G^s(O_+(z))$ is a single point. Define $\mathcal{I} = \mathcal{I}(n) \subset \mathcal{T}_l^s$ by $\mathcal{I}(n) = G_l^s(\hat{I}'(n))$, where $\{\hat{I}'(n)\}$ is that collection of intervals in $W_l^s \cap F_l^s$ that generates $\hat{S}_l^u(n)$. Hence each leaf $\mathcal{L}_z \subset \hat{A}_z(n_1, n_2, b)$ corresponds to some $\bar{z} \in \mathcal{I}(n_2)$. In fact the whole orbit of $\mathcal{L}_z$ in W_1 corresponds to $\bar{z}$. Likewise define $G_1^s\colon W_1^{s+} \to \mathcal{T}_1^s$ by $G_1^s(x) = e^{-2\pi i x/\ln \mu_1}$ and $G_1^u\colon W_1^{u+} \to \mathcal{T}_1^u$ by $G_1^u(x) = e^{2\pi i x/\ln \lambda_1}$. Let $\mathcal{J}(n) = G_1^u(\hat{I}(n))$, where $\hat{I}(n)$ is that collection of intervals in $F_1^u \cap W_1^u$ that generate $\hat{S}_1^u(n)$. We shall measure distance in $\mathcal{T}_l^s, \mathcal{T}_1^s$ and $\mathcal{T}_1^n$ by arc length. Observe that for y_1 and $y_2 \in W_1^{u+}$, for example, $d(G_1^u(y_2), G_1^u(y_1)) = 2\pi \ln(y_2/y_1) < 2\pi(y_2 - y_1)/y_1$ if $y_2 > y_1$ and $y_2 - y_1$ is small. By $\operatorname{card}(\mathcal{I}(n))$ and $\operatorname{card}(\mathcal{J}(n))$ we mean the number of component intervals in $\mathcal{I}(n)$ and $\mathcal{J}(n)$, respectively. We know that $m_1(n) = \operatorname{card}(\mathcal{J}(n))$, $m_l(n) = \operatorname{card}(\mathcal{I}(n))$, $m_1(n) < \eta(n)$ and $m_l(n) < \eta(n)$ from Lemma 3.6 where η is a polynomial. By $\mathrm{l}(\mathcal{J}(n))$ and $\mathrm{l}(\mathcal{I}(n))$ we mean the length of the largest component interval. Observe that Lemma 3.6 implies $\mathrm{l}(\mathcal{J}(n)), \mathrm{l}(\mathcal{I}(n)) \le K\sigma^n$ for some $K > 0$ and $0 < \sigma < 1$.

REMARK 4.2. If D is any fundamental domain of W_1^{s+} and $\mathcal{L}_z(b)$ is critical at $q \in W_1$, then some iterate of q, say $g_b^k(q)$, is a critical point of $\mathcal{L}_{\mu_l^k z}(b)$ and $\pi_s \circ g_b^k(q) \in D$.

LEMMA 4.3. *There exists $b_0 > 0$ arbitrarily small so that for large n_2,*

(1) *all leaves of $\hat{A}_2(n_1, n_2, b_0)$ are transverse to W_1^s, and*

(2) *$\mathcal{C}(n_1, n_2, b_0)$ consists entirely of nondegenerate critical points.*

PROOF OF LEMMA 4.3. In light of Remark 4.2, it suffices to show that conditions (1) and (2) are satisfied over any fundamental domain of W_1^s, so we choose $[\mu_1, 1]$.

Suppose $\mathcal{L}_z(b)$ is critical at q with $\pi_s(q) = t_0 \in [\mu_1, 1]$. Then $b/z = h(t_0)$. If $q \in W_1^s$, then also

$$0 = b/t_0^p - a(t_0) \cdot z \quad \text{or} \quad b/z = a(t_0) \cdot t_0^p,$$

so

$$h(t_0) = a(t_0) \cdot t_0^p.$$

But $h(t) = a(t) \cdot t^p$ has generically only finitely many solutions on $[\mu_1, 1]$. Hence for a given b there are at most finitely many $\bar{z} \in \mathcal{I}(n_2)$ so that some leaf $\mathcal{L}_z$, $z \in G_l^{s^{-1}}(\bar{z})$, may be tangent to W_1^s. Denote this collection of $\bar{z}$'s by $\mathcal{E}_1$. Now suppose $\mathcal{L}_z(b)$ is critical at q with $\pi_s(q) = t_0 \in [\mu_1, 1]$ and q is a degenerate critical point. Then $h'(t_0) = 0$. Once again, $h'(t) = 0$ has at most finitely many solutions on $[\mu_1, 1]$. So for any b there are only finitely many $\bar{z} \in \mathcal{I}(n_2)$ so that some leaf $\mathcal{L}_z$, with $G_1^s(z) = \bar{z}$, has a degenerate critical point. Call this collection of z's $\mathcal{E}_2$. $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ is a finite set of points in $\mathcal{T}_l^s$. $\mathcal{E} = \mathcal{E}(b)$. Now fix some small $b' > 0$ and consider the map $[0,1] \to \mathbf{R}$ given by $t \mapsto \mu_l^t b'$. $\mathcal{E}(\mu_l^t b') = R_{2\pi t}\mathcal{E}(b')$, where $R_\alpha \colon \mathcal{T}_l^s \to \mathcal{T}_l^s$ is rotation by α. Let $B = \{t \in [0,1] | \mathcal{E}(\mu^t b') \cap \mathcal{I}(n) \neq \emptyset\}$. If $c = \operatorname{card} \mathcal{E}$, then $m(B) \leq c \cdot m_l(n_2) \cdot K\sigma^n$. Thus for sufficiently large n_2, $m(B) < 1$, so there is some $t' \in [0,1]$, so that for $b_0 = \mu_l^{t'} b'$ conditions (1) and (2) are satisfied. This proves the lemma.

Let $b_k = b_0 \mu_l^k$. From the above proof we see that conclusions (1) and (2) of the lemma are valid for b_k, $k \in \mathbf{N}$. It is straightforward to verify that there exists $N_0 \in \mathbf{N}$ so that when $n_1 = N_0 + k$, $A_1(n_1, b_k)$ has no critical points. Hereafter we shall write $A_1(k) = A_1(N_0 + k, b_k)$ and merely write $\hat{A}_2(n, b_k)$ for $\hat{A}_2(N_0 + k, n, b_k)$. We will write $\mathcal{C}(n, b_k)$ for $\mathcal{C}(N_0 + k, n, b_k)$. Let $J'(n, b_k) = G_1^u \circ \pi_u \mathcal{C}(n, b_k)$ and $\mathcal{H}(n, b) = G_1^s \circ \pi_s \mathcal{C}(n, b_k)$. We will see that $J'(n_2, b_k)$ and $\mathcal{H}(n_2, b_k)$ are finite unions of intervals in $\mathcal{T}_1^u$ and $\mathcal{T}_1^s$, respectively. Toward this end consider $\bar{z} \in \bar{I} \subset \mathcal{I}(n)$. Let $z \in W_l^s \cap F_l^s$ be such that $G_l^s(z) = \bar{z}$. Let $C(\bar{z}, b_k)$ be the collection of critical points of $\mathcal{L}_{O_+(z)}$ in W_1.

CLAIM 4.4. $\operatorname{card}[G_1^s \circ \pi_s C(\bar{z}, b_k)] = \operatorname{card}[G_1^u \circ \pi_u C(\bar{z}, b_k)] < \infty$.

It suffices to consider the critical points over any fundamental domain of W_1^{s+} so we choose $[\mu_1, 1]$. Each critical point of $\mathcal{L}_{O_+(z)}$ over $[\mu_1, 1]$ corresponds to a solution of

$$h(t) = b_k / \mu_l^j z, \qquad j \in \mathbf{N}, t \in [\mu_1, 1].$$

Since h is bounded above, this equation has finitely many solutions $\{t_0, t_1, \ldots, t_r\}$. Each t_i corresponds to exactly one critical point of $\mathcal{L}_{O_+(z)}$ over $[\mu_1, 1]$ and thus to some $y_i = b_k / t_i^p - a(t_i) \cdot \mu_l^j z$. Each y_i corresponds to $\bar{y}_i = G_1^u \circ \pi_u(y_i) \in G_1^u \circ \pi_u(C(\bar{z}, b_k))$. This establishes the claim. See Figure 4.5.

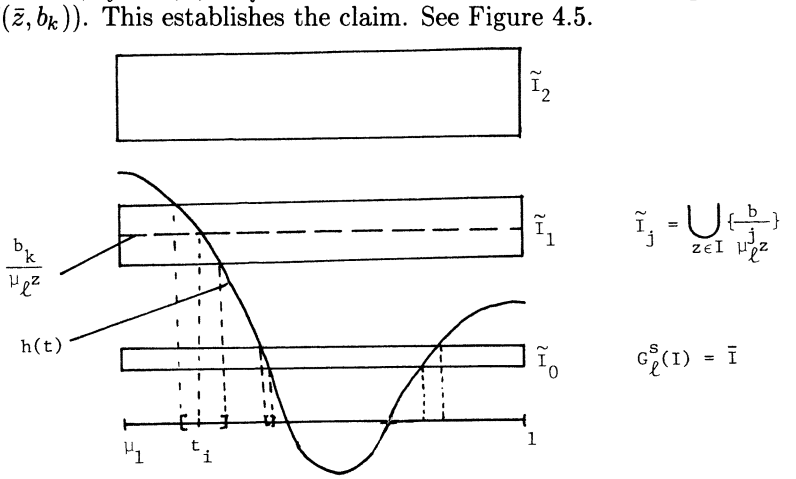

FIGURE 4.5

In a similar fashion for b_k, $k \in \mathbf{N}$, and large n, each interval $\overline{I} \subset \mathcal{I}(n)$ corresponds to finitely many intervals in $\mathcal{H}(n, b_k)$. Since the number of intervals in $\mathcal{I}(n)$ is finite and bounded by $\eta(n)$, $m'(n) \leq K\eta(n)$ for some $K > 0$, where $m'(n)$ is the number of intervals in $\mathcal{H}(n, b_k)$ and $\mathcal{J}'(n, b_k)$. We may (and will) assume w.l.o.g. that $m'(n) \leq \eta(n)$. The reader may verify that $\mathcal{H}(n, b_k)$ is the same set for all $k \in \mathbf{N}$, but how does $\mathcal{J}'(n, b_k)$ compare to $\mathcal{J}'(n, b_0)$? This is answered by the following lemma.

LEMMA 4.6. *$\mathcal{J}'(n, b_k) = R_\alpha^k \mathcal{J}'(n, b_0)$ where $R_\alpha \colon \mathcal{T}_1^u \to \mathcal{T}_1^u$ is rotation by $\alpha = -2\pi(\ln \mu_l)/(\ln \lambda_1)$.*

PROOF OF LEMMA 4.6. It will suffice to show that $\mathcal{J}'(n, b_1) = R_\alpha \mathcal{J}'(n, b_0)$. Suppose $\bar{y} \in \mathcal{J}'(n, b_1)$; then for some $z \in G_l^{s^{-1}}(\mathcal{I}(n))$ and $t_0 \in G_1^{s^{-1}}(\mathcal{H}(n, b))$,

$$y_1 = b_1/t_0^p - a(t_0)z \quad \text{and} \quad G_1^u(y_1) = \bar{y}.$$

But

$$\begin{aligned} y_1 &= b_1/t_0^p - a(t_0) \cdot z = \mu_l b_0/t_0^p - a(t_0) \cdot \mu_l \mu_l^{-1} z \\ &= \mu_l((b_0/t_0^p)a(t_0) \cdot \mu_l^{-1} z) = \mu_l y_0 \end{aligned}$$

and since $G_l^s(\mu_l^{-1} z) = G_l^s(z) \in \mathcal{I}(n)$,

$$G_1^u(y_0) \in \mathcal{J}'(n, b_0) \quad \text{and} \quad G_1^u(\mu_l y_0) = R_\alpha G_1^u(y_0),$$

so $\mathcal{J}'(n, b_1) \subset R_\alpha \mathcal{J}'(n, b_0)$. The other inclusion is proved similarly, so the lemma is established.

The following lemma gives some control on the thickness of the intervals of $\mathcal{J}'(n, b_k)$.

LEMMA 4.7. *$l(\mathcal{J}'(n, b_k)) \leq K\sigma^n$ for some $K > 0$, $\sigma > 1$ and large n.*

PROOF OF LEMMA 4.7. Let $\overline{J}_0'$ be any component interval of $\mathcal{J}'(n, b_k)$. Take any $\bar{y}_2, \bar{y}_1 \in \overline{J}_0'$ and $y_2, y_1 \in J_0' \in W_1^{u+}$ with $G_1^u(y_i) = \bar{y}_i$, $i = 1, 2$, and $G_1^u(J_0') = \overline{J}_0'$. Assume $y_2 > y_1$. Now

$$y_i = b_k/t_i^p - a(t_i) \cdot z_i$$

for some t_i, z_i, $i = 1, 2$, with z_1 and z_2 in the same interval $J_0 \subset G_l^{s^{-1}}(\mathcal{I}(n))$. $d(\bar{y}_2, \bar{y}_1) = 2\pi \ln(y_2/y_1) < 2\pi(y_2 - y_1)/y_1$. But the map defined locally by $z_i \to y_i$ is Lipschitz (in fact at least C^1), so $|y_2 - y_1| \leq L_0|z_2 - z_1| \leq L_0 K' \sigma^n$ by Lemma 3.6. Further, we may assume that $y_1 \in F_1^u$, so in particular $1/\lambda_1 \leq y_1$. Hence $d(\bar{y}_2, \bar{y}_1) \leq K_0 \sigma^n$ for some $K_0 > 0$. So $l(J_0') \leq K_0 \sigma^n$. Since the number of intervals in $\mathcal{J}'(n, b_k)$ is finite, there exists $K > 0$ so that $l(\mathcal{J}'(n, b_k)) \leq K\sigma^n$, which was to be shown.

LEMMA 4.8. *There are an increasing sequence of positive integers $\overline{K} = \{k_0, k_1, k_2, \ldots\}$ and a real number $c_0 > 0$ so that $d(\mathcal{J}'(n, b_{k_i}), \mathcal{J}(n)) \geq c_0$ for all $k_i \in \overline{K}$ and large n.*

Before proving this lemma we pause to observe that the lemma implies that for b_{k_i}, with $k_i \in \overline{K}$, $TM = E^1 \oplus E^2$ over $V_2'(n, b_{k_i})$ with large n. This is guaranteed

since $d(\mathcal{J}'(n,b_{k_i}),\mathcal{J}(n)) > 0$ implies that the critical points of $\hat{A}_2(n,b_{k_i})$ must all miss the disks of $O_-(S_1^s(n),g_{b_{k_i}})$ in W_1. So we will have our splitting, but we still need a bit more.

PROOF OF LEMMA 4.8. $\mathcal{J}'(n,b_k)$ and $\mathcal{J}(n)$ are compact subsets of $\mathcal{T}_1{}^u$. For fixed n, $\mathcal{J}(n)$ is fixed on $\mathcal{T}_1{}^u$. As k varies, $\mathcal{J}'(n,b_k)$ is rigidly (and discretely) rotated in $\mathcal{T}_1{}^u$. We wish to show that, for large n and infinitely many k_i's, $\mathcal{J}'(n,b_{k_i})$ and $\mathcal{J}(n)$ are not only disjoint but are separated by at least some constant $c_0 > 0$. First consider the map $\psi\colon [0,1] \to C^\infty(\mathcal{T}_1{}^u, \mathcal{T}_1{}^u)$ given by $\psi_t = R_{2\pi t}$. So, for each $t \in [0,1]$, ψ_t is rotation by $2\pi t$. $\psi_1 = \psi_0 = \mathrm{id}$. As t varies through $[0,1]$, the set $\psi_t\mathcal{J}'(n,b_0)$ rotates through one complete cycle in $\mathcal{T}_1{}^u$. As this happens, for certain values of t, the intervals of $\psi_t\mathcal{J}'(n,b_0)$ will cross the intervals of $\mathcal{J}(n)$. We are interested in these interval crossings. For large n, define

$$\mathcal{E}(n) = \{t \in [0,1]\colon \psi_t\mathcal{J}'(n,b_0) \cap \mathcal{J}(n) \neq \varnothing\}.$$

$\mathcal{E}(n)$ clearly corresponds to those values of t at which interval crossing is taking place. Since $\operatorname{card}\mathcal{J}'(n,b_0) = m'(n)$ and $\operatorname{card}\mathcal{J}(n) = m_1(n)$, there are exactly $m'(n)\cdot m_1(n)$ such crossings. Since $l(\mathcal{J}'(n,b_0)) < K\sigma^n$ and $l(\mathcal{J}(n)) < K\sigma^n$ for some $K > 0$ and $\sigma < 1$, the amount of parameter used up in each crossing is at most $2K\sigma^n$. Thus

$$m(\mathcal{E}(n)) \le m'(n)\cdot m_1(n)\cdot 2K\sigma^n \le K_0[\eta(n)]^2\sigma^n.$$

But $\eta(n)$ is a polynomial so $\lim_{n\to\infty} m(\mathcal{E}(n)) = 0$. In particular there exists $n_0 > 0$ so that $n \ge n_0 \Rightarrow m(\mathcal{E}(n)) < 1 \Rightarrow$ there exists $t_0 \in [0,1]$ and $c_0' > 0$ so that $\mathcal{I}(\psi_{t_0}\mathcal{J}'(n,b_0),\mathcal{J}(n)) \ge c_0'$. As k varies through $\mathbf{N}$, however, $\mathcal{J}'(n,b_k)$ rotates discretely in $\mathcal{T}_1{}^u$ by an amount $2\pi|\ln\mu_l|k/\ln\lambda_1$. We assume that $|\ln\mu_l|/\ln\lambda_1$ is irrational (by slight perturbation if necessary) so there are infinitely many k_i's, $k_i \in \mathbf{N}$, so that $\mathcal{J}'(n,b_{k_i})$ is as close as we want to $\psi_{t_0}\mathcal{J}'(n,b_0)$. Let $C_0 = c_0'/2$ and let

$$\overline{K} = \{k \in \mathbf{N}\colon d(\mathcal{J}'(n,b_k),\psi_{t_0}\mathcal{J}(n,b_0)) \le c_0\}.$$

Then for large n,

$$d(\mathcal{J}'(n,b_{k_i}),\mathcal{J}(n)) \ge c_0 \quad \text{for } k_i \in \overline{K}.$$

This proves the lemma.

As a consequence of the proof, observe that for each $k \in \overline{K}$ there corresponds a unique integer $\bar{n}_3$ so that $\lambda_1^{-\bar{n}_3} \approx \mu_l^k$. Let $\bar{n}_3 = \bar{n}_3(k)$ be this integer. Now $pk/\varepsilon - 1 \le \bar{n}_3(k) \le pk/\varepsilon + 1$, where $\mu_1^p = \lambda_1^{-1}$ and $\mu_1^\varepsilon = \mu_l$. Furthermore, it is easy to check that there exists $\overline{N}_0 \in \mathbf{N}$ so that if $n_3(k) = \overline{N}_0 + \bar{n}_3(k)$, then $B_1(n_3)$ contains no critical points of $\hat{A}_2(n,b_k)$. We will write $B_1(k)$ for $B_1(n_3(k))$.

We also want $A_2(n,b_k) \subset \hat{A}_2(n,b_k)$. This is just the requirement that $W^s(p_1,g_{b_k}) \cap F^s \subset T^s(n)$ and $W^u(p_l,g_{b_k}) \cap F_1^u \subset T_1^u(n)$. One can check that there exists $K_1 > 0$ so that if $[\cdot]\colon \mathbf{R} \to \mathbf{Z}$ is the greatest integer function and $n_2(k) = [K_1\cdot k]$, then $A_2(n_2,b_k) \subset \hat{A}_2(n_2,b_k)$. Write $A_2(k)$ for $A_2(n_2(k),b_k)$ and $B_2(k)$ for $B_2(n_3(k),n_2(k))$. Define $V_2'(k)$ depending only on k as

$$V_2'(k) = (A_1(k)\cup A_2(k)) \cap (B_1(k)\cup B_2(k)) \cap V_2.$$

Since $n_2(b) \ge K_1\cdot k - 1$, it is clear that for sufficiently large $k \in \overline{K}$, n_2 will be sufficiently large for Lemma 4.8 to hold. Define $\tilde{K} \subset \overline{K}$ by $\tilde{K} = \{k \in \overline{K} | n_2(k)$ is

large enough for Lemma 4.8}. We conclude that for $k \in \tilde{K}$, $A_2(k)$ is foliated by leaves $\mathcal{L}_z$ with $G^s(z) \in \mathcal{I}(n_2(t))$ (i.e., $A_2(k) \subset \hat{A}_2(k)$), and we have the required splitting $T_{V_2'}M = E^1_{V_2'} \oplus E^2_{V_2'}$.

The following lemma provides an estimate of the u-slopes of E^2_x for $x \in V_2'(x)$.

LEMMA 4.9. *There exists a constant $C > 0$ so that if $k \in \tilde{K}$ and $x \in V_2'(k)$, then $\rho_u(E^2_x) \geq C\mu_l^k$.*

PROOF OF LEMMA 4.9. There are two cases: $x \in A_1(k)$ and $x \in A_2(k)$. First take $x \in A_1(k)$; then $x \in \mathcal{L}_z$ and $|z| \leq \mu_l^{N_0+k}$. Let $t = \pi_s(x)$ so

$$\begin{aligned}\rho_u(E^2_x) &= |-pb_k/t^{p+1} - a'(t)\cdot z| = |-pb_0\mu_l^k/t^{p+1} - a'(t)\cdot z_0\mu_l^k| \\ &= |-pb_0/t^{p+1} - a'(t)\cdot z_0|\mu_l^k \quad \text{with } |z_0| \leq \mu_l^{N_0}.\end{aligned}$$

But $|-pb_0/t^{p+1} - a'(t)\cdot z_0| = \rho_u(E^2_{x_0})$ for some $x_0 \in A_1(0)$.

Let $\bar{c} = \inf_{x_0 \in A_1(0)\cap V_2} \rho_u(E^2_{x_0}) > 0$. Then $\rho_u(E^2_x) \geq \bar{c}\mu_l^k$.

To estimate $\rho_u(E^2_x)$ for $x \in V_2'(k) \cap A_2(k)$, first consider $A_2(k_0)$, where k_0 is the least element of $\tilde{K}$. Each $x_0 \in A_2(k_0) \cap V_2'(k_0)$ is a particular point in $\pi_u^{-1}(y_0) \cap \mathcal{L}_{z_0}$ for some $y_0 \in G_1^{u^{-1}}(\mathcal{J}(n_2(k)))$ or $|y_0| \leq \lambda_1^{-n_3(k_0)}$ and $z_0 \in G_l^{s^{-1}}(\mathcal{I}(n_2(k_0)))$. (See Figure 4.10.) Now, in light of Lemma 4.8, if $\tilde{K} = \{k_0, k_1, \ldots\}$ then for $k_i \in \tilde{K}$ there corresponds exactly one point $x_i \in \pi_u^{-1}(y_i) \cap \mathcal{L}_{z_i}(b_{k_i})$, where $y_i = y_0\lambda_1^{-(n_3(k_i)-n_3(k_0))}$ and $z_i = \mu_l^{k_i-k_0}z_0$. Let $\psi_i : V_2'(k_0) \cap A_2(k_0) \to V_2$ be the map $x_0 \mapsto x_i$. Let $D_i = \psi_i(V_2'(k_0) \cap A_2(k_0))$. By construction of $V_2'(k)$ it may be seen that $V_2'(k_i) \cap A_2(k_i) \subset D_i$. This is not an equality since larger k forces large n_2, and if $n_2' < n_2''$ then $\mathcal{I}(n_2'') \subset \mathcal{I}(n_2')$ and $\mathcal{J}(n_2'') \subset \mathcal{J}(n_2')$. However, this inclusion is all we need. Let S be the π_s projections of the critical points of $\mathcal{L}_{z_i}(b_{k_i})$ in $\pi_s(V_2)$ (S is the same for all i).

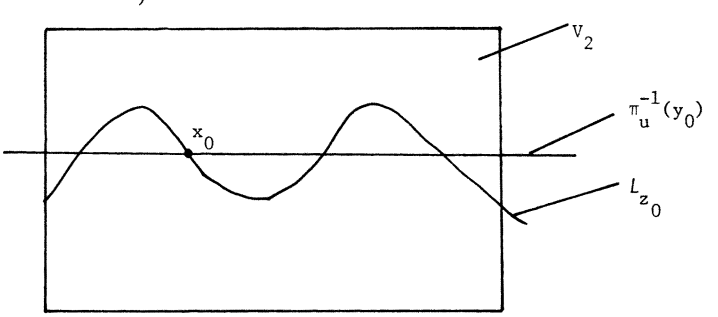

FIGURE 4.10

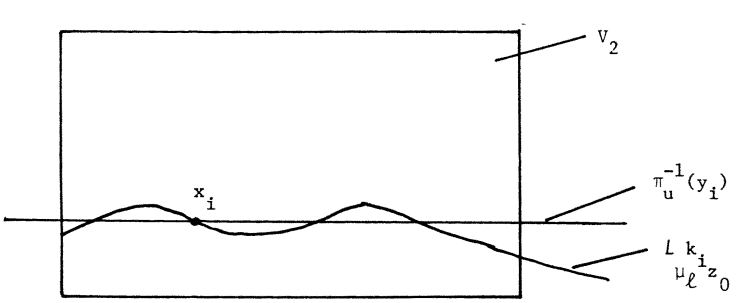

FIGURE 4.11

CLAIM 4.12. $\overline{\bigcup_{i\in\mathbf{N}} \pi_s(x_i)} \cap S = \emptyset$.

PROOF. Let $\pi_s(x_i) = t_i$ and suppose $t_i \to s$ as $i \to \infty$ for some $s \in S$, taking subsequences if necessary. Let q_i be the critical point of $\mathcal{L}_{z_i}$ which corresponds to s and let c_i be the distance from q_i to $\pi_u^{-1}(y_i)$. We assume that $\pi_u(q_i) > y_i$, though this is not necessary. (See Figure 4.13.)

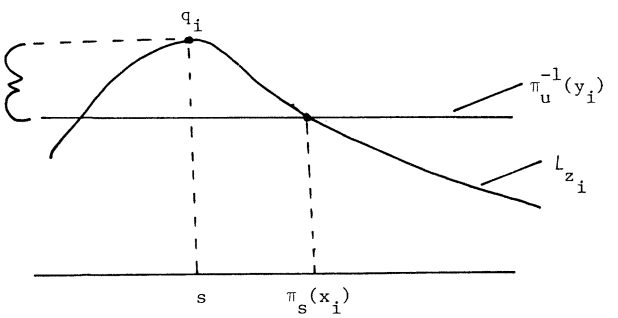

FIGURE 4.13

SUBCLAIM 4.14. There exists $K_1 > 0$ so that $c_i > K_1\mu_l^{k_i}$.

PROOF. Let $\tilde{y}_i = \pi_u(p_i)$. From construction of the correspondence $x_0 \mapsto x_i$ (i.e., the choice of x_i as the "corresponding" intersection of $\pi_u^{-1}(y_i)$ with $\mathcal{L}_{z_i}$), there is a constant $\tilde{c} = \tilde{c}(x_0)$ such that $d(G_1^u(\tilde{y}_i), G_1^u(y_i)) < \tilde{c}$. From the previous lemma we have $c_0 < d(G_1^u(\tilde{y}_i), G_1^u(y_i))$. We already know that $d(G_1^u(\tilde{y}_i), G_1^u(y_i)) = 2\pi \ln(\tilde{y}_i/y_i)$, and $c_0 < 2\pi \ln(\tilde{y}_i/y_i) < \tilde{c}$. Now

$$\tilde{y}_i = b_{k_i}/s^p - a(s)\cdot z_{k_i} = \mu_l^{k_i} b_0/s^p - a(s)\cdot \mu_l^{k_i} z_0 = \mu_l^{k_i}\tilde{y}_0,$$

so $2\pi \ln(\tilde{y}_i/y_i) < \tilde{c}$ gives $y_i \geq K_0\mu_l^{k_i}$ for some K_0. From this and $c_0 < 2\pi \ln(\tilde{y}_i/y_i) < 2\pi(\tilde{y}_i - y_i)/y_i = 2\pi c_i/y_i$ we get

$$c_0 y_i < 2\pi c_i \quad \text{and} \quad c_i > K_1\mu_l^{k_i}.$$

This proves the subclaim.

Returning to the proof of the claim, let $\mathcal{L}_{z_i}(b_{k_i})$ be the graph of

$$\xi_i(t) = b_{k_i}/t^p - a(t)\cdot z_i = \mu_l^{k_i}[b_0/t^p - a(t)\cdot z_0] = \mu_l^{k_i}\xi_0(t).$$

Near q_i, $\xi_i(t)$ may be written

$$\begin{aligned}\xi_i(t) &= \alpha_i + \tfrac{1}{2}\xi_i''(s)(t-s)^2 + O((t-s)^2)\\ &= \mu_l^{k_i}[\alpha_0 + \tfrac{1}{2}\xi_0''(s)(t-s)^2 + O((t-s)^2)].\end{aligned}$$

Since $\xi_i(t_i) = y_i$,

$$|\mu_l^{k_i}[\tfrac{1}{2}\xi_0''(s)(t_i-s)^2 + O(t_i-s)^2]| = c_i$$

(see Figure 4.15) or

$$\mu_l^{k_i}|\tfrac{1}{2}\xi''(s)(t_i-s)^2 + O(t_i-s)^2| = c_i \geq K_0\mu_l^{k_i},$$

so

$$\tfrac{1}{2}|\xi''(s)(t_i-s)^2 + O(t_i-s)^2| > K_0.$$

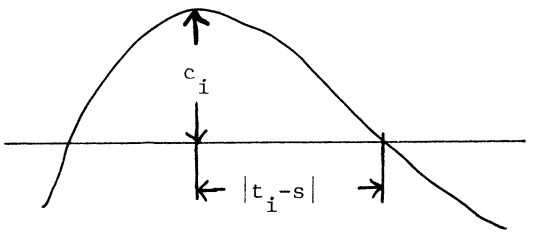

FIGURE 4.15

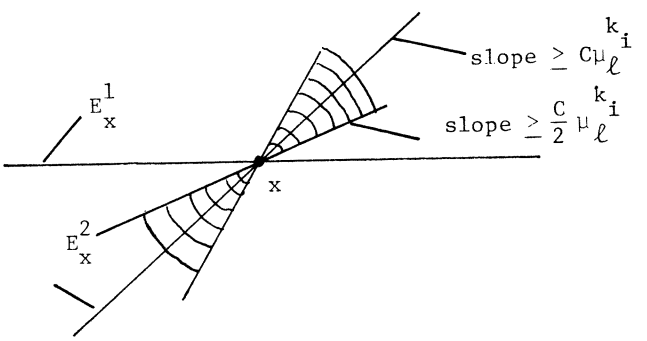

FIGURE 4.16

But $t_i \to s$ implies that $|\xi''(s)(t_i - s) + O(t_i - s)^2| \to 0$, which is a contradiction, so $\overline{\bigcup_{i\in\mathbf{N}} \pi_s(x_i)} \cap S = \varnothing$ and Claim 4.10 is established.

We see that for any $x_0 \in V_2'(k)$ there is a compact set $\tilde{S} = \tilde{S}(x_0) \subset [\mu_1, 1]$ so that $\pi_s(x_i) \in \tilde{S}$ for all i and $\tilde{S} \cap S = \varnothing$. Now

$$\rho_u E^2_{x_i} = | - b_{k_i} p/t_i^{p+1} - a'(t_i) \cdot z_i| = \mu_l^{k_i} | - pb_0/t_i^p - a(t_i) \cdot z_0|,$$

but since $t_i \in \tilde{S}$, $\tilde{S}$ is compact and $\tilde{S} \cap S = \varnothing$,

$$C_{x_0} \equiv \inf_{t_i \in S} | - pb_0/t_i^p - a'(t_i) z_0| > 0.$$

So $\rho_u E^2_{x_i} \geq C_{x_0} \mu_l^{k_i}$. Let $\tilde{C} = \inf_{x_0 \in V_2'(k_0)} C_{x_0} > 0$, so

$$\rho_u E^2_x \geq \tilde{C} \mu_l^{k_i} \quad \text{for all } x \in D_i.$$

But $V_2'(k_i) \cap A_2(k_i) \subset D_i$, so if $C = \min\{\tilde{C}, \overline{C}\}$ the lemma is proved.

We may now define the sector field $S_{\varepsilon(x)} E^2_x$ on $V_2'(k)$. For each $k \in \tilde{K}$, and all $x \in V_2'(k)$, let $\varepsilon(x) \equiv c$ be the greatest real number so that

$$\rho_u(v) \geq (c/2)\mu_l^{k_i} \quad \text{for all } v \in SE^2_x, x \in V_2'(n, b_{k_i}).$$

From the previous lemma and compactness of V_2', $c > 0$. See Figure 4.16.

5. Proof of Theorem 2.1. We are now in a position to prove the theorem.

PROOF OF THEOREM 2.1. We show that for arbitrarily large $k \in \tilde{K}$ the splitting and sector field satisfy the conditions of Lemma NP and conclude that $\Lambda_{g_{b_k}}$ is hyperbolic. Let $g = g_{b_k}$. First consider condition A: $\bigcap_{j\in\mathbf{Z}} g^j(V_1 \cup V_2) \cap V_2 \subset V_2'$.

Suppose $x \in \bigcap_{j\in\mathbf{Z}} g^j(V_1 \cup V_2) \cap V_2$. We may assume $g^{-2}(x) \subset V_1$.

Either:

(1) $O_-(g^{-2}(x)) \subset V_1$ which implies $x \in W^u(\Lambda_1)$, so $x \in W^u(p_i)$ for some $i \in \{1, \dots, l\}$, or

(2) there exists some $j > 2$ so that $g^{-j}(x) \subset V_2$.

For case (1), $W^u(P_i) \subset O_+(T_i^u(n_2(k)))$, so $x \in A_1(k) \cup A_2(k)$. For case (2), since $x \in \bigcap_{i \in \mathbf{Z}} g^i(V_1 \cup V_2)$, $g^{-j}(x)$ must lie between $[W^u(P_l)]^* \cap V_2$ and $W_1^s \cap V_2$. But $n_2(k)$ was chosen so that $O_-(x)$ must pass through $T_1^u(n_2)$, so $x \in O_+(T_1^u(n_2)) \subset A_1(k) \cup A_2(k)$. Similar arguments show that $x \in B_1(k) \cup B_2(k)$ so $x \in V_2'(k)$, and we conclude that condition A is satisfied.

Condition B is satisfied by virtue of Lemma 4.8.

Consider condition

(C)(1): There exists $\lambda > 1$ and $Q \in \mathbf{N}$ so that for each $x \in V_2'(k)$ there exists $j = j(x)$ so that $0 < j(x) < Q$ and

(a) $T_x g^j(S_{\varepsilon(x)} E_x^2) \subset S_{\varepsilon(g^j(x))} E_{g^j(x)}^2$,

(b) $m(T_x g^j | S_{\varepsilon(x)} E_x^2) \geq \lambda$ and

(c) $g^j(x) \in g^{-1}(V_1) \cap V_1 \cap g(V_1)$

where $g = g_{b_k}$ and $\varepsilon(x) \equiv c$ is as above.

Let $N = N(k)$ be the greatest integer so that $g^N(V_2') \subset W_1$. It is easy to check that $N(k) \geq \varepsilon k/p - 1$. Now choose any $v \in S_{\varepsilon(x)} E_x^2$ with $\|v\| = 1$ for any $x \in V_2'(k)$. Since x spends the next N iterates in W_1,

$$\rho_u T_x g^N(v) = \left(\frac{\lambda_1}{\mu_1}\right)^N \rho_u(v) \geq \left(\frac{\lambda_1}{\mu_1}\right)^{\varepsilon k/p} \cdot \frac{c}{2} \mu_l^k = \left(\frac{1}{\mu_1}\right)^{\varepsilon k/p} \cdot \frac{c}{2}.$$

We conclude that for large k (independent of v and x),

$$T_x g^N(v) \subset S_{c_1} \hat{E}_{g^N(x)}^2 \subset S_{\varepsilon(g^N(x))} \tilde{E}_{g^N(x)}^2. \tag{5.1}$$

So if we were to let $Q = N$ (which we will not), condition (C)(1)(a) would be satisfied. We must choose Q so that (C)(1)(b) and (C)(1)(c) are also satisfied. Using the max-norm on $T_{W_1} M$,

$$\begin{aligned} \|T_x g^N(v)\| &= |\pi_2 T_x g^N(v)| = \lambda_1^N \pi_2(v) \\ &= \lambda_1^N \rho_u(v) \quad \text{(we may assume that } \rho_u(v) < 1) \\ &\geq \lambda_1^{\varepsilon k/p} \cdot (c/2) \mu_l^k = c/2. \end{aligned}$$

We conclude that $m(T_x g^N | S_{\varepsilon(x)} E_x^2) \geq c/2$. Now if $c/2 > 1$ we let $\lambda = c/2$, and (C)(1)(b) and (C)(2)(b) are satisfied. If $c/2 \leq 1$, then we do not have the required expansion, but the contraction is uniformly bounded. By construction of $V_2'(n, b_k)$, each $g^N(x)$ is destined, after a bounded number of iterates N_1, to spend $n_2(k)$ consecutive iterates in some W_i. Furthermore, by (5.1) and Lemma 3.0(4), $T_x g^N(v)$ (or more properly some iterate of it) will be expanded by a factor of $\lambda_i^{n_2(k)}$ while $g^N(x)$ (or its iterates) passes through W_i. We conclude that for large enough k, v will be appropriately expanded within $N + N_1 + n_2$ iterates, so if $Q_1 = N + N_1 + n_2$, condition (C)(1) is satisfied for $Q = Q_1$. A similar argument produces Q_2 so the (C)(2) is satisfied. We conclude that $\Lambda_{g_{b_k}}$ is hyperbolic, so $L(g_{b_k})$ is hyperbolic. It is clear that g_{b_k} has the strong transversality property so g_{b_k} is structurally stable. In fact $\Lambda_{g_{b_k}}$ is a single basic set of $L(g_{b_k})$.

The theorem would be proved, but we have assumed that $\mathcal{L}$ has the special form

$$F(x,z) = b/x^p - a(x)\cdot z$$

instead of the general form

$$\hat{F}(x,z) = b/x^p - a(x) + r(x,z).$$

Now we consider the question as to whether the general foliation $\hat{\mathcal{L}}$ and those structures generated by it have the same behavior as $\mathcal{L}$ and its structures for large k. Recall that the leaves $\hat{\mathcal{L}}_z$ of $\hat{\mathcal{L}}$ are curves of the form

$$y = b/x^p - a(x)\cdot z + r(x,z),$$

where

$$\lim_{z\to 0}\frac{r(x,z)}{z} = 0 \quad\text{and}\quad \lim_{z\to 0}\frac{\partial r(x,z)/\partial x}{z} = 0.$$

We shall write $r'(x,z) \equiv \partial r(x,z)/\partial x$. $\hat{\mathcal{L}}_z(b)$ is critical over t iff $\tilde{h}(t,z) = b/z$, where

$$\tilde{h}(t,z) = h(t) + (q'(x,z)\cdot x^{p+1})/p$$

and

$$q'(x,z) = r'(x,z)/z.$$

It will suffice to compare $\{\hat{\mathcal{L}}_{z_k}(b_k)\}$ to $\{\mathcal{L}_{z_k}(b_k)\}$, where $z_k = z_0\mu_l^k$ and $z_0 \in \mathcal{G}_l^{s^{-1}}(\mathcal{J}(n_2))$. Let $z_0 \in \mathcal{G}_l^{u^{-1}}(\mathcal{J}(n_2))$. We are interested in the critical behavior of the sequence $\{\hat{\mathcal{L}}_{z_k}(b_k)\}$ where $z_k = \mu_l^k z_0$. We will show that the critical behavior of $\hat{\mathcal{L}}_{z_k}(b_{z_k})$ approaches the critical behavior of $\{\mathcal{L}_{z_k}(b_k)\}$ as $k\to\infty$. Let $\{t_k\}$ be a collection of π_s-projections of critical points of $\{\hat{\mathcal{L}}_{z_k}(b_k)\}$; i.e., $\hat{\mathcal{L}}_{z_k}(b_k)$ is critical over t_k. Suppose $t_k \to t^*$ and $t^* \notin S(z_0) = \{\pi_s$ projections of critical points of $\{\mathcal{L}_{z_k}(b_k)\}$. Recall that $S(z_0) = S(z_k)$ for all $k\in\mathbf{N}$. Let I be a closed neighborhood of t^* such that $|h(t) - b_0/z_0| > 0$ on I. Let $c > 0$ satisfy $|h(t) - b_0/z_0| \geq c > 0$ for $t\in I$. Now $\hat{\mathcal{L}}_{z_k}(b_k)$ is critical at t_k iff $\hat{h}(t_k,z_k) - b_0/z_0 = 0$. But

$$\hat{h}(t_k,z_k) = h(t_k) + \frac{q'(t_k,z_k)}{p}t_k^{p+1},$$

so

$$\hat{h}(t_k,z_k) - \frac{b_0}{z_0} = h(t_k) - \frac{b_0}{z_0} + \frac{q'(t_k,z_k)}{p}t_k^{p+1}$$

when $t_k\in I$,

$$\left|\hat{h}(t_k,z_k) - \frac{b_0}{z_0}\right| \geq \left|\left|h(t_k) - \frac{b_0}{z_0}\right| - \left|\frac{q'(t_k,z_k)}{p}t_k^{p+1}\right|\right| \geq \left|c - \left|\frac{q'(t_k,z_k)}{p}t_k^{p+1}\right|\right|.$$

But $|q'(t_k,z_k)t_k^{p+1}/p| \to 0$ as $k\to\infty$ for large k, so $|\hat{h}(t_k,z_k) - b_0/z_0| > 0$, contradicting $\hat{\mathcal{L}}_{z_k}(b_k)$ being critical over t_k.

Now let $\{t_k\}$ be as before and suppose $t_k \to t_0$ with $\mathcal{L}_{z_k}(b_k)$ critical over t_0. Let $\hat{y}_k$ and y_k be the π_u projections of the critical points of $\hat{\mathcal{L}}_{z_k}(b_k)$ and $\mathcal{L}_{z_k}(b_k)$, so

$$\hat{y}_k = b_k/t_k^p - a(t_k)\cdot z_k + r(t_k,z_k)$$

and

$$y_k = b/t_0^p - a(t_0)\cdot z_k.$$

Now

$$|\hat{y}_k - y_k| = \left|\left(\frac{b_k}{t_k^p} - a(t_k)\cdot z_k\right) - \left(\frac{b_k}{t_0^p} - a(t_0)\cdot z_k\right) + r(t_k, z_k)\right|$$
$$= \mu_l^k \left|\left(\frac{b_0}{t_k^p} - a(t_k)\cdot z_0\right) - \left(\frac{b_0}{t_0^p} - a(t_0)\cdot z_0\right) + \frac{r(t_k, z_k)}{z_k}\cdot z_0\right|.$$

So

$$\frac{|\hat{y}_k - y_k|}{y_k} = \frac{1}{\xi(t_0)}\left|(\xi(t_k) - \xi(t_0)) + \frac{r(t_k, z_k)}{z_k}\cdot z_0\right|,$$

where $\xi(t) = b_0/t^p - a(t)\cdot z_0$. But continuity of ξ and $t_k \to t_0$ force $\xi(t_k) - \xi(t_0) \to 0$ as $k \to \infty$, and we also have

$$\frac{r(t_k, z_k)}{z_k}\cdot z_0 \to 0 \quad \text{as } k \to \infty,$$

so

$$|\hat{y}_k - y_k|/y_k \to 0 \quad \text{as } k \to \infty.$$

Now $d(\mathcal{G}_1^u(\hat{y}_k) - \mathcal{G}_1^u(y_k)) \le |\hat{y}_k - y_k|/y_k$, so $\mathcal{G}_1^u(\hat{y}_k) \to \mathcal{G}_1^u(y_k)$ as $k \to \infty$.

Now take $t_0 \in [\mu_1, 1]$ so that $\mathcal{L}_{z_k}(b_k)$ is *not* critical over t_0, so $|h(t_0) - b_0/z_0| > 0$. Let $p_k \in \mathcal{L}_{z_k}(b_k)$ with $\pi_s(p_k) = t_0$ and $\hat{p}_k \in \hat{\mathcal{L}}_{z_k}(b_k)$ with $\pi_s(\hat{p}_k) = t_0$. Let $\tilde{E}^2_{\hat{p}_k} = T_{\hat{p}_k}\hat{\mathcal{L}}_{z_k}(b_k)$, so

$$\rho_u(\tilde{E}_{p_k}) = \left|\frac{pb_k}{t_0^{p+1}} + a'(t_0)\cdot z_k - \frac{r'(t_0, z_k)}{z_k}\cdot z_k\right|$$
$$\ge \left|\rho_u E^2_{p_k} - \left|\frac{r'(t_0, z_k)}{z_k}\cdot z_0\right| \mu_l^k\right|.$$

But we have seen that $\rho_u E^2_{p_k} \ge c\mu_l^k$ for some positive c, and since

$$|r'(t_0, z_k)/z_k \cdot z_0| \to 0 \quad \text{as } k \to \infty,$$

for large k, $|r'(t_0, z_k)/z_k \cdot z_0| < c/2$. Thus for such k, $\rho_u \tilde{E}_{p_k} \ge (c/2)\mu_l^k$. So we see that for sufficiently large k, all of our arguments work for the general foliation $\hat{\mathcal{L}}$.

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