THE SINGULARITIES OF THE 3-SECANT CURVE ASSOCIATED TO A SPACE CURVE

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ABSTRACT. Let $C$ be a curve in $P^3$ over an algebraically closed field of characteristic zero. We assume that $C$ is nonsingular and contains no plane component except possibly an irreducible conic.

In [GP] one defines closed $r$-secant varieties to $C$, $r \in N$. These varieties are embedded in $G$, the Grassmannian of lines in $P^3$. Denote by $T$ the 3-secant variety (curve), and assume that the set of 4-secants is finite. Let $\tilde{T}$ be the curve obtained by blowing up the ideal of 4-secants in $T$. The curve $\tilde{T}$ is in general not in $G$.

We study the local geometry of $\tilde{T}$ at any point whose fibre of the blowing-up map is reduced at the point. The multiplicity of $\tilde{T}$ at such a point is determined in terms of the local geometry of $C$ at certain chosen secant points. Furthermore we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at a singular point. We also give a criterion for whether all the tangential directions are distinct or not.

1. Introduction. In this paper we consider a nonsingular curve $C$ in $P^3_K$, where $K$ is an algebraically closed field of characteristic 0. We assume that $C$ contains no plane component except possibly an irreducible conic.

There are various ways of studying the scheme of 3-secants to $C$. In [La1] one defines secant schemes from a functorial point of view (see also [La2] where a generalized trisecant lemma is given). In this paper we follow the approach of [GP]. There one studies a curve $T$ in the Grassmannian of lines in $P^3$ whose points correspond to lines intersecting $C$ at least three times, counted with multiplicity.

We will always assume that the set of 4-secants is finite. Let $\tilde{T}$ denote the curve obtained by blowing up $T$ in the scheme of 4-secants. The goal of this paper is to study the local nature of $\tilde{T}$. If a fibre of this blowing-up map is reduced at a point, the dimension of the tangent space of $\tilde{T}$ at this point is at most 2. We give conditions on $C$ that determine the multiplicity of $\tilde{T}$ at such a point (Theorem 2.3.1), and we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at the point (Theorem 2.3.2). We also give a criterion for whether a singularity of $\tilde{T}$ is ordinary or not (Theorem 2.3.3). These results are stated without proofs in §2.

In §3 we consider 3-secants that are not 4-secants, and we sketch a proof of the results of §2 in this case. We show in §4 that at any point of $T$, contained in a fibre of the blowing-up map which is reduced at this point, we are essentially in a situation covered by the discussion in §3.

In §5 we give an application of Theorems 2.3.1, 2.3.2 and 2.3.3 in the case where $C$ is a smooth complete intersection of two cubic surfaces in $P^3$. 

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107
In [WL] Tore Wentzel-Larsen studies the local geometry of secant varieties using
the set-up of [La1] where a 3-secant curve Sec3(C) is defined, which corresponds
bijectively with our curve T. The results in [WL] partly overlap with those in this
paper. In particular [WL] contains the same multiplicity formula as the one in our
Theorem 2.3.1.

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2. Definitions and main results.

2.1. Let G be the Grassmannian of lines in P^3. Given a line L \subset P^3, we denote
by l(L) (or l) the corresponding point in G. Let

\[ F = \{(P, l(L)) \in P^3 \times G | P \in L \}. \]

Let \( p: F \to P^3 \) and \( q: F \to G \) be the natural projection maps, and set \( C = p^{-1}(C) \).

The j-secant scheme associated to C is defined by \( F^{j-1}(q_* \mathcal{O}_C) \), the \((j-1)\)th
Fitting ideal of the \( \mathcal{O}_G \)-module \( q_* \mathcal{O}_C \). For a variety \( X \) over \( K \), and an \( \mathcal{O}_X \)-module
\( \mathcal{F} \) we recall that

\[ V(F^r(\mathcal{F})) = \{ x \in X | \dim_K(\mathcal{F} \otimes K(x)) > r \} \]

and that \( X \setminus V(F^r(\mathcal{F})) \) is the largest open subscheme, where \( \mathcal{F} \) can be generated
locally by \( r \) elements.

We then see that the support of \( F^{j-1}(q_* \mathcal{O}_C) \) is \( \{ l(L) | \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{P^3} \mathcal{O}_L) \geq j \} \). We
denote the 3-secant scheme by \( T \). By the trisecant lemma (see e.g. [M, Ab, An,
Sa]) \( T \) is a curve in \( G \). We always assume that the scheme of 4-secants is finite.

Let \( \phi: \tilde{T} \to T \) denote the blowing-up of the sheaf of ideals \( F^3(q_* \mathcal{O}_C)\mathcal{O}_T \).

2.2. Assume \( \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{P^3} \mathcal{O}_L) = n \geq 3 \) and \( L \cap C = \{P_1, \ldots, P_k\} \). Let

\[ n_i = \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{P^3} \mathcal{O}_L)_{P_i}, \quad i = 1, \ldots, k. \]

Then \( n = \sum_{i=1}^{k} n_i \). It is known [GP, p. 21] that the points of \( \tilde{T} \) in the fibre of \( \phi 
over l(L) \) are in one-to-one correspondence with the set of \( k \)-tuples \((n_{1x}, \ldots, n_{kx})\)
such that \( \sum_{i=1}^{k} n_{ix} = 3 \), and such that \( 0 \leq n_{ix} \leq n_i \) for \( i = 1, \ldots, k \). The notation
\((n_{1x}, \ldots, n_{kx})\) is explained in [GP, p. 21]. Briefly each point \( \phi^{-1}(l(L)) \) corresponds
to an \( x = (x_0, \ldots, x_{n-4}) \in K^{n-3} \) such that \( \Omega_x = x_0 + x_1Z_1 + \cdots + x_{n-4}Z_{n-4} + Z_{n-3} \)
divides \( \prod_{i=1}^{k}(Z - z_i)^{n_i} \) in \( K[Z] \), where \( P_i = (0, 0, z_i, 1), \quad i = 1, \ldots, k \). We have

\[ \prod_{i=1}^{k}(Z - z_i)^{n_{ix}} = \prod_{i=1}^{k}(Z - z_i)^{n_i}/\Omega_x. \]

We denote the point in \( \phi^{-1}(l(L)) \) corresponding to \( x \), or \((n_{1x}, \ldots, n_{kx})\), by \( l_x \). For
a curve \( X \) and a surface \( F \) in \( P^3 \) we denote by \( I(Q, X \cap F) \) the intersection number
of \( X \) and \( F \) at the point \( Q \). The following is well known.

**Proposition 2.2.1** [GP, Proposition 3.4]. \( \tilde{T} \) is singular at a point \( l_x \) iff
there exists a plane \( H \) such that \( I(P_i, C \cap H) \geq 2n_{ix} \) for \( i = 1, \ldots, k \).

We shall show in §4 that for a point \( l_x \) in the fibre \( \phi^{-1}(l) \), the fibre \( \phi^{-1}(l) \) is
reduced at \( l_x \) iff in the corresponding \( k \)-tuple \((n_{1x}, \ldots, n_{kx})\) we have that \( n_{ix} \) is
either 0 or $n_i$ for $i = 1, \ldots, k$. When $\tilde{T}$ is singular at such a point, the plane $H$ described in Proposition 2.2.1 is uniquely determined.

To see this we study the three possible cases:

Case 1: $n_{1x} = n_{2x} = n_{3x} = 1$. $H$ must contain $L$ and the tangent line to $C$ at say $P_1$. This tangent line is not equal to $L$ because that would imply $n_1 \geq 2$.

Case 2: $n_{1x} = 2$, $n_{2x} = 1$. $H$ must contain $L$ and the tangent line to $C$ at $P_2$.

Case 3: $n_{1x} = 3$. $H$ must intersect $C$ at least six times at $P_1$, i.e. $H$ must pass through $P$ and contain the $j$th derivative vector of $C$ at $P_1$, $j = 1, \ldots, 5$, for a suitable local parametrization of $C$ at $P_1$. Since $n_1 = 3$, the first and second derivative vectors are contained in $L$, while the third derivative vector is not.

We then have that the first and third derivative vectors of $C$ at $P_1$ span a unique plane.

2.3. Throughout this section we assume that a point $l_x \in T$ is contained in the fibre $\phi^{-1}(l(L))$ of the blowing-up map, that $\phi^{-1}(l(L))$ is reduced at the point $l_x$, and that $L \cap C = \{P_1, \ldots, P_k\}$. We also assume that $\tilde{T}$ is singular at $l_x$. Let $m_i$ be the intersection number $I(P_i, C \cap H)$ for $i = 1, \ldots, k$, where $H$ is the special plane described in Proposition 2.2.1. Then we have the following.

**THEOREM 2.3.1.** The dimension of the tangent space of $\tilde{T}$ at $l_x$ is 2, and the multiplicity of $\tilde{T}$ at $l_x$ is $m$, where

$$m = \min_{i, n_{ix} \neq 0} \{m_i/n_i\}.$$  

([x] means the integral part of the real number x.)

**DEFINITION.** Let $X$ be a variety, $P$ a point of $X$ and $m$ the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Then the tangent cone of $X$ at $P$ is defined as

$$\tau_P(X) = \text{Spec } \left( \bigoplus_{i=0}^{\infty} \frac{m^i}{m^{i+1}} \right).$$

The projectivized tangent cone of $X$ at $P$ is defined as

$$P\tau_P(X) = \text{Proj } \left( \bigoplus_{i=0}^{\infty} \frac{m^i}{m^{i+1}} \right).$$

By Theorem 2.3.1 we can embed $\tau_{l_x}(\tilde{T})$ in a plane as $m$ lines (counted with multiplicity) through a point.

We show in §4, (a) and (b), that in our situation there is a natural isomorphism

$$\psi: \hat{\mathcal{O}}_{T, l_x} \simeq \mathcal{O}_{G, l(L)}/\mathfrak{A}_x$$

for some ideal $\mathfrak{A}_x$.

Hence we can embed $\tau_{l_x}(\tilde{T})$ as $m$ lines through $l(L)$ in (an affine part of) some plane in $P^5$ via $\psi$ and the Plücker embedding $G \subset P^5$.

For a point $l_x$ satisfying the assumptions at the beginning of §2.3 the isomorphism $\psi$ corresponds to “forgetting what happens outside those $P_i$ such that $n_{ix} = n_i$, and thus pretend that $L$ is a 3-secant but not a 4-secant”. $\psi$ reduces to the obvious isomorphism $\hat{\mathcal{O}}_{T, l_x} \simeq \hat{\mathcal{O}}_{T, l(L)}$ when $L$ is a 3-secant, but not a 4-secant, and the embedded (compactified) tangent cone in $P^5$ is simply the union of the $m$-tangents to $T$ at $l(L)$. 
All of §4 is devoted to giving a precise formulation of the last, rather loose statements, and to proving the precise statements (i.e. (a) and (b) at the beginning of §4).

Let \( L_1, \ldots, L_m \) be the \( m \) (not necessarily distinct) lines in \( P^5 \) determined by the embedding of \( \mathcal{T}_x(T) \) via \( \psi \). We recall the \( H \) is the special plane described in Proposition 2.2.1.

**DEFINITION.** \( \hat{H} \) is the plane in \( G \subset P^5 \), such that the points of \( H \) parametrize the lines in \( H \subset P^3 \).

We will show that each of the lines \( L_1, \ldots, L_m \) are contained in \( \hat{H} \). By plane duality we then have that to a line \( L \) in \( \hat{H} \) through the point \( l(L) \) there corresponds a unique point \( Q \in L \subset P^3 \), such that the points of \( L \) parametrize the lines in \( H \) through \( Q \).

**DEFINITION.** We say that a point \( Q \in L \subset P^3 \) corresponds \( H \)-dually to a tangential direction of \( T \) at \( l_x \) if one of the lines \( L_1, \ldots, L_m \), say \( L_1 \), is contained in \( \hat{H} \), and if the points of \( L_1 \) parametrize the lines in \( H \) through the point \( Q \).

If \( L_1 \) appears at least twice among \( L_1, \ldots, L_m \), we say that \( Q \) corresponds \( H \)-dually to a multiple tangential direction of \( T \) at \( l_x \).

Under the assumptions made at the beginning of §2.3 we now give a closer description of the tangential directions of \( \hat{T} \) at a point \( l_x \). Let the multiplicity of \( T \) at \( l_x \) be \( m \).

**THEOREM 2.3.2.** (i) A tangential direction of \( \hat{T} \) at \( l_x \) always corresponds \( H \)-dually to some point \( Q \in L \).

(ii) \( Q \in L \) corresponds \( H \)-dually to a tangential direction iff there exists a surface \( M \) in \( P^3 \) such that

(a) \( \deg M = m + 1 \), and \( M \) has a singularity of order at least \( m \) at \( Q \).
(b) \( I(P_i, C \cap M) \geq (m + 1)n_{ix} \) for \( i = 1, \ldots, k \).
(c) \( L \not\subset \text{Sing}(M) \), and \( m \cdot L \subset M \cap H_1 \), i.e. \( I(M) \subset (I(H) + I(L)^m) \).
(d) Modulo the square of the ideal defining \( L \), the equation defining \( M \) is equal to the equation of a cone of degree \( m + 1 \) with vertex at \( Q \).

**DEFINITION.** If a curve \( X \) is singular at a point \( P \), we say that the singularity is nonordinary if the projectivized tangent cone \( PTP(X) \) is singular (i.e. if it contains a multiple point).

**THEOREM 2.3.3.** The singularity of \( \hat{T} \) at the point \( l_x \) is nonordinary with \( Q \in L \) corresponding \( H \)-dually to a multiple tangential direction iff there exists a surface \( N \) such that

(a) \( N \) is a cone of degree \( m \) with vertex at \( Q \).
(b) \( I(P_i, C \cap N) \geq (m + 1)n_{ix} \) for \( i = 1, \ldots, k \).
(c) \( L \not\subset \text{Sing}(N) \).

**REMARK.** Theorem 2.3.3 says that a tangential direction is multiple iff some surface \( M \) satisfying the properties of Theorem 2.3.2 breaks up into the union of a cone of degree \( m \) and a plane not containing any of the points of \( L \cap C \).

3. **Proof of the main results for 3-secants that are not 4-secants.**

3.1. If \( l_x \in \hat{T} \) is the unique point in \( \phi^{-1}(l(L)) \), where \( L \) is a 3-secant which is not a 4-secant, then \( \hat{\mathcal{O}}_{\hat{T}, l_x} \) is isomorphic to \( \hat{\mathcal{O}}_{T, l} \). In this situation we prefer to work with \( T \), hence in \( G \) where the computations have natural geometric interpretations.
We shall use the notation of [GP, pp. 15–16], and consider first a secant \( L \) such that \( \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{p_3} \mathcal{O}_L) = n \geq 3 \). (This notation will be used also in §4.) We choose coordinates \( X', Y', Z', W' \) for \( P^3 \), such that \( L \) has equations \( X' = Y' = 0 \), and such that \( L \cap C \) is contained in the affine space \( A^3 \) with equation \( W' \neq 0 \). Let \( A^3 \) be \( \text{Spec} K[X,Y,Z] \), where \( X = X'/W', Y = Y'/W', Z = Z'/W' \). We set \( L \cap C = \{P_1, \ldots, P_k\} \), where \( P_i = (0,0,z_i) \) for \( i = 1, \ldots, k \), and choose the plane \( Z = 0 \) such that

\[
X = \sum_{j \geq n_i} \alpha_{i,j} (Z - z_i)^j, \quad Y = \sum_{j \geq n_i} \beta_{i,j} (Z - z_i)^j
\]

are local parametrizations of \( C \) at \( P_i \), where \( n_i = \text{rk}(\mathcal{O}_C \otimes \mathcal{O}_{p_3} \mathcal{O}_L)_{P_i} \) for \( i = 1, \ldots, k \).

For each \( P_i \in L \cap C \) there is a unique plane \( H_i \) containing \( L \), such that

\[
\text{I}(P_i, C \cap H_i) \geq n_i + 1.
\]

One uses the same argument as at the end of §2.2 to prove this. We can assume that none of these finitely many planes has equation \( X = 0 \), i.e. we assume \( \alpha_{i,n_i} \neq 0 \) for \( i = 1, \ldots, k \). We assume that \( T \) is singular at the point \( l_x \), and that the fibre \( \phi^{-1}(l(L)) \) is reduced at \( l_x \). We choose \( Y = 0 \) as the equation of the unique plane \( H \) described in Proposition 2.2.1.

Let \( R \) be \( \hat{\mathcal{O}}_{G,l} \). We recall the point-line incidence variety \( F \) introduced in §2.1. There exists a regular system of parameters \( (a, b, c, d) \) of \( R \) such that the inverse image of \( F \) in \( A^3 \times \text{Spec} R = \text{Spec} R[X,Y,Z] \) is defined by \( X = a + bZ, Y = c + dZ \). The completion of the local ring of \( C = p_1(C) \) at \( (l, P_i) \) is \( A_i = R[[Z - z_i]]/\mathfrak{a}_i \), where \( \mathfrak{a}_i \) is generated by

\[
f_i = a + bZ - \sum_{j \geq n_i} \alpha_{i,j} (Z - z_i)^j, \quad g_i = c + dZ - \sum_{j \geq n_i} \beta_{i,j} (Z - z_i)^j
\]

for \( i = 1, \ldots, k \). When \( n = 3 \), we have

\[
F^2(q_\ast \mathcal{O}_C)\hat{\mathcal{O}}_{G,l} = F^2 \left( \bigoplus_{i=1}^{k} A_i \right) \hat{\mathcal{O}}_{G,l} = \sum_{i=1}^{k} F^{n_i - 1}(A_i) \hat{\mathcal{O}}_{G,l}.
\]

The last equality follows from the general

\[
F^{r-1} \left( \bigoplus_{i=1}^{k} A_i \right) = \sum_{j_1 + \cdots + j_k = r-1} F^{\chi_1}(A_1) \times \cdots \times F^{\chi_k}(A_k)
\]

(see e.g. [GP, p. 16]). When \( r = \sum_{i=1}^{k} n_i = 3 \), the right-hand side reduces to \( \sum_{i=1}^{k} F^{n_i - 1}(A_i) \) since the \( \hat{\mathcal{O}}_{G,l} \)-module \( A_i \) is generated by \( n_i \) elements for \( i = 1, \ldots, k \).

By Weierstrass’ Preparation Theorem [ZS, pp. 140–141, 145], there is a distinguished polynomial \( S_i \) of degree \( n_i \) in \( Z - z_i \) such that \( S_i \) generates the same ideal as \( f_i \) in \( R[[Z - z_i]] \). There is also a polynomial \( T_i \) of degree \( n_i - 1 \) with coefficients in \( M = (a, b, c, d) \), such that modulo \( S_i \) we have \( T_i \equiv g_i \). The coefficients of the \( T_i \) generate the ideal

\[
\sum_{i=1}^{k} F^{n_i - 1}(A_i) \hat{\mathcal{O}}_{G,l}.
\]
To see this one uses the resolution

\[ R[[Z - z_i]]/S_i \xrightarrow{\mathcal{T}_i} R[[Z - z_i]]/S_i \to R[[Z - z_i]]/(S_i, T_i) \to 0. \]

We obtain a matrix description of the map to the left by choosing \((Z_i - z_i)^j, j = 0, \ldots, n_i - 1\), as an \(R\)-basis of \(R[[Z - z_i]]/S_i\) on both sides. Then the first column of the matrix consists of the coefficients of \(T_i\), and the other entries of the matrix are contained in the ideal generated by the entries in the first column. Hence when \(n = 3\) and \(\phi(l_x) = l(L)\), we have

\[ \hat{\mathcal{O}}_{T,l_x} \cong \mathcal{O}_T,l(L) \cong R/(\text{the coefficients of the } T_i). \]

We have 3 different cases when \(n = 3\):

- **Case 1.** \(L \cap C = \{P_1, P_2, P_3\}\), \(n_1 = n_2 = n_3 = 1\).
- **Case 2.** \(L \cap C = \{P_1, P_2\}\), \(n_1 = 2, n_2 = 1\).
- **Case 3.** \(L \cap C = \{P_1\}\), \(n_1 = 3\).

We prove our main results in Case 1. The proofs of the main results in the two other cases follow the same lines and are contained in [J].

3.2. **Proof of Theorem 2.3.1 in Case 1.** Let \(m = \min_i m_i\), where \(m_i = I(P_i, C \cap H), i = 1, \ldots, k\), and \(H\) is the special plane described in Proposition 2.2.1. Since the equation of \(H\) is \(Y = 0\), \(m_i = \min\{j|\gamma_{ij} \neq 0\}\). We will show that the multiplicity of \(T\) at \(l\) is \(m\). Using Weierstrass’ Preparation Theorem [ZS, pp. 140–141, 145] we obtain

\[ S_i = Z - z_i + s_i,0(a, b), \quad \text{where } s_i,0 \equiv -\frac{a + z_i b}{\alpha_{i,1}} \text{ modulo}(a, b)^2, \]

\[ T_i \equiv c + z_i d - \left(\frac{a + z_i b}{\alpha_{i,1}}\right)^m \beta_{i,m} \text{ modulo}((a, b) \cdot (c, d), (a, b, c, d)^{m+1}). \]

We have \(\hat{\mathcal{O}}_T,l = R/(T_1, T_2, T_3)\), where \(R = k[[a, b, c, d]]\). We denote by \(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\) the images of \(a, b, c, d\) in \(\hat{\mathcal{O}}_T,l\). Let \(\mathcal{M} = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})\). It is clear that \(\mathcal{M} = (\tilde{a}, \tilde{b})\). A little calculation gives that the unique relation between \(\tilde{a}\) and \(\tilde{b}\) modulo \(\mathcal{M}^{m+1}\) is

\[ \sum_{j=0}^{m} \binom{m}{j} \gamma_j \tilde{a}^{m-j} \tilde{b}^j \equiv 0, \]

where

\[ \gamma_j = \alpha_{1,1}^{m_j}(z_2 - z_3)z_1^j + \alpha_{2,1}^{m_j}(z_3 - z_1)z_2^j + \alpha_{3,1}^{m_j}(z_1 - z_2)z_3^j. \]

This implies that the multiplicity of \(T\) at \(l\) is at least \(m\) since the unique relation between \(\tilde{a}\) and \(\tilde{b}\) modulo \(\mathcal{M}^{m+1}\) contains no term of degree less than \(m\).

To prove that the multiplicity is exactly \(m\) we must show that the relation (*) does not vanish identically, i.e., we must show that some \(\gamma_j\) is not equal to zero.

By assumption, \(m \geq 2\). If \(\gamma_0 = \gamma_1 = \gamma_2 = 0\), then \(\beta_{1,m} = \beta_{2,m} = \beta_{3,m} = 0\). By the definition of \(m\) at least one of the \(\beta_i,m\) is not equal to zero. Hence the relation (*) does not vanish identically, and the multiplicity of \(T\) at \(l\) is exactly \(m\).

This proves Theorem 2.3.1 in Case 1.

**Remark 3.2.1.** We also see that \(\tilde{b}^3\) is not a factor in (*). This implies that a point \(Q \in L\), where \(Q \notin C\), does not correspond \(H\)-dually to a triple tangential direction (see §3.3).
3.3. In Cases 1, 2 and 3 we say that a $j'$-fold tangential direction of $T$ at $l$

in Case 1, 2 and 3 we say that a $j'$-fold tangential direction of $T$ at $l$
corresponds $\mathcal{H}$-dually to the point $(0,0,0,1) \in L \subset P^3$ if

$(w' \vec{a} + z' \vec{b})^j$ is a factor in the unique relation between

$\vec{a}$ and $\vec{b}$ modulo $\mathcal{M}_{m+1}$. It is easy to see that for a

point $Q \in L$ this property is independent of how we choose our coordinate system

within the frame given in §3.1 (see §2.3).

**Proof of Theorem 2.3.2 in Case 1.** Since $c \equiv d$ modulo $\mathcal{M}^2$,

all the tangents to $T$ at $l$ are contained in $\mathcal{H}$, the plane in $G$

consisting of all lines in the special plane $H$. This proves (i).

For the proof of (ii) we may assume $Q = (0,0,0,1)$ without loss of generality.

Then a $2$-fold tangential direction to $T$ at $l$ corresponds $\mathcal{H}$-dually to $Q$

iff the coefficient $\gamma_m$ of $(\ast)$ is zero.

$$\gamma_m = \frac{\beta_{1,m}}{\alpha_{1,1}^m} (z_2 - z_3)z_1^m + \frac{\beta_{2,m}}{\alpha_{2,1}^m} (z_3 - z_1)z_2^m + \frac{\beta_{3,m}}{\alpha_{3,1}^m} (z_1 - z_2)z_3^m = 0$$

is equivalent to

$$\text{det} \begin{bmatrix} \beta_{1,m} z_1^m & \alpha_{1,1}^m z_1 & \alpha_{1,1}^m \\ \beta_{2,m} z_2^m & \alpha_{2,1}^m z_2 & \alpha_{2,1}^m \\ \beta_{3,m} z_3^m & \alpha_{3,1}^m z_3 & \alpha_{3,1}^m \end{bmatrix} = 0.$$

Using the local parametrizations of $C$ at the $P_i$ in $L \cap C$ we see that this is equivalent
to the existence of a triple $(r_1, r_2, r_3)$ with $r_1 \neq 0$, such that the surface with affine equation

$$r_1 YZ^m + r_2 XZ^m + r_3 X^m = 0$$

intersects $C$ at least $m + 1$ times at $P_i$ for $i = 1, 2, 3$. On the other hand any surface satisfying property (c) of Theorem 2.3.2(ii) has affine equation

$$Y \cdot f(Z) + XY \cdot g(X, Y, Z) + X^m \cdot h(X, Z) + Y^2 i(Z) = 0$$

for polynomials $f, g, h, i$. By property (a) we can write $f(Z)$ as

$$r_0 Z^{m-1} + r_1 Z^m$$

for scalars $r_0, r_1$, and we can write $h(X, Z)$ as

$$r_2 Z + r_3 + r_4 X$$

for scalars $r_2, r_3, r_4$.

By property (d) we have $r_0 = 0$, and then $r_1 \neq 0$ since $L \not\subset \text{Sing}(M)$ by property (c).

Hence any surface $M$ in question has affine equation

$$r_1 YZ^m + r_2 XZ^m + r_3 X^m = 0 \quad \text{modulo}(XY, Y^2, X^{m+1}).$$

This completes the proof of Theorem 2.3.2.

3.4. **Proof of Theorem 2.3.3.** Again we may assume $Q = (0,0,0,1)$. A

2-fold tangential direction of $T$ at $l$ corresponds $\mathcal{H}$-dually to $Q$

iff $\gamma_{m-1} = \gamma_m = 0$, i.e.

$$\frac{\beta_{1,m}}{\alpha_{1,1}^m} (z_2 - z_3)z_1^m + \frac{\beta_{2,m}}{\alpha_{2,1}^m} (z_3 - z_1)z_2^m + \frac{\beta_{3,m}}{\alpha_{3,1}^m} (z_1 - z_2)z_3^m = 0$$

for $j = m - 1, m$. This is equivalent to

$$\frac{\beta_{1,m}}{\alpha_{1,1}^m} z_1^{m-1} = \frac{\beta_{2,m}}{\alpha_{2,1}^m} z_2^{m-1} = \frac{\beta_{3,m}}{\alpha_{3,1}^m} z_3^{m-1} \quad (=r)$$
which is equivalent to the existence of a scalar \( r \) such that the surface with (affine) equation \( YZ^{m-1} - rX^m = 0 \) intersects \( C \) at least \( m + 1 \) times at \( P_1, P_2, P_3 \). It is enough to show that any surface \( N \) satisfying (a), (b), (c) of Theorem 2.3.3 has equation \( YZ^{m-1} - rX^m = 0 \mod(XY, Y^2) \) for some \( r \in K \). The equation of such a surface \( N \) is obviously homogeneous of degree \( m \). The image of the function \( X^jZ^{m-j} \) is of order \( j \), for \( j = 0, \ldots, m \), in \( O_{C, P_i} \), for each \( P_i \) different from \( Q \). Hence no term of the form \( X^jZ^{m-j} \) occurs in the equation of \( N \) for \( j = 0, 1, \ldots, m - 1 \). Hence the equation of \( N \) is \( r_1 YZ^{m-1} + r_2 X^m = 0 \mod(XY, Y^2) \). Since \( L \not\subset \text{Sing}(N) \), \( r_1 \neq 0 \). Hence we have proved Theorem 2.3.3 in Case 1.

3.5. In Case 2 we use Weierstrass’ Preparation Theorem to obtain the unique relation between \( \ddot{a} \) and \( \ddot{b} \) modulo \( M^{m+1} \):

\[
\begin{align*}
(**) & \quad \det \begin{bmatrix} 1 & z_1 & f_1(\ddot{a}, \ddot{b}) \\ 0 & 1 & f_2(\ddot{a}, \ddot{b}) \\ 0 & z_2 & f_3(\ddot{a}, \ddot{b}) \end{bmatrix} = 0,
\end{align*}
\]

where

\[
\begin{align*}
f_1(\ddot{a}, \ddot{b}) &= \left( \frac{\ddot{a} + z_1 \ddot{b}}{\alpha_{1,2}} \right)^m \cdot \beta_{1,2m}, \\
f_2(\ddot{a}, \ddot{b}) &= \left( \frac{\ddot{a} + z_1 \ddot{b}}{\alpha_{1,2}} \right)^m \cdot \beta_{1,2m+1} + m \left( \frac{\ddot{a} + z_1 \ddot{b}}{\alpha_{1,2}} \right)^{m-1} \\
&\quad \cdot \left( \frac{\ddot{b}}{\alpha_{1,2}} - \frac{\ddot{a} + z_1 \ddot{b}}{\alpha_{1,2}^2} \cdot \alpha_{1,3} \right) \cdot \beta_{1,2m}, \\
f_3(\ddot{a}, \ddot{b}) &= \left( \frac{\ddot{a} + z_2 \ddot{b}}{\alpha_{2,1}} \right)^m \cdot \beta_{2,m},
\end{align*}
\]

and where \( m \) is the number given in Theorem 2.3.1.

In Case 3 the corresponding relation is

\[
\begin{align*}
(\frac{\ddot{a} + z_1 \ddot{b}}{-\alpha_3})^m \cdot \beta_{3m+2} + m \left( \frac{\ddot{a} + z_1 \ddot{b}}{-\alpha_3} \right)^{m-1} \cdot \left( \frac{\alpha_4}{\alpha_3^2} (\ddot{a} + z_1 \ddot{b}) - \frac{\ddot{b}}{\alpha_3} \right) \cdot \beta_{3m+1} \\
&\quad + m \left( \frac{\ddot{a} + z_1 \ddot{b}}{-\alpha_3} \right)^{m-1} \cdot \left( \frac{\alpha_5 \alpha_3 - \alpha_4^2}{\alpha_3^2} (\ddot{a} + z_1 \ddot{b}) + \frac{\alpha_4 \ddot{b}}{\alpha_3^2} \right) \cdot \beta_{3m} \\
&\quad + \left( \frac{m}{2} \right) \left( \frac{\ddot{a} + z_1 \ddot{b}}{-\alpha_3} \right)^{m-2} \cdot \left( \frac{\ddot{a} + z_1 \ddot{b}}{\alpha_3^2} - \frac{\ddot{b}}{\alpha_3} \right)^2 \cdot \beta_{3m} = 0.
\end{align*}
\]

Treating (**) and (***), the same way as (*) in Case 1, we obtain Theorems 2.3.1, 2.3.2 and 2.3.3 in Cases 2 and 3.

4. The last part of the proofs of the main results. We now consider an arbitrary point \( l_x \in \mathcal{T} \), reduced in its fibre over \( l(L) \in \mathcal{T} \). We keep the notation from §§2.1, 2.2 and 3.1, and assume that \( \text{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_p} \mathcal{O}_L) = n \geq 3 \). As before we identify points \( l_x \) in the fibre \( \phi^{-1}(l(L)) \) with \( k \)-tuples \( (n_{1,x}, \ldots, n_{k,x}) \) such that \( 0 \leq n_{i,x} \leq n_i \) for \( i = 1, \ldots, k \), and \( \sum_{i=1}^k n_{i,x} = 3 \), where \( L \cap C = \{P_1, \ldots, P_k\} \) and \( \text{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_p} \mathcal{O}_L)_p = n_i \).

To prove Theorems 2.3.1, 2.3.2 and 2.3.3 it is enough (see next Remark) to show
(a) The fibre $\bar{T} \times_T \text{Spec} \, K(l(L)) = \phi^{-1}(l(L))$ is reduced at $l_x$ if and only if in the corresponding $k$-tuple $(n_{1x}, \ldots, n_{kx})$ we have

$$n_{ix} \text{ is either } 0 \text{ or } n_i \quad \text{for } i = 1, \ldots, k.$$ 

(b) If $n_{ix}$ is either 0 or $n_i$ for $i = 1, \ldots, k$, then $\bar{T}_{l_x} = R/\mathfrak{a}_x$, where $R = \mathcal{O}_G,l$ and $\mathfrak{a}_x$ is the ideal in $R$ generated by those $T_i$ (see §3.1) such that $n_{ix} = n_i$.

REMARK. It is enough to show (a) and (b) because together these statements imply that the local geometry of $T$ at $l_x$ is determined only by the local parametrizations of $C$ at those $P_i$ such that $n_{ix} = n_i \neq 0$, and it is determined in exactly the same way as "if $L$ was a 3-secant only intersecting $C$ in those points $P_i$ where $n_{ix} = n_i \neq 0".

The last is true because the coefficients of each separate $T_i$ depend only on the local behaviour of $C$ at $P_i$. Since $\sum_{i=1}^k n_{ix} = 3$, we are then in a situation already covered by the discussion in §3, i.e. in one of the Cases 1, 2, or 3.

PROOF OF (a) AND (b). We shall use the local description in [GP, pp. 19-23]. Consider the semilocal ring $\bigoplus_{i=1}^k A_i$, where $A_i = \mathcal{O}_C(l_i(P_i))$. We have $\bigoplus_{i=1}^k A_i = R[Z]/(S, D)$, where $S = \prod_{i=1}^k S_i$ and $D = \sum_{i=1}^k T_i \prod_{i \neq j} S_j$. In particular $D$ is a polynomial of degree $n - 1$ in $Z$, with coefficients in $(a, b, c, d) \cdot K[[a, b, c, d]]$. (In [GP] one denotes this polynomial by $T$.)

Let $B$ be the semilocal ring obtained by blowing up $F^3(\bigoplus_{i=1}^k A_i)\bar{T}_{l_i}$ in $\bar{T}_{l_i}$. It is shown in [GP, Proposition 3.3], that $B = R[X_0, \ldots, X_{n-4}]/(V_0, V_1, V_2, r_0, \ldots, r_{n-4})$, where $r_0 + r_1 Z + \cdots + r_{n-4} Z^{n-4}$ is the rest of the Euclidean division of $S$ by $\Omega = Z^{n-3} + X_{n-4} Z^{n-4} + \cdots + X_1 Z + X_0$.

We now describe the $V_i$: For an element $P$ of $R[X_0, \ldots, X_{n-4}][Z]$ denote by $P$ its image in $R[X_0, \ldots, X_{n-4}][Z]/(S)$. The $R[X_0, \ldots, X_{n-4}]$-module $R[X_0, \ldots, X_{n-4}][Z]/(S)$ is free of rank $n$, and it is generated by $Z^j$, $j = 0, \ldots, n - 1$. Let

$$\tau \in \text{Hom}_{R[X_0, \ldots, X_{n-4}]}(R[X_0, \ldots, X_{n-4}][Z]/(S), R[X_0, \ldots, X_{n-4}])$$

be defined by

$$\tau(Z^j) = \delta_{j,n-1} \quad \text{(Kronecker delta function)}$$

for $j = 0, \ldots, n - 1$. Then we have $V_i = \tau(D\Omega Z^i)S$ for $i = 0, 1, 2$. The maximal ideals of $B$ correspond to those $(x_0, \ldots, x_{n-4}) \in K^{n-3}$ such that $\Omega_x = x_0 + x_1 Z + \cdots + x_{n-4} Z^{n-4} + Z^{n-3}$ divides $\prod_{i=1}^n (Z - z_i)^{n_i}$. For such an $(n - 3)$-tuple $(x_0, \ldots, x_{n-4})$ the maximal ideal $\mathfrak{n}_x$ is the image of $(a, b, c, d, X_0 - x_0, \ldots, X_{n-4} - x_{n-4})$ in $B$. When $\Omega_x = \prod_{i=1}^n (Z - z_i)^{n_i - n_{ix}}$, the $k$-tuple corresponding to $l_x$ is $(n_{1x}, \ldots, n_{kx})$. Denote by $r_j$ the image of $r_j$ in $R[X_0, \ldots, X_{n-4}]/(a, b, c, d)$, $j = 0, \ldots, n - 4$. By construction each $r_j$ is contained in $(X_0 - x_0, \ldots, X_{n-4} - x_{n-4})$. Since the coefficients of $D$ are nonunits in $R$, each $V_i$ is a nonunit in $R$, $i = 0, 1, 2$. This implies...
that the fibre $\phi^{-1}(l(L))$ is reduced at $l_x$ if and only if the images of $\bar{r}_0, \ldots, \bar{r}_{n-4}$ span the vector space

$$(X_0 - x_0, \ldots, X_{n-4} - x_{n-4})/(X_0 - x_0, \ldots, X_{n-4} - x_{n-4})^2.$$ 

By Lemma 3.5 of [GP, p. 22], this happens if and only if the polynomials

$$\prod_{i=1}^{k} (Z - z_i)^{n_i} \quad \text{and} \quad \prod_{i=1}^{k} (Z - z_i)^{n_i - n_{ix}}$$

have no common factor, i.e. $n_{ix}$ is either 0 or $n_i$, $i = 1, \ldots, k$. This proves (a).

Under the assumption of (b) this lemma also gives

$$R \cong R_x,$$

where $R_x$ is the ring

$$[R[X_0, \ldots, X_{n-4}]/(r_0, r_1, \ldots, r_{n-4})] \quad \text{localized in the ideal}$$

$$(a, b, c, d, X_0 - x_0, \ldots, X_{n-4} - x_{n-4}).$$

Hence $B_{n_x} \cong R_x/(V_0, V_1, V_2)$. Since $R_x$ is a regular local ring, $R_x[Z]$ is a UFD. Identifying all polynomials with their images in $R_x[Z]$, we have in $R_x[Z]$

$$S = S'S'' = \Omega \Delta,$$

where $S' = \prod_{i,n_{ix}=0} S_i$, $S'' = \prod_{i,n_{ix}\neq 0} S_i$, and $\Delta$ is some power series. Since $\Omega_x$ and $\Delta_x = \prod \lambda(z - z_i)^{n_{ix}}$ have no common factors, we obtain $S' = \Omega$ in $R_x[Z].$

Hence $V_i = \tau(DS'Z')$, $i = 0, 1, 2$, and one easily obtains that $(V_0, V_1, V_2)$ generates the same ideal as the coefficients of those $T_i$ where $n_{ix} \neq 0$. Since $\sum_{i=1}^{k} n_{ix}$, we are in a situation already covered by the discussion in §3.

A more detailed proof of (a) and (b) is contained in [J].

5. Examples and remarks. Let $C$ be a smooth complete intersection of two cubic surfaces. Then $C$ contains no line since a complete intersection is connected, and since a reducible connected curve is singular.

Furthermore $C$ possesses no 4-secants since a 4-secant would have been contained in both cubic surfaces by Bezout’s theorem, and then $C$ would have contained a line. Assume $L$ is a 3-secant corresponding to a point $l \in T$.

As a consequence of Proposition 2.2.1, the following three statements are equivalent:

(a) $T$ is singular at $l$.
(b) There exists a plane $H$ and a cubic surface $F_3$ containing $C$ such that $H \cap F \supseteq \Omega \Delta$, where $L_2$ is the double line in $H$ with support $L$.
(c) There exists a cubic surface $F_3$ containing $C$ and having two (possibly coinciding) singularities on $L$.

If $T$ is singular at $l$, the surface $F_3$ of (b) is the same as the surface $F_3$ of (c).

Furthermore, as a consequence of the results in §2, we have: The multiplicity of $T$ at the point $l$ is 3 iff there exists a plane $H$ and a cubic surface $F_3$ containing $C$ such that $H \cap F_3 = L_3$, where $L_3$ is the triple line in $H$ with support $L$. The multiplicity of $T$ at any point $l$ is at most 3.

Assume that the multiplicity of $T$ at $l$ is 2. Then $l$ is a nonordinary singularity on $T$ iff $H \cap F = L_2 \cup L'$, where $L'$ intersects $L$ in one of the singular points of $F_3$ on $L$. Then the unique double tangential direction of $T$ at $l$ corresponds $H$-dually to the other singular point of $F_3$ on $L$. 

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If the two singularities of $F_3$ on $L$ coincide at a point $Q$, but $L' \cap L \neq \{Q\}$, then a single tangential direction corresponds $H$-dually to $Q$.

Assume the multiplicity of $T$ at $l$ is 3. Then $l$ is a nonordinary singularity on $T$ if the two singular points of $F_3$ on $L$ coincide. In this case a double, but not a triple tangential direction of $T$ at $l$ corresponds $H$-dually to the unique singular point of $F_3$ on $L$.

**Remarks.** (1) If we study "the initial forms" (*), (**) and (***) of §3 more closely, we can say more about the local geometry of $\tilde{T}$ at a point $l_x$ whose fibre $\phi^{-1}(l(l_x))$ is reduced at $l_x$. When $m \geq 3$, we have:

The singularity is totally nonordinary, i.e. all tangential directions coincide iff

$$\sum_{i,n_{ix} = n_i} [(m + 1)n_{ix} - \min(m_i, (m + 1)n_{ix})] = 1,$$

where $m_i = I(P_i, C \cap H)$ for the special plane $H$. Then the point corresponding $H$-dually to the $m$-fold tangential direction is the unique $P_i \in L \cap C$ such that $m_i = (m + 1)n_{ix} - 1$.

(2) Using essentially the same methods as in the proofs of the results in §2.3 we may examine the local geometry of $T$ at nonsingular points of $T$. Let $\mathcal{L}$ be the tangent line to $T$ at a nonsingular point $l(l)$. Then we have, for example, $l$ is a flex on $T$, i.e. $\text{rk}(\mathcal{O}_T \otimes \mathcal{O}_T) \geq 3$, iff $L \cap C = \{P_1, P_2, P_3\}$, $n_1 = n_2 = n_3 = 1$, and $I(P_1, C \cap H) = 1$, $I(P_j, C \cap H) \geq 3$, $j = 2, 3$, for some plane $H$, or: $L \cap C = \{P_1, P_2\}$, $n_1 = 1$, $n_2 = 2$, and $I(P_1, C \cap H) = 1$, $I(P_2, C \cap H) \geq 6$, for some plane $H$.

(3) When the fibre $\phi^{-1}(l)$ is not reduced at a point $l_x \in \tilde{T}$, the multiplicity formula given in Theorem 2.3.1 is no longer valid, although Proposition 2.3.1 holds. In addition the dimension of the tangent space of $\tilde{T}$ at $l_x$ may be 3.

As mentioned in §1 there is a (set-theoretical) bijection between $\tilde{T}$ and the curve $\text{Sec}_3(C)$ defined in [Lai]. In [WL] one shows that the same multiplicity formula as given in our Theorem 2.3.1 holds for points of $\text{Sec}_3(C)$ corresponding to the points of $\tilde{T}$, at which their respective fibres of the blowing-up map $\phi: \tilde{T} \to T$ are reduced. In fact the initial forms are essentially the same.

It is an open question whether the same is true for the remaining points (if any) of $\text{Sec}_3(C)$ and $\tilde{T}$.

**References**


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