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EMBEDDING STRICTLY PSEUDOCONVEX DOMAINS INTO BALLS

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ABSTRACT. Every relatively compact strictly pseudoconvex domain D with $\mathbf{C}^2$ boundary in a Stein manifold can be embedded as a closed complex submanifold of a finite dimensional ball. However, for each $n \geq 2$ there exist bounded strictly pseudoconvex domains D in $\mathbb{C}^n$ with real-analytic boundary such that no proper holomorphic map from D into any finite dimensional ball extends smoothly to $\overline{D}$.

0. Introduction. In this paper we study the representations of bounded strictly pseudoconvex domains $D \subset \mathbb{C}^n$. If the boundary of D is of class $\mathbf{C}^k$, $k \in \{2, 3, \ldots, \infty\}$, then, by a theorem of Fornaess [**8**] and Khenkin [**13**], D can be mapped biholomorphically onto the intersection $X \cap \Omega$ of a bounded strictly convex domain $\Omega \subset \mathbb{C}^N$ with $\mathbf{C}^k$ boundary and a closed complex submanifold X defined in a neighborhood of $\overline{\Omega}$ in $\mathbb{C}^N$, X intersecting the boundary of Ω transversally. Moreover, the map $f: D \to X \cap \Omega$ extends to a holomorphic map on a neighborhood of $\overline{D}$. The convex domain Ω depends on D; hence a natural question is whether a similar result holds with Ω replaced by the unit ball

$$\mathbb{B}^N = \left\{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\|^2 = \sum_{j=1}^{N} |z_j|^2 < 1 \right\},$$

provided that N is sufficiently large. Since the sphere $b\mathbb{B}^N$ is real-analytic and the intersection $X \cap b\mathbb{B}^N$ is transversal, the boundary of D must necessarily be real-analytic in order for such a representation to exist. Thus our question is:

(Q) *Given a bounded domain D in $\mathbb{C}^n$ with real-analytic and strictly pseudoconvex boundary, does there exist a complex submanifold X of $\mathbb{C}^N$ for some $N \geq n$ that intersects $b\mathbb{B}^N$ transversally such that D is biholomorphically equivalent to $X \cap \mathbb{B}^N$?*

This question has been mentioned by Lempert [**14**], Pinčuk [**20**], Bedford [**2**] and others. Our main result is that the answer to this question is negative in general. If D is as above, then every biholomorphism of D onto $X \cap \mathbb{B}^N$ extends smoothly to $\overline{D}$ according to [**3**]. However, we will show that not all such domains D admit a proper holomorphic map into a finite dimensional ball that is smooth on $\overline{D}$ (Theorem 1.1). A similar local result was obtained independently by Faran [**7**]. We shall show that the answer to the question (Q) is positive if we allow the intersections of complex submanifolds with strictly convex domains $\Omega \subset \mathbb{C}^N$ with real-analytic boundaries (Theorem 1.2). The theorems of Fornaess [**8**] and Khenkin [**13**] only give an Ω with smooth boundary.

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If we do not require that the embedding be smooth to the boundary of D, then the answer is positive: Every relatively compact strictly pseudoconvex domain with $\mathbf{C}^2$ boundary in a Stein manifold can be embedded into a ball (Corollary 1.4). Theorem 1.3 shows that, for every sufficiently big N, there are plenty of proper maps from D into $\mathbf{B}^N$, and these maps need not extend continuously to the boundary of D. The same result was proved simultaneously by Løw [**16**], who showed that the embedding can be made continuous on $\overline{D}$. He also showed that every such domain can be embedded into a polydisk [**15, 16**].

The paper is organized as follows. In §1 we state our main results. In §2 we prove a local version of the nonembedding result, and in §3 we prove Theorem 1.1. Theorem 1.2 is proved in §4. In §§5–7 we prove Theorem 1.3.

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1. Results. Our first main result is that not all strictly pseudoconvex domains with real-analytic boundary can be mapped properly into a finite dimensional ball if the map is to be smooth up to the boundary. More precisely, we have

1.1 THEOREM. *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with $\mathbf{C}^s$ boundary, $s \geq 1$. In every neighborhood of Ω in the $\mathbf{C}^s$ topology on domains there exists a domain D with smooth real-analytic boundary such that no proper holomorphic map $f: D \to \mathbf{B}^N$, $N \geq n$, extends smoothly to $\overline{D}$.*

Here, smooth means $\mathbf{C}^\infty$. Note that if a proper map $f: D \to \mathbf{B}^N$ is to exist, D must be pseudoconvex.

This theorem shows that the answer to the question (Q) above is negative in general. The main idea involved in the proof of Theorem 1.1 is to study the conditions on the power series defining a germ of a real hypersurface M in $\mathbb{C}^n$ which imply that M admits a C-R embedding into a sphere $b\mathbf{B}^N$. For each fixed integer N there are obstructions for such embeddings in terms of the Taylor polynomial of M of sufficiently high order. We then use the Baire category theorem to find a germ M that does not embed into any sphere (Theorem 2.2). A similar method was used by Faran [**7**]. It is not clear whether any of these germs extends to a compact real analytic hypersurface in $\mathbb{C}^n$, but we can use the same idea to prove the global nonembedding result (§3). The obstructions on the power series defining M do not require convergence, and hence they also apply to the $\mathbf{C}^\infty$ case. This idea goes back to Poincaré [**21**], who used it to argue that there are biholomorphic invariants of real hypersurfaces in $\mathbb{C}^n$, $n \geq 2$. He found these invariants in the case $n = 2$. Chern and Moser found the invariants in the general case [**6**].

We could seek a positive answer to the question (Q) for a more restricted class of domains. In particular, we do not know what the answer to (Q) is for domains in $\mathbb{C}^n$ with strictly pseudoconvex real-algebraic boundary, i.e., for domains whose defining function is a real polynomial.

A result of Pinčuk [**20**, Theorem 6.2] implies that for a bounded domain D in $\mathbb{C}^n$, $n \geq 2$, with real-analytic, strictly pseudoconvex and simply connected boundary,

the following assertions are equivalent:

(i) *there exist a neighborhood U of a point $p \in bD$ in $\mathbb{C}^n$ and a smooth map $f: U \to \mathbb{C}^N$ that is holomorphic and nonconstant on $U \cap D$ such that*

$$f(U \cap D) \subset \mathbb{B}^N \quad \text{and} \quad f(U \cap bD) \subset b\mathbb{B}^N,$$

(ii) *there exists a proper holomorphic map $f: D \to \mathbb{B}^N$ that is smooth on $\overline{D}$.*

In fact, Pinčuk proved that a map f satisfying (i) continues analytically along every path in bD starting at p. Since bD is simply connected, we get a well-defined C-R map $f: bD \to b\mathbb{B}^N$ which extends to a holomorphic map on D. Thus the local embedding problem is equivalent to the global problem (Q) when the boundary of D is simply connected. One can see that the same is true if D is simply connected [**9**, pp. 37–39].

If we allow the intersections with arbitrary strictly convex domains with real-analytic boundary, then the answer to the question (Q) is positive. The following theorem is an extension to the real-analytic case of the embedding theorem of Fornaess [**8**, p. 543] and Khenkin [**13**, p. 668].

1.2 THEOREM. *Let D be a relatively compact strictly pseudoconvex domain in a Stein space S, and suppose that the boundary bD is a real-analytic submanifold of S consisting of regular points of S. Then there exist an open neighborhood U of $\overline{D}$ in S, a holomorphic map $F: U \to \mathbb{C}^N$ for some integer N, and a bounded strictly convex domain Ω in $\mathbb{C}^N$ with smooth real-analytic boundary such that the following hold:*

(i) *F is biholomorphic onto a closed subvariety X of $\mathbb{C}^N$,*

(ii) *$F(D) = X \cap \Omega$ and $F(U \backslash \overline{D}) = X \backslash \overline{\Omega}$, and*

(iii) *X intersects $b\Omega$ transversally.*

We now turn to the problem of embedding strictly pseudoconvex domains into balls, dropping the requirement that the map be regular on the boundary. Our main result is the following.

1.3 THEOREM. *If D is a bounded strictly convex domain with $\mathbf{C}^2$ boundary in $\mathbb{C}^n$, there is a positive integer s with the following property. If*

$$h = (h_1, h_2, \ldots, h_p): D \to \mathbb{C}^p$$

is a holomorphic map such that $\|h\|^2 = \sum_{j=1}^p |h_j|^2$ extends continuously to $\overline{D}$ and $\|h(z)\| < 1$ for all $z \in \overline{D}$, then there is a holomorphic map $f = (f_1, \ldots, f_{2s})$ of D to $\mathbb{C}^{2s}$ such that $F = (f, h): D \to \mathbb{C}^{2s+p}$ maps D properly into the unit ball $\mathbb{B}^{2s+p}$.

If we apply Theorem 1.3 to the map $h(z) = \varepsilon z$ for a small $\varepsilon > 0$, we obtain a proper holomorphic embedding of D into $\mathbb{B}^N$, $N = n + 2s$. The embedding theorem of Fornaess [**8**, p. 543] and Khenkin [**13**, p. 668] implies the following result.

1.4 COROLLARY. *If X is a Stein space and D a relatively compact strictly pseudoconvex domain in X whose boundary bD is of class $\mathbf{C}^2$ and is contained in the set of smooth points of X, then D can be mapped biholomorphically onto a closed complex subvariety of a ball $\mathbb{B}^N$.*

In particular, every bounded strictly pseudoconvex domain D of class $\mathbf{C}^2$ in $\mathbb{C}^n$ can be embedded properly into a high dimensional ball. Previously Lempert

proved [**14**] that every such D can be embedded into the unit ball of the infinite dimensional Hilbert space l_2.

The integer N depends on the domain D. More precisely, if D is strictly convex, N depends on the curvature of bD, and N will blow up if we deform D to a weakly pseudoconvex domain. However, if we have a family of strictly convex domains that form a compact set in the $\mathbf{C}^2$ topology on domains, all domains in the family embed in the same ball $\mathbb{B}^N$ (see §5). For example, all domains D which are sufficiently close to the ball $\mathbb{B}^n$ in the $\mathbf{C}^2$ sense embed in the same ball. It is known that most such domains are pairwise biholomorphically inequivalent [**4**].

The idea of this construction of proper mappings is due to Løw [**15**]. It is very similar to the construction of inner functions due to Hakim and Sibony [**12**] and Løw [**17**]. After this work has been completed I received the preprint [**16**] from Løw which contains a similar result.

Using our Theorem 1.3 one can construct proper maps of balls that do not extend continuously to the boundary. The question whether such proper maps exist was mentioned by Bedford [**2**, p. 171].

1.5 COROLLARY. *For each integer $n \geq 1$ there is a proper holomorphic embedding $F: \mathbb{B}^n \to \mathbb{B}^N$, $N = n + 1 + 2s$, where $s = s(n)$ is as in Theorem* 1.3, *such that F does not extend continuously to $\overline{\mathbb{B}}^n$.*

To construct such an embedding, we choose a continuous real-valued function u on the circle $b\Delta$ whose harmonic conjugate v is not continuous, and we let g be the holomorphic function on Δ with boundary values e^{u+iv}. We extend g to a holomorphic function on $\Delta \times \mathbb{C}^{n-1}$ by letting g be constant in the other variables, and we put $h(z) = \varepsilon(z, g(z))$ for $z \in \mathbb{B}^n$ and $\varepsilon > 0$ sufficiently small. Then $\|h\|$ extends continuously to $\overline{\mathbb{B}}^n$ but h does not. We apply Theorem 1.3 to obtain a holomorphic map $f: \mathbb{B}^n \to \mathbb{B}^{2s}$ such that $F = (h, f): \mathbb{B}^n \to \mathbb{B}^N$ is proper. Clearly F is an embedding that does not extend continuously to $\overline{\mathbb{B}}^n$.

It would be interesting to know the minimal dimension $N(n)$ for which there exist proper holomorphic maps $F: \mathbb{B}^n \to \mathbb{B}^{N(n)}$ which do not extend continuously to $\overline{\mathbb{B}}^n$. Recently Hakim and Sibony announced that they have constructed such maps for $N(n) = 4n$. Could it be $N(n) = n + 1$?

Globevnik and Stout [**10**] constructed a proper map $g: \Delta \to \mathbb{B}^2$ whose cluster set $E_g = \overline{g(\Delta)} \backslash g(\Delta) \subset b\mathbb{B}^2$ is the entire three-sphere $b\mathbb{B}^2$. More generally, E_g can be the closure of any open connected subset Ω of $b\mathbb{B}^2$. There exist similar proper maps from Δ to higher dimensional balls. If we apply Theorem 1.3 to the map g, we obtain a proper map from $\mathbb{B}^n$ into a ball whose restriction to the disk $\Delta \times (0) \subset \mathbb{B}^n$ has large cluster set.

It is an interesting question how large can the cluster set $E_f = \overline{f(\mathbb{B}^n)} \backslash f(\mathbb{B}^n)$ of a proper map $f: \mathbb{B}^n \to \mathbb{B}^N$ be. Can we have $E_f = b\mathbb{B}^N$ when $n > 1$?

In a different direction we can ask whether every bounded strictly pseudoconvex domain can be embedded into a ball by a holomorphic map that is of class $\mathbf{C}^k$ on $\overline{D}$ for some low value of k, say $k = 1$ or $k = 2$.

In view of Corollary 1.4 we may ask whether smoothly bounded *weakly* pseudoconvex domains also embed into balls. If we remove the smoothness hypothesis, the answer is no. The domain $D = \{(z, w) \in \mathbb{C}^2 : 0 < |z| < |w| < 1\}$ is pseudoconvex, but every bounded holomorphic function on D extends to a holomorphic function

on the bidisk $\Delta^2 = \{(z,w): |z|, |w| < 1\}$. This implies that D does not embed properly into a ball.

A necessary condition for a complex manifold M to embed into a ball is that the bounded holomorphic functions $\mathbf{H}^\infty(M)$ separate points in M, that each point $z \in M$ has a local chart $(f_1, \ldots, f_n)$ consisting of functions $f_j \in \mathbf{H}^\infty(M)$, and that the $\mathbf{H}^\infty(M)$-hull

$$\hat{K} = \left\{ z \in M: |f(z)| \leq \sup_k |f| \text{ for all } f \in \mathbf{H}^\infty(M) \right\}$$

of each compact subset K of M is itself compact. These are precisely the conditions which define a Stein manifold, except that we have replaced the algebra of all holomorphic functions on M by the algebra of bounded functions. Does every manifold of this kind embed into a ball?

2. A local nonembedding result. Let $M \subset \mathbb{C}^n$, $n \geq 2$, be a germ of a real hypersurface at the point 0 in $\mathbb{C}^n$. We choose the coordinates $z = (z_1, \ldots, z_{n-1})$ and $w = u + iv$ in $\mathbb{C}^n$ such that the tangent space of M at 0 and its maximal complex subspace are

$$\mathbf{T}_0 M = \{v = 0\} \quad \text{and} \quad \mathbf{T}_0^{\mathbb{C}} M = \{w = 0\},$$

respectively.

We shall assume that M is of class $\mathbf{C}^\infty$. The equation of M may be written in the form

$$v = F(z, \overline{z}, u), \tag{2.1}$$

where F is a real-valued smooth function that vanishes at 0 together with its first order partial derivatives. The germ of $M = M_F$ is strictly pseudoconvex at 0 if and only if the $(n-1) \times (n-1)$ Hermitian matrix

$$\left(\frac{\partial^2 F}{\partial z_j \partial \overline{z}_k}(0) \right)_{j,k=1}^{n-1} \tag{2.2}$$

is positive definite.

Since F is smooth, we may associate to it the Taylor series

$$F \sim \sum_{\substack{\alpha,\beta \in \mathbf{Z}_+^{n-1} \\ j \in \mathbf{Z}_+}} F_{\alpha,\beta,j} z^\alpha \overline{z}^\beta u^j, \qquad F_{\alpha,\beta,j} \in \mathbb{C}, \tag{2.3}$$

satisfying the conditions:

$$F_{\alpha,\beta,j} = 0 \quad \text{if } |\alpha| + |\beta| + j \leq 1, \tag{2.4}$$

$$\overline{F}_{\alpha,\beta,j} = F_{\beta,\alpha,j}. \tag{2.5}$$

Conversely, a theorem of Borel [**19**, p. 30] asserts that for each formal power series of the form (2.3) there exists a smooth function F whose Taylor series at 0 is given by (2.3). If the series (2.3) also satisfies the condition (2.5), the corresponding F may be chosen real-valued. Such an F is not uniquely determined by its Taylor series at 0. However, a convergent series (2.3) uniquely determines a real-analytic function in a neighborhood of 0.

We denote by $\mathcal{F}$ the algebra of all formal power series of the form (2.3) that satisfy the conditions (2.4) and (2.5). Let $\mathcal{F}^\nu$, $\nu \in \mathbb{Z}_+$, be the subset consisting of all $F \in \mathcal{F}$ for which $F_{\alpha,\beta,j} = 0$ whenever $|\alpha| + |\beta| + j > \nu$. In other words, $\mathcal{F}^\nu$ is the set of all polynomials of order not exceeding ν contained in $\mathcal{F}$. We denote by P^ν the canonical projection of $\mathcal{F}$ onto $\mathcal{F}^\nu$.

As a set we may identify $\mathcal{F}$ with a subset of $\mathbb{C}^{\mathbb{Z}_+^{2n-1}}$, and we can use the product topology to induce a topology on $\mathcal{F}$. It is the metrizable topology induced by the seminorms d_{a_0} given, for each multi-index $a_0 = (\alpha_0, \beta_0, j_0)$, by

$$d_{a_0}\left(\sum F_{\alpha,\beta,j} z^\alpha \overline{z}^\beta u^j\right) = |F_{\alpha_0,\beta_0,j_0}|.$$

Thus, $\mathcal{F}$ is a Fréchet space [**22**, p. 8]. The relative topology induced on $\mathcal{F}^\nu$ agrees with the standard topology of a finite dimensional vector space. Each projection $P^\nu\colon \mathcal{F} \to \mathcal{F}^\nu$ is continuous and open.

For each $t = (t_1, \dots, t_{2n-1}) \in \mathbb{R}_+^{2n-1}$ we define the t-norm of

$$F = \sum F_{\alpha,\beta,j} z^\alpha \overline{z}^\beta u^j$$

to be the nonnegative number

$$\|F\|_t = \sum |F_{\alpha,\beta,j}| t^{(\alpha,\beta,j)}.$$

We denote by $\mathcal{F}_t$ the set of all $F \in \mathcal{F}$ for which $\|F\|_t < \infty$. Then $\|\cdot\|_t$ is a norm on the vector space $\mathcal{F}_t$ that makes $\mathcal{F}_t$ into a Banach algebra [**11**, p. 15]. If $t \in \mathbb{R}_+$, we shall also write $\mathcal{F}_t = \mathcal{F}_{(t,\dots,t)}$. The inclusions $\mathcal{F}^\nu \hookrightarrow \mathcal{F}_t$ and $\mathcal{F}_t \hookrightarrow \mathcal{F}$ are continuous. For each integer ν the projection $P^\nu\colon \mathcal{F}_t \to \mathcal{F}^\nu$ is onto, continuous and open. The condition of strict pseudoconvexity (2.2) defines an open subset in $\mathcal{F}$ and in each $\mathcal{F}_t$.

We shall now explain what we mean by an embedding of a hypersurface into a sphere. We will distinguish three types of embeddings: *holomorphic*, $\mathcal{A}^\infty$ and *formal* embeddings.

2.1 DEFINITION. (a) The germ at 0 of a real hypersurface $M = \{v = F(z, \overline{z}, u)\}$ admits a *holomorphic embedding* into a sphere if there are a neighborhood U of 0 in $\mathbb{C}^n$, an integer $k \in \mathbb{Z}_+$ and a holomorphic map $f\colon U \to \mathbb{C}^k$ such that the following hold:

(i) $f(U \cap M) \subset b\mathbb{B}^k$,

(ii) the rank of $df(0)$ equals n, and

(iii) f is transversal to $b\mathbb{B}^k$ at 0.

(b) The germ of M at 0 admits an $\mathcal{A}^\infty$*-embedding* into a sphere if there exists a neighborhood U of 0 in $\mathbb{C}^n$ and a $\mathbf{C}^\infty$ map $f\colon U \to \mathbb{C}^k$ that is holomorphic on either $\{v > F\} \cap U$ or on $\{v < F\} \cap U$ such that the conditions (i)–(iii) of part (a) hold.

(c) Let $F \in \mathcal{F}$ be a formal power series. The formal hypersurface M_F admits a *formal embedding* into a sphere if there exists a formal power series $f = (f_1, \dots, f_k)$ in the variables $(w, z_1, \dots, z_{n-1})$ for some $k \in \mathbb{Z}_+$ such that

$$\sum_{j=1}^k f_j(u + iv, z) \cdot \overline{f_j(u + iv, z)} = 1$$

is a formal power series identity when we replace v by the series $F(z, \overline{z}, u)$, and the conditions (ii) and (iii) of part (a) hold.

Notice that the conditions (ii) and (iii) in part (a) are meaningful in the context of formal power series, as is the condition (2.2) of strong pseudoconvexity. Clearly a holomorphic embedding is also an $\mathcal{A}^\infty$ embedding, and an $\mathcal{A}^\infty$ embedding gives rise in a natural way to a formal embedding. A germ M that admits a holomorphic embedding is necessarily real-analytic and strictly pseudoconvex, and one that admits an $\mathcal{A}^\infty$ embedding is of class $\mathbf{C}^\infty$ and strictly pseudoconvex.

Our aim is to show that very few germs of hypersurfaces admit even a formal embedding into a sphere. The theorem below is similar to a result of Faran [**7**].

2.2 THEOREM. *The set $\mathcal{S}$ of all $F \in \mathcal{F}$ for which the germ at 0 of the formal hypersurface $M_F = \{v = F(z, \overline{z}, u)\}$ admits a formal embedding into a sphere is a set of the first category in the Fréchet space $\mathcal{F}$. Moreover, for each $t \in \mathbf{R}_+^{2n-1}$, $\mathcal{S} \cap \mathcal{F}_t$ is a set of the first category in the Banach space $\mathcal{F}_t$.*

PROOF. Write $k = n + l$, $l \in \mathbf{Z}_+$, and let the coordinates on $\mathbf{C}^k$ be $w = u + iv$, $z = (z_1, \ldots, z_{n-1})$, $\varsigma = (\varsigma_1, \ldots, \varsigma_l)$. Instead of the unit sphere $b\mathbf{B}^k$ it will be more convenient for us to use the Heisenberg group $S \subset \mathbf{C}^k$ given by the equation

$$v = \sum_{j=1}^{n-1} |z_j|^2 + \sum_{j=1}^{l} |\varsigma_j|^2 \tag{2.6}$$

that is biholomorphically equivalent to $b\mathbf{B}^k$ minus a point via the Cayley transform [**22**, p. 31].

Let $f = (f_1, \ldots, f_k)$ be a formal power series in the variables (w, z) that represents a formal embedding of a real hypersurface $M \in \mathbf{C}^n$ into S. Composing f with an automorphism of S we may assume that

$$f(0) = 0 \quad \text{and} \quad df(0)(\mathbf{C}^n) = \{\varsigma_1 = \cdots = \varsigma_l = 0\}.$$

(See [**1**, p. 277] for a description of the automorphism group of S.) After a formal holomorphic change of coordinates on $\mathbf{C}^n$ at 0 we can represent the same embedding in the form

$$w = w, \qquad z = z, \qquad \varsigma_j = f_j(w, z), \quad 1 \le j \le l, \tag{2.7}$$

where the f_j's are formal power series in (w, z) with no constant or linear terms. Substituting (2.7) into (2.6) gives the equation

$$v = \sum_{j=1}^{n-1} |z_j|^2 + \sum_{j=1}^{l} |f_j(u + iv, z)|^2. \tag{2.8}$$

This is not of the form (2.1) since the right-hand side contains v. However, it does not contain any linear terms in v, and hence the tangent space $\mathbf{T}_0 M$ of the formal hypersurface defined by (2.8) is $\{v = 0\}$. We can bring the equation (2.8) in the form (2.1) by solving (2.8) for v using iteration:

$$v_0 = 0, \qquad v_{r+1} = \sum_{j=1}^{n-1} |z_j|^2 + \sum_{j=1}^{l} |f_j(u + iv_r, z)|^2, \quad r \in \mathbf{Z}_+. \tag{2.9}$$

Notice that for each $\nu \in \mathbf{Z}_+$ we ony need finitely many iterations (2.9) to find the terms of order not exceeding ν in the series (2.1). Thus, the iteration converges in the Fréchet space $\mathcal{F}$.

We choose an $F \in \mathcal{F}$ and ask whether we can find a formal power series $f = (f_1, \dots, f_l)$ such that the hypersurface (2.8) is equivalent to the one given by (2.1) under a change of coordinates

$$w = h_0(w^*, z^*), \qquad z_j = h_j(w^*, z^*), \quad 1 \le j \le n-1, \tag{2.10}$$

where $h = (h_0, h_1, \dots, h_{n-1})$ are formal power series in the coordinates w^*, z^* with no constant terms and rank of $dh(0)$ equal to n. Since the change of coordinates (2.10) should preserve the tangent space of M at 0 given by $\{v = 0\}$, h_0 is of the form

$$h_0(w^*, z^*) = cw^* + g(w^*, z^*), \tag{2.11}$$

where $c \in \mathbf{R}\backslash\{0\}$ and g has no constant or linear terms. We factor h as $h = \phi \circ \tilde{h}$, where ϕ is given by

$$w \to cw, \qquad z \to \sqrt{|c|}z, \tag{2.12}$$

and $\tilde{h}_0$ is of the form (2.11) with $c = 1$.

Substituting (2.12) into (2.9) we obtain

$$cv = |c| \sum_{j=1}^{n-1} |z_j|^2 + \sum_{j=1}^{l} \left| f_j\left(cw, \sqrt{|c|}z\right)\right|^2.$$

Divide by c and let $\varepsilon = |c|/c = \pm 1$:

$$v = \varepsilon \sum_{j=1}^{n-1} |z_j|^2 + \varepsilon \sum_{j=1}^{l} \left| \frac{1}{\sqrt{|c|}} f_j\left(cw, \sqrt{|c|}z\right)\right|^2.$$

This shows that up to the sign $\varepsilon = \pm 1$ we may absorb c into f. It suffices to consider the strictly pseudoconvex hypersurfaces, so we replace w by $-w$ if necessary and assume that $\varepsilon = 1$. Therefore we start with the equation

$$v = \sum_{j=1}^{n-1} |z_j|^2 + \sum_{j=1}^{l} |f_j(w, z)|^2$$

and substitute

$$w = w^* + g(w^*, z^*), \qquad z_j = h_j(w^*, z^*), \quad 1 \le j \le n-1,$$

where h_j contains no constant term and g contains no constant or linear terms. Dropping the stars we have

$$v = \sum_{j=1}^{n-1} |h_j(w, z)|^2 - \operatorname{Im} g(w, z) + \sum_{j=1}^{l} |f_j(w + g(w, z), h(w, z))|^2. \tag{2.13}$$

We solve this equation for v by iteration of the form (2.9), again convergent in $\mathcal{F}$, and obtain an equation

$$v = F(z, \overline{z}, u), \qquad F \in \mathcal{F},$$

where the power series $F \in \mathcal{F}$ depends on f, g and h.

Choose an integer $\nu \in \mathbb{Z}_+$. We remark that the coefficients of F of order $\leq \nu$ are polynomial functions of the coefficients of (f, g, h) of order $\leq \nu$ and their conjugates, and they do not depend on the higher order terms in the power series (f, g, h). To see this, we observe that each term in F is obtained from those in (f, g, h) by a finite succession of the following operations with formal power series:

—conjugation,

—addition,

—multiplication,

—substitution of one power series into another.

The last operation consists of successive applications of the first three. Each of these operations is given by a polynomial map of the coefficients and their conjugates, and none of them lowers the order of terms. This justifies the remark above.

Denote by $\mathcal{H}_k$ the set of all formal power series (f, g, h) with complex coefficients in the variables (w, z) such that $f = (f_1, \ldots, f_l)$ and $h = (h_1, \ldots, h_{n-1})$, $n + l = k$, have no constant term, and g has no constant or linear terms. Let $\mathcal{H}_k^\nu$ be the set of all $(f, g, h) \in \mathcal{H}_k$ that contain no nonzero terms of order $> \nu$, and let $Q_k^\nu \colon \mathcal{H}_k \to \mathcal{H}_k^\nu$ be the canonical projection of $\mathcal{H}_k$ onto $\mathcal{H}_k^\nu$. The equation (2.13), when solved for v, determines a map $\Phi_k \colon \mathcal{H}_k \to \mathcal{F}$. Since the terms of order lower or equal ν of $F \in \mathcal{F}$ do not depend on terms of order higher than ν of $(f, g, h) \in \mathcal{H}_k$, we have a commuting diagram:

$$\begin{array}{ccc} \mathcal{H}_k & \xrightarrow{\Phi_k} & \mathcal{F} \\ \downarrow Q_k^\nu & & \downarrow P^\nu \\ \mathcal{H}_k^\nu & \xrightarrow{\Phi_k^\nu} & \mathcal{F}^\nu \end{array} \tag{2.14}$$

By the remark we made above, Φ_k^ν is a polynomial map on coefficients.

We shall determine the dimensions of the vector spaces $\mathcal{H}_k^\nu$ and $\mathcal{F}^\nu$. In $\mathcal{H}_k^\nu$ we have $n+l = k$ polynomials of order ν with complex coefficients in n variables, so the leading term in the expression for $\dim_{\mathbf{R}} \mathcal{H}_k^\nu$ is $Ck\nu^n$ for some constant $C = C(n)$ depending only on n. In $\mathcal{F}^\nu$ we have a single polynomial of order ν with real coefficients in $2n - 1$ variables, so the leading term in $\dim_{\mathbf{R}} \mathcal{F}^\nu$ is $C'\nu^{2n-1}$ for some $C' = C'(n)$. Since $2n - 1 > n$, we have for each fixed k

$$\dim_{\mathbf{R}} \mathcal{F}^\nu > \dim_{\mathbf{R}} \mathcal{H}_k^\nu \tag{2.15}$$

for all sufficiently big $\nu \in \mathbb{Z}_+$.

Fix a $\nu = \nu(k)$ for which (2.15) holds and consider the diagram (2.14). Because of (2.15) the image of the polynomial map Φ_k^ν is a set of the first category in $\mathcal{F}^\nu$. Since the projection $P^\nu \colon \mathcal{F} \to \mathcal{F}^\nu$ is continuous and open, its preimage $K = (P^\nu)^{-1}(\operatorname{Im} \Phi_k^\nu)$ is a set of the first category in the Fréchet space $\mathcal{F}$. The commutativity of (2.14) implies $\operatorname{Im} \Phi_k \subset K$, and hence the set $\operatorname{Im} \Phi_k$ of all $F \in \mathcal{F}$ for which the hypersurface M_F admits a formal embedding in $b\mathbf{B}^k$ is of the first category in $\mathcal{F}$. Taking the countable union over all $k \in \mathbb{Z}_+$ we still have a set of the first category in $\mathcal{F}$. The same argument applies to each space $\mathcal{F}_t$, $t \in \mathbf{R}_+^{2n-1}$. This concludes the proof of Theorem 2.2.

3. Proof of the global nonembedding result. In this section we give a proof of Theorem 1.1 that was stated in §1. We shall use the notation introduced

in §2. Let $s \in \mathbf{Z}_+$, $s \geq 1$. Given a bounded domain D in $\mathbb{C}^n$ with $\mathbf{C}^s$ boundary, we may approximate its defining function by a real polynomial arbitrarily close in the $\mathbf{C}^s$-sense [**19**, p. 33] and thus obtain a domain with real-analytic boundary which is a small $\mathbf{C}^s$ perturbation of D. Therefore we may assume that bD is real-analytic to begin with.

Assume that there exists a proper map $f: D \to \mathbf{B}^k$ for some integer k. Then $\sum_1^k |f_j|^2$ is a bounded plurisubharmonic exhaustion function on D and hence D is pseudoconvex. Fix a point $p \in bD$ and choose coordinates $w = u + iv$, $z = (z_1, \dots, z_{n-1})$ on $\mathbb{C}^n$ such that

$$p = 0 \quad \text{and} \quad \mathbf{T}_0 bD = \{v = 0\}.$$

Let ρ be a real-analytic defining function for D. Multiplying ρ with a constant we may assume that near 0 we have

$$\rho(w, z, \overline{w}, \overline{z}) = -v + r(u, v, z, \overline{z}),$$

where r is a convergent power series that contains no constant or linear terms. Since D is pseudoconvex, the complex Hessian $(\partial^2 r / \partial z_i \partial \overline{z}_j(0))$ has nonnegative eigenvalues. Replacing r by $r + \delta \sum_1^{n-1} |z_j|^2$ for a small δ we may assume that the Hessian of r is positive definite whence bD is *strictly pseudoconvex* at 0.

Choose $t \in \mathbf{R}_+$ large enough such that $\overline{D}$ is contained in the open polydisk

$$|w| < t, \qquad |z_j| < t, \quad 1 \leq j \leq n-1.$$

For each power series $F(z, \overline{z}, u) \in \mathcal{F}_t$ we consider the germ at 0 of the real-analytic hypersurface defined by $\rho + F = 0$. We write this equation in the form

$$v = r(z, \overline{z}, u, v) + F(z, \overline{z}, u). \tag{3.1}$$

Since the right-hand side of (3.1) contains no constant or linear terms, we can solve (3.1) for v by iteration as explained in §2. We obtain an equation

$$v = \tilde{F}(z, \overline{z}, u), \tag{3.2}$$

where $\tilde{F} \in \mathcal{F}$ depends on $F \in \mathcal{F}_t$. This defines a mapping $\Phi: \mathcal{F}_t \to \mathcal{F}$ sending F to $\tilde{F}$. The terms of order $\leq \nu$ of $\tilde{F}$ depend only on the terms of order $\leq \nu$ of F (see the remark in §2). Hence we get an induced map $\Psi^\nu: \mathcal{F}^\nu \to \mathcal{F}^\nu$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_t & \xrightarrow{\Psi} & \mathcal{F} \\ \downarrow P^\nu & & \downarrow P^\nu \\ \mathcal{F}^\nu & \xrightarrow{\Psi^\nu} & \mathcal{F}^\nu \end{array}$$

3.1 LEMMA. *The map Ψ^ν is a real-analytic diffeomorphism onto $\mathcal{F}^\nu$.*

PROOF. We choose for the basis of the vector space $\mathcal{F}^\nu$ the monomials $z^\alpha \overline{z}^\beta u^j$, $|\alpha| + |\beta| + j \leq \nu$, and we denote the corresponding coefficients of $F \in \mathcal{F}^\nu$ by $F_{\alpha,\beta,j}$. Similarly we denote by $r_{\alpha,\beta,j}$ the coefficient of the term $z^\alpha \overline{z}^\beta u^j$ in r.

Consider the equation for the term $z^\alpha \overline{z}^\beta u^j$ in (3.1) and (3.2). It is of the form

$$F_{\alpha,\beta,j} + r_{\alpha,\beta,j} + \cdots = \tilde{F}_{\alpha,\beta,j}, \tag{3.3}$$

where the dots indicate the terms which depend polynomially on the coefficients of F and r of order less than $|\alpha| + |\beta| + j$. For each $\tilde{F}$ we can solve (3.3) uniquely for F

by induction on the order $|\alpha|+|\beta|+j$. Thus Ψ^ν is one-to-one and onto. Moreover, (3.3) shows that the determinant of the derivative $d\Psi^\nu$ is identically equal to 1 in the chosen basis. This proves Lemma 3.1.

Choose an integer k and consider for each $\nu \in \mathbb{Z}_+$ the commuting diagram:

$$\begin{array}{ccccc} \mathcal{F}_t & \xrightarrow{\Psi} & \mathcal{F} & \xleftarrow{\Phi_k} & \mathcal{H}_k \\ \downarrow P^\nu & & \downarrow P^\nu & & \downarrow P^\nu \\ \mathcal{F}^\nu & \xrightarrow{\Psi^\nu} & \mathcal{F}^\nu & \xleftarrow{\Phi_k^\nu} & \mathcal{H}_k^\nu \end{array} \tag{3.4}$$

We have seen in §2 that for all sufficiently big integers ν we have $\dim \mathcal{F}^\nu > \dim \mathcal{H}_k^\nu$. Hence the image of Φ_k^ν is a subset of the first category of $\mathcal{F}^\nu$, and by Lemma 3.1 the set $(\Psi^\nu)^{-1}(\operatorname{Im}\Phi_k^\nu)$ is a subset of the first category of $\mathcal{F}^\nu$. The same is then true for the set $K_k = (P^\nu)^{-1}(\Psi^\nu)^{-1}(\operatorname{Im}\Phi_k^\nu)$ in $\mathcal{F}_t$. The commutativity of (3.4) implies

$$\Psi^{-1}(\operatorname{Im}\Phi_k) \subset K_k.$$

Taking the countable union over all $k \geq n$ we see that the set

$$\mathcal{S}_t = \bigcup_{k=n}^{\infty} \Psi^{-1}(\operatorname{Im}\Phi_k)$$

is of the first category in $\mathcal{F}_t$. Notice that $\mathcal{S}_t$ consists of precisely those $F \in \mathcal{F}_t$ for which the germ at 0 of the real-analytic hypersurface $v = \tilde{F}(z,\bar{z},u)$ with $\tilde{F} = \Psi(F)$ admits a formal embedding into a finite dimensional sphere. In particular, we may choose an $F \in \mathcal{F}_t \backslash \mathcal{S}_t$ arbitrarily close to 0 in the $\mathbf{C}^s$ sense on a neighborhood of $\overline{D}$, and therefore $\rho + F$ is a real-analytic defining function for the domain $D_F = \{\rho + F < 0\}$ which is a small $\mathbf{C}^s$ perturbation of D. Moreover, we may choose an $F \in \mathcal{F}_t \backslash \mathcal{S}_t$ such that D_F is strictly pseudoconvex at the point 0.

Suppose that for some integer k there exists a proper holomorphic map $f\colon D_F \to \mathbb{B}^k$, $f \in \mathbf{C}^\infty(\overline{D}_F)$. The strict pseudoconvexity of D_F at 0 implies that rank of $df(0)$ equals n and that f intersects the sphere $b\mathbb{B}^k$ transversally at 0. Thus f induces a formal embedding of the germ $v = \tilde{F}$ with $\tilde{F} = \Psi(F)$ into the sphere $b\mathbb{B}^k$. This contradicts our choice of F, and hence there is no proper holomorphic map, smooth on $\overline{D}_F$, from the domain D_F into any ball. This concludes the proof of Theroem 1.1.

4. Embedding into convex domains with real-analytic boundaries. In this section we shall prove Theorem 1.2. Let the notation be as in the statement of the theorem. The embedding theorem of Fornaess [**8**, p. 543, Theorem 9] and Khenkin [**13**, Russian p. 112 or English p. 668] gives U, F and Ω as in Theorem 1.2, with $b\Omega$ of class $\mathbf{C}^\infty$. The real-analytic boundary bD is mapped by F onto the real-analytic submanifold $\Sigma = F(U) \cap b\Omega$ of $\mathbb{C}^N$, which is contained in the $\mathbf{C}^\infty$ boundary $b\Omega$. We shall show how to perturb $b\Omega$ slightly in the $\mathbf{C}^2$-sense while keeping it fixed along Σ to obtain a strictly convex domain $\tilde{\Omega}$ with smooth real-analytic boundary satisfying the properties (i)–(iii) of Theorem 1.2.

Let ρ be a $\mathbf{C}^\infty$ strictly convex defining function for Ω:

$$\Omega = \{z \in \mathbb{C}^N | \rho(z) < 0\}, \qquad d\rho \neq 0 \text{ on } b\Omega.$$

The first step in the proof is to find real-valued real-analytic functions $f_1, f_2, \ldots, f_k$ on $\mathbb{C}^N$ such that $f_j = 0$ on Σ for $1 \leq j \leq k$ and such that on a neighborhood of $\overline{\Omega}$

the function ρ can be written in the form $\rho = \sum_{j=1}^{k} \rho_j f_j$ for some $\mathbf{C}^\infty$ functions ρ_j. The second step is to approximate each ρ_j by a real-analytic function $\tilde{\rho}_j$, to let $\tilde{\rho} = \sum \tilde{\rho}_j f_j$ and let $\tilde{\Omega} = \{\tilde{\rho} < 0\}$ be the strictly convex domain with real-analytic boundary that satisfies the conclusions of Theorem 1.2.

To find the functions f_j we consider the sheaf $\mathfrak{R}$ of germs of real-valued real-analytic functions on $\mathbf{R}^{2N} = \mathbf{C}^N$ and its subsheaf $\mathcal{J}$ of all germs that vanish on Σ. Since Σ is a real-analytic submanifold of $\mathbf{R}^{2N}$, the sheaf $\mathcal{J}$ is a coherent sheaf of ideals in $\mathfrak{R}$. Choose a large ball $B \subset \mathbf{R}^{2N}$ contaning $\overline{\Omega}$. Since $\overline{B}$ is compact, Cartan's Theorem A for real-analytic manifolds [**5**, p. 710, Théorème 2] implies that there are finitely many sections $f_1, \dots, f_k$ of $\mathcal{J}$ that generate the ideal $\mathcal{J}_x$ as an $\mathfrak{R}_x$-module for each point $x \in B$. In other words, the map $\Phi\colon \mathfrak{R}^k \to \mathcal{J}$ given by

$$\Phi(\mathbf{c}_{1,x}, \dots, \mathbf{c}_{k,x}) = \sum_{j=1}^{k} \mathbf{c}_{j,x} \mathbf{f}_{j,x} \tag{4.1}$$

is *onto* on each stalk $\mathfrak{R}_x^k$, $x \in B$.

Let $\mathcal{E}$ be the sheaf of germs of real-valued $\mathbf{C}^\infty$ functions on $\mathbf{R}^{2N}$, and let $\tilde{\mathcal{J}} \subset \mathcal{E}$ be the subsheaf of $\mathcal{E}$ consisting of all germs that vanish on Σ. We define the map $\tilde{\Phi}\colon \mathcal{E}^k \to \tilde{\mathcal{J}}$ by the formula (4.1), where the f_j's are the same sections as above. We claim that $\tilde{\Phi}$ is onto on each stalk $\mathcal{E}_x^k$, $x \in B$. To see this, we fix a point $x \in \Sigma$ and choose a set $(h_1, \dots, h_s)$ of real-analytic functions near x such that near x we have

$$\Sigma = \{h_1 = \cdots = h_s = 0\} \quad \text{and} \quad dh_1 \wedge \cdots \wedge dh_s \neq 0.$$

Every germ $\mathbf{h}_x \in \tilde{\mathcal{J}}_x$ can be expressed in the form

$$\mathbf{h}_x = \sum_{j=1}^{s} \mathbf{d}_{j,x} \mathbf{h}_{j,x}, \qquad \mathbf{d}_{j,x} \in \mathcal{E}_x, \tag{4.2}$$

where $\mathbf{h}_{j,x}$ is the germ of h_j at x [**18**, p. 32, Lemma 2.1]. Moreover, since $\mathbf{h}_{j,x} \in \mathcal{J}_x$ and since the germs $\mathbf{f}_{j,x}$ of f_j at x $(1 \le j \le k)$ generate $\mathcal{J}_x$ over $\mathfrak{R}_x$, each germ $\mathbf{h}_{j,x}$ can be expressed in the form

$$\mathbf{h}_{j,x} = \sum_{l=1}^{k} \mathbf{c}_{l,x}^{(j)} \mathbf{f}_{l,x}, \qquad \mathbf{c}_{l,x}^{(j)} \in \mathfrak{R}_x. \tag{4.3}$$

From (4.2) and (4.3) we get an expression for $\mathbf{h}_x$ in terms of the germs $\mathbf{f}_{j,x}$, with coefficients in $\mathcal{E}_x$. This proves that the map $\tilde{\Phi}$ is onto at point $x \in \Sigma$. Clearly $\tilde{\Phi}$ is onto at the points $x \in B \backslash \Sigma$.

This gives a short exact sequence of sheaves on B:

$$0 \to \ker \tilde{\Phi} \to \mathcal{E}^k \xrightarrow{\tilde{\Phi}} \tilde{\mathcal{J}} \to 0$$

and the corresponding long exact sequence at the cohomology level:

$$\mathbf{H}^0(B, \mathcal{E}^k) \xrightarrow{\tilde{\Phi}_*} \mathbf{H}^0(B, \tilde{\mathcal{J}}) \to \mathbf{H}^1(B, \ker \tilde{\Phi}) \to \cdots. \tag{4.4}$$

The sheaf $\ker \tilde{\Phi}$ admits a smooth partition of unity, i.e., it is a *fine sheaf*, and consequently $\mathbf{H}^1(B, \ker \tilde{\Phi}) = 0$. From (4.4) it follows that the map $\tilde{\Phi}_*$ is onto, and

hence we can write the defining function ρ of the domain Ω in the form

$$\rho = \sum_{j=1}^{k} \rho_j f_j$$

on B, where $\rho_j \in \mathbf{C}^\infty(B)$ for $1 \le j \le k$. Shrinking the ball B slightly we may assume that the functions ρ_j are $\mathbf{C}^\infty$ on a neighborhood of $\overline{B}$.

By the generalized theorem of Weierstrass [**19**, p. 33] we can approximate each ρ_j by a real polynomial $\tilde{\rho}_j$ such that $\rho_j - \tilde{\rho}_j$ is arbitrarily small in the $\mathbf{C}^2$ sense on B. The function

$$\tilde{\rho} = \sum_{j=1}^{k} \tilde{\rho}_j f_j$$

is still strictly convex provided that our approximations are close enough, and it is real-analytic and equal to 0 on Σ. The domain $\tilde{\Omega} = \{z \in B | \tilde{\rho}(z) < 0\}$ is therefore strictly convex with real-analytic boundary, $\Sigma \subset b\tilde{\Omega}$, and $F(U)$ intersects $b\tilde{\Omega}$ transversally along Σ. Hence $\tilde{\Omega}$ satisfies the conclusions of Theorem 1.2, and the proof is complete.

5. Embedding into balls: Some lemmas. Let $z = (z_1, \ldots, z_n)$ be coordinates on $\mathbf{C}^n$. We denote by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ the usual inner product on $\mathbf{C}^n$ and by $\|z\| = \langle z, z\rangle^{1/2}$ the usual norm.

If Ω is an open subset of $\mathbf{C}^n$ and $\rho\colon \Omega \to \mathbf{R}$ is a $\mathbf{C}^2$ function, we define

$$N(z) = \left(\frac{\partial \rho}{\partial \overline{z}_1}(z), \ldots, \frac{\partial \rho}{\partial \overline{z}_n}(z)\right), \qquad z \in \Omega,$$

and

$$Q_z(w) = \operatorname{Re}\left(\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(z) w_i w_j\right) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(z) w_i \overline{w}_j, \qquad z \in \Omega,\ w \in \mathbf{C}^n.$$

The quadratic form Q_z is the real Hessian of ρ.

Throughout this section D is a bounded strictly convex domain in $\mathbf{C}^n$ and $S = bD$ is its boundary. There is a $\mathbf{C}^2$ defining function $\rho\colon \mathbf{C}^n \to \mathbf{R}$, $D = \{z \in \mathbf{C}^n | \rho(z) < 0\}$, such that the vector $N(z)$ does not vanish for any z in S and the Hessian of ρ is positive definite:

$$Q_w(z) \neq 0 \quad \text{for all } z, w \in \mathbf{C}^n,\ z \neq 0.$$

This implies that for each compact set K there are constants $c_1, c_2 > 0$ such that

$$c_1 \|z\|^2 \le Q_w(z) \le c_2 \|z\|^2, \qquad w \in K,\ z \in \mathbf{C}^n. \tag{5.1}$$

If we write $N(w) = \|N(w)\| \cdot \nu(w)$, then $\nu(w)$ is the outward unit normal vector to S at the point $w \in S$.

5.1 LEMMA. *There are constants $\alpha_1, \alpha_2 > 0$ such that for all points $z, w \in S$ we have*

$$\alpha_1 \|w - z\|^2 \le \operatorname{Re}\langle w - z, \nu(w)\rangle \le \alpha_2 \|w - z\|^2. \tag{5.2}$$

Note. If D is the unit ball $\mathbf{B}^n$ and $S = b\mathbf{B}^n$, then (5.2) holds with $\alpha_1 = \alpha_2 = \frac{1}{2}$.

PROOF. Put $h(t) = \rho((1-t)w + tz)$, $t \in \mathbf{R}$. Then $h(0) = \rho(w) = 0$ and $h(1) = \rho(z) = 0$. By the chain rule,

$$h'(0) = 2\,\mathrm{Re}\langle z - w, N(w)\rangle \quad \text{and} \quad h''(t) = 2Q_u(w - z),$$

where $u = (1-t)w + tz$. We insert these data into the Taylor formula

$$h(1) = h(0) + h'(0) + \tfrac{1}{2}h''(t)$$

which holds for some $t \in (0,1)$ to obtain

$$2\,\mathrm{Re}\langle w - z, N(w)\rangle = Q_u(w - z)$$

for some point $u \in D$. We used here the hypothesis that D is convex. If $c_1, c_2 > 0$ are such that (5.1) holds, then (5.2) will hold if we put

$$\alpha_1 = c_1(2\max\{\|N(z)\|: z \in \overline{D}\})^{-1}, \qquad \alpha_2 = c_2(2\min\{\|N(z)\|: z \in \overline{D}\})^{-1}. \tag{5.3}$$

This proves Lemma 5.1.

For each point a in $\mathbf{R}^n$ or $\mathbf{C}^n$ and each $r > 0$ we denote by $B(a,r)$ the open unit ball of radius r centered at a. We shall need the following covering lemma.

5.2 LEMMA. *If S is a compact hypersurface of class $\mathbf{C}^1$ in $\mathbf{R}^{l+1}$ and if $\lambda > 1$, there exists an integer s with the following property. For each $r > 0$ there are s families of balls $\mathcal{F}_1, \dots, \mathcal{F}_s$, $\mathcal{F}_i = \{B(a_{i,j}, \lambda r) | 1 \le j \le N_i\}$, with centers $a_{i,j}$ in S, such that the balls in each family are pairwise disjoint and $S \subset \bigcup B(a_{i,j}, r)$, i.e., the small balls of radius r cover S.*

PROOF. Let $U_1, \dots, U_k$ be open subsets of $\mathbf{R}^{l+1}$ such that $S \subset \bigcup_{j=1}^k U_j$. If we can solve the covering problem for each $S_j = S \cap U_j$ with some integer s_j, then we can solve the problem for S with $s = s_1 + \cdots + s_k$ since we can put the balls corresponding to different S_j's into different families. Thus, by compactness of S, it suffices to solve the covering problem locally on S.

Choose a point $a \in S$ and make an affine change of coordinates in $\mathbf{R}^{l+1}$ such that $a = 0$ and $\mathbf{T}_0 S = \mathbf{R}^l \times \{0\}$. Choose a constant $0 < \delta < 1$. Let $x' = (x_1, \dots, x_l)$ and $x = (x', x_{l+1})$ be coordinates on $\mathbf{R}^l$ and $\mathbf{R}^{l+1}$, respectively. There is a small neighborhood $U = \{x \in \mathbf{R}^{l+1}: |x_j| < \varepsilon,\ 1 \le j \le l+1\}$ of 0 and a $\mathbf{C}^1$ function $h: U' = U \cap \mathbf{R}^l \to \mathbf{R}$ such that

$$S \cap U = \{x \in U | x_{l+1} = h(x')\}$$

and the gradient ∇h of h vanishes at the point $0'$. Since ∇h is continuous, we can choose U so small that

$$1 + \|\nabla h(x')\|^2 < (1+\delta)^2 \tag{5.4}$$

for all $x' \in U'$.

Given points $a', b' \in U'$ we estimate the distance between the corresponding points $a = (a', h(a'))$ and $b = (b', h(b'))$ on S. By the Taylor formula we have

$$\begin{aligned} \|a - b\|^2 &= \|a' - b'\|^2 + |h(a') - h(b')|^2 \\ &= \|a' - b'\|^2 + |\nabla h(u') \cdot (a' - b')|^2 \\ &\le \|a' - b'\|^2 (1 + \|\nabla h(u')\|^2) \end{aligned}$$

for some point u' on the line segment between a' and b'. Thus (5.4) implies

$$\|a' - b'\| \le \|a - b\| < (1+\delta)\|a' - b'\|, \qquad a, b \in S \cap U. \tag{5.5}$$

If we have a family of balls $B(a'_j, r/(1+\delta))$ with centers a'_j in U' that cover $\overline{U}'$, then by the right estimate in (5.5) the balls $B(a_j, r)$, $a_j = (a'_j, h(a'_j))$, cover $S \cap \overline{U}$. Similarly, if the balls $B(a'_j, \lambda r)$ are pairwise disjoint, then by the left estimate in (5.5) the balls $B(a_j, \lambda r)$ are also disjoint.

This shows that it suffices to solve the covering problem for an open ball U' in $\mathbf{R}^l$, with λ replaced by $\lambda_1 = \lambda(1+\delta)$. It is easier to work with cubes than with balls. Let $C(a, r)$ be the open cube with side $2r$ centered at a. For each $r > 0$ and $a \in \mathbf{R}^l$ we have

$$C(a, rl^{-1/2}) \subset B(a, r) \subset B(a, \lambda_1 r) \subset C(a, \lambda_1 r). \tag{5.6}$$

We start in one vertex of the cube U' and put small open cubes of side $2r/\sqrt{l}$ next to each other so that they overlap just barely, and they cover $\overline{U}'$. One can see that the large cubes of side $2\lambda_1 r$ can be arranged into

$$s = ([\lambda(1+\delta)\sqrt{l}] + 1)^l$$

families such that the cubes in each family are pairwise disjoint. Because of (5.6) the same constant s is good for balls. Lemma 5.2 is proved.

Let α_1 and α_2 be as in Lemma 5.1. We determine the number $\lambda > 1$ by

$$\lambda = 2\sqrt{\alpha_2/\alpha_1}. \tag{5.7}$$

If D is the ball $\mathbf{B}^n$, then $\lambda = 2$. Choose an integer s such that Lemma 5.2 holds for S and λ. Thus for each $r > 0$ we can find s families $\mathcal{F}_1, \dots, \mathcal{F}_s$ of balls with radii λr, $\mathcal{F}_i = \{B(z_{i,j}, \lambda r) \mid j = 1, \dots, N_i\}$, $z_{i,j} \in S$, satisfying

(P) The balls in each family $\mathcal{F}_i$ are pairwise disjoint, and $S \subset \bigcup B(z_{i,j}, r)$, i.e., the small balls of radius r cover S.

For each $1 \le i \le s$ we let $z_{i+s,j} = z_{i,j}$ and $\mathcal{F}_{i+s} = \mathcal{F}_i$, so that we have $2s$ families of balls. For $z \in S$ and $1 \le i \le 2s$ we define

$$V_{i,k}(z) = \{z_{i,j} \mid k\lambda r \le \|z - z_{i,j}\| < (k+1)\lambda r\}, \qquad k = 0, 1, 2, \dots.$$

For large k, $V_{i,k}(z)$ is empty. Denote by $N_{i,k}(z)$ the number of points in $V_{i,k}(z)$. Since the balls in $\mathcal{F}_i$ are disjoint, it follows that

$$N_{i,k}(z) \le C_1 k^{2n} \tag{5.8}$$

for some constant C_1 independent of λ and r.

For each fixed point w in $S = bD$ and $m > 0$ the entire function

$$\phi_w(z) = e^{-m\langle w - z, \nu(w)\rangle} \tag{5.9}$$

is a peaking function on $\overline{D}$ for the point w. Using (5.2) we can estimate its modulus

$$|\phi_w(z)| = e^{-m\,\mathrm{Re}\langle w - z, \nu(w)\rangle}$$

for each $z, w \in S$ by

$$e^{-\alpha_2 m\|z-w\|^2} \le |\phi_w(z)| \le e^{-\alpha_1 m\|z-w\|^2}. \tag{5.10}$$

For each $i \in \{1,\dots,2s\}$ and $j \in \{1,\dots,N_i\}$ we write $\phi_{i,j} = \phi_{z_{i,j}}$. Notice that $\phi_{i,j} = \phi_{i+s,j}$ for each $1 \le i \le s$. Let g_i be an entire function of the form

$$g_i(z) = \sum_{j=1}^{N_i} \beta_{i,j}\phi_{i,j}(z), \qquad |\beta_{i,j}| \le 1. \tag{5.11}$$

5.3 LEMMA. *For each sufficiently small $\eta > 0$ there are $m, r > 0$ such that for each family of balls $\{\mathcal{F}_i\}$ satisfying* (P) *above and for each function g_i of the form* (5.11) *the following hold.*

(a) *If a point $z \in S$ lies in no ball in $\mathcal{F}_i$, then $|g_i(z)| < \eta$.*

(b) *If $z \in S \cap B(z_{i,j}, \lambda r)$ for some j, then $|g_i(z) - \beta_{i,j}\phi_{i,j}(z)| < \eta$.*

(c) *If $z \in S \cap B(z_{i,j}, r)$, then $|\phi_{i,j}(z)| \ge C\eta^{1/4}$, where the constant $C > 0$ is independent of r and η.*

PROOF OF (a). If $z \in S$ lies in no ball in $\mathcal{F}_i$, then (5.7), (5.8) and (5.10) imply

$$|g_i(z)| \le \sum_{k=1}^{\infty} \sum_{z_{i,j} \in V_{i,k}(z)} |\phi_{i,j}(z)| < C_1 \sum_{k=1}^{\infty} k^{2n} e^{-k^2\lambda^2\alpha_1 m r^2} = C_1 \sum_{k=1}^{\infty} k^{2n} e^{-k^2 4\alpha_2 m r^2}.$$

Put $\beta = 4\alpha_2 m r^2$. We have

$$\sum_{k=1}^{\infty} k^{2n} e^{-k^2\beta} = e^{-\beta}\left(1 + \sum_{k=2}^{\infty} k^{2n} e^{-k^2\beta+\beta}\right).$$

If $\beta \ge \frac{4}{3}$, then $-k^2\beta + \beta \le -k^2$ for each $k \ge 2$ and hence $k^{2n}e^{-k^2\beta+\beta} \le k^{2n}e^{-k^2}$. If we put

$$C_2 = C_1\left(1 + \sum_{k=2}^{\infty} k^{2n} e^{-k^2}\right),$$

then

$$|g_i(z)| < C_2 e^{-4\alpha_2 m r^2} \tag{5.12}$$

provided that $\beta = 4\alpha_2 m r^2 \ge \frac{4}{3}$.

Given an $\eta > 0$, we adjust m and r such that $C_2 e^{-\beta} = \eta$. This is equivalent to

$$m r^2 = \frac{1}{4\alpha_2} \log\left(\frac{C_2}{\eta}\right). \tag{5.13}$$

Given a fixed r, m is determined by (5.13). We can choose r arbitrarily small and make m as large as we want. If we choose $\eta \le C_2 e^{-4/3}$, then $\beta \ge \frac{4}{3}$ so that the estimate (5.12) holds. This proves (a).

PROOF OF (b). If $z \in B(z_{i,j}, \lambda r)$, then z lies in no other ball in $\mathcal{F}_i$ since these are disjoint. Therefore we can estimate the difference $g_i(z) - \beta_{i,j}\phi_{i,j}(z)$ by part (a) above. This proves (b).

PROOF OF (c). Suppose that m and r satisfy (5.13). If $z \in B(z_{i,j}, r) \cap S$, the left estimate in (5.10) implies

$$|\phi_{i,j}(z)| \ge e^{-\alpha_2 m r^2} = C_2^{-1/4}\eta^{1/4}.$$

Thus (c) holds with $C = C_2^{-1/4}$. This proves Lemma 5.3.

6. Embedding into balls: The main lemma. Let D be a bounded $\mathbf{C}^2$ strictly convex domain in $\mathbb{C}^n$ and $S = bD$. Determine λ by (5.7), and choose an integer s satisfying Lemma 5.2. Given some functions $f_1, \dots, f_{2s}$, we write $F = (f_1, \dots, f_{2s+p})$, where $f_{2s+i} = h_i$ for $1 \le i \le p$. The following lemma is similar to Lemma 1 in [**15**].

6.1 LEMMA. *There is an $\varepsilon_0 > 0$ such that if*
(i) $\frac{1}{2} < a < 1$, $\varepsilon < \varepsilon_0$, $\varepsilon < 1 - a$, *and K is a compact subset of D,*
(ii) $F = (f_1, \dots, f_{2s+p})$ *is holomorphic in D,*
(iii) *the functions f_i, $1 \le i \le 2s$, extend continuously to $\overline{D}$, and*
(iv) $\|F\|$ *extends continuously to $\overline{D}$ and satisfies $\|F\| < a$ there,*
then there is an entire mapping $G = (g_1, \dots, g_{2s}, 0, \dots, 0)$ such that
(a) $\|F(z) + G(z)\| < a + \varepsilon$ *for each point $z \in S$,*
(b) *if $\|F(z) + G(z)\| \le a - \varepsilon^{1/7}$ for some $z \in S$, then $\|F(z) + G(z)\| \ge \|F(z)\| + \varepsilon^{2/3}$,*
(c) $\|G(z)\| < \varepsilon$ *if $z \in K$,*
(d) $\|G(z)\|^2 < 1 - \|F(z)\|$ *if $z \in S$.*

PROOF. Let $\eta = \varepsilon/120s$. Since the functions f_i, $1 \le i \le 2s$, and $\|F\|$ are continuous on $\overline{D}$, we can find an $r_0 > 0$ such that whenever $z, w \in \overline{D}$ satisfy $\|z - w\| < 2\lambda r_0$, we have

$$|f_i(z) - f_i(w)| < \eta, \quad 1 \le i \le 2s, \qquad \big|\, \|F(z)\| - \|F(w)\| \,\big| < \eta. \tag{6.1}$$

Given a number $r < r_0$ we choose s families of balls $\mathcal{F}_1, \dots, \mathcal{F}_s$,

$$\mathcal{F}_i = \{B(z_{i,j}, \lambda r) \mid 1 \le j \le N_i\}, \qquad z_{i,j} \in S,$$

such that the balls in each family are pairwise disjoint and the small balls $B(z_{i,j}, r)$ cover S (see §5). Let $z_{i+s,j} = z_{i,j}$ and let $\mathcal{F}_{i+1} = \mathcal{F}_s$ for $1 \le i \le s$, so that we have $2s$ families of balls. We shall find entire functions $g_1, \dots, g_{2s}$ of the form (5.11) such that the map $G = (g_1, \dots, g_{2s}, 0, \dots, 0)$ will satisfy Lemma 6.1.

We now determine the coefficients $\beta_{i,j}$ of g_i. Let $1 \le i \le s$ and $1 \le j \le N_i$. We choose $\beta_{i,j}$ and $\beta_{i+s,j}$ such that

$$f_i(z_{i,j})\overline{\beta_{i,j}} + f_{i+s}(z_{i,j})\overline{\beta_{i+s,j}} = 0 \tag{6.2}$$

and

$$|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 = (a^2 - \|F(z_{i,j})\|^2)/2s. \tag{6.3}$$

The condition (6.2) means that the vector $(\beta_{i,j}, \beta_{i+s,j}) \in \mathbb{C}^2$ is orthogonal to the vector $(f_i(z_{i,j}), f_{i+s}(z_{i,j})) \in \mathbb{C}^2$. Clearly $|\beta_{i,j}| < 1$ for all i and j.

We shall now prove that the entire map $G = (g_1, \dots, g_{2s}, 0, \dots, 0)$ satisfies properties (a)–(d) of Lemma 6.1, provided that the constant m in (5.9) is chosen sufficiently large.

PROOF OF (a). For each point $z \in S$ we let $I(z)$ be the set of indices $1 \le i \le 2s$ such that the point z lies in some ball $B(z_{i,j}, \lambda r) \in \mathcal{F}_i$. Since the balls in $\mathcal{F}_i$ are pairwise disjoint, there is at most one such ball and so j is uniquely determined. Notice that i and $i + s$ are in $I(z)$ simultaneously for every i, $1 \le i \le s$. Consider the function $f_i + g_i$ for $1 \le i \le 2s$. If i is not in $I(z)$, then $|g_i(z)| < \eta$ by Lemma 5.3(a), and therefore

$$|f_i(z) + g_i(z)|^2 < |f_i(z)|^2 + 2|f_i(z)|\eta + \eta^2 < |f_i(z)|^2 + 3\eta. \tag{6.4}$$

If on the other hand $i \in I(z)$, we have

$$|f_i(z) + g_i(z)| \le |f_i(z) - f_i(z_{i,j})| + |f_i(z_{i,j}) + \beta_{i,j}\phi_{i,j}(z)| + \left|\sum_{k\ne j} \beta_{i,k}\phi_{i,k}(z)\right|.$$

The first term on the right is at most η by (6.1), and the last term is at most η by Lemma 5.3(b). Therefore

$$|f_i(z) + g_i(z)|^2 < |f_i(z_{i,j}) + \beta_{i,j}\phi_{i,j}(z)|^2 + 8\eta.$$

We assume $1 \le i \le s$ and group together the terms with indices i and $i+s$. Using (6.2) and (6.3) we have

$$\begin{aligned}
&|f_i(z) + g_i(z)|^2 + |f_{i+s}(z) + g_{i+s}(z)|^2 \\
&\quad < |f_i(z_{i,j}) + \beta_{i,j}\phi_{i,j}(z)|^2 + |f_{i+s}(z_{i,j}) + \beta_{i+s,j}\phi_{i,j}(z)|^2 + 16\eta \\
&\quad \le |f_i(z_{i,j})|^2 + |f_{i+s}(z_{i,j})|^2 + |\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 + 16\eta \\
&\quad = |f_i(z_{i,j})|^2 + |f_{i+s}(z_{i,j})|^2 + (a^2 - \|F(z_{i,j})\|^2)/2s + 16\eta.
\end{aligned}$$

Summing over all indices $1 \le i \le 2s+p$ and using (6.4) we obtain

$$\begin{aligned}
\|F(z) + G(z)\|^2 < &\sum_{\substack{i\in I(z)\\ i\le s}} \left(|f_i(z_{i,j})|^2 + |f_{i+s}(z_{i,j})|^2 + (a^2 - \|F(z_{i,j})\|^2)/2s\right) \\
&+ \sum_{i\notin I(z)} |f_i(z)|^2 + 22s\eta.
\end{aligned}$$

The points z and $z_{i,j}$, $i \in I(z)$, in the above estimate are distinct. However, since $z \in B(z_{i,j}, \lambda r)$ for each $i \in I(z)$, (6.1) implies that we may replace these points by the point z, making an error not exceeding $8s\eta$. Doing this we have

$$\begin{aligned}
\|F(z) + G(z)\|^2 &< \sum_{i=1}^{2s+p} |f_i(z)|^2 + \frac{\operatorname{card} I(z)}{2s}(a^2 - \|F(z)\|^2) + 30s\eta \\
&\le \|F(z)\|^2 + (a^2 - \|F(z)\|^2) + 30s\eta \le a^2 + 30s\eta.
\end{aligned}$$

Since $a > \frac{1}{2}$, we have

$$\sqrt{a^2 + 30s\eta} < a\sqrt{1 + 120s\eta} < a + 120s\eta = a + \varepsilon,$$

and hence $\|F(z) + G(z)\| < a + \varepsilon$. This proves (a).

PROOF OF (b). Fix a point $z \in S$ and let $I(z)$ be as in (a) above. For each i, $1 \le i \le s$, we will estimate from below the difference

$$D_i(z) = |f_i(z) + g_i(z)|^2 + |f_{i+s}(z) + g_{i+s}(z)|^2 - |f_i(z)|^2 - |f_{i+s}(z)|^2.$$

If z is not in $I(z)$, Lemma 5.3(a) implies $|g_i(z)| < \eta$, $|g_{i+s}(z)| < \eta$, and hence $D_i(z) = O(\eta) = O(\varepsilon)$. If on the other hand $i \in I(z)$, then $z \in B(z_{i,j}, \lambda r)$ for some j, and (6.1) and Lemma 5.3(b) imply

$$\begin{aligned}
D_i(z) &= |f_i(z_{i,j}) + g_i(z)|^2 + |f_{i+s}(z_{i,j}) + g_{i+s}(z)|^2 \\
&\quad - |f_i(z_{i,j})|^2 - |f_{i+s}(z_{i,j})|^2 + O(\varepsilon) \\
&= |f_i(z_{i,j}) + \beta_{i,j}\phi_{i,j}(z)|^2 + |f_{i+s}(z_{i,j}) + \beta_{i+s,j}\phi_{i,j}(z)|^2 \\
&\quad - |f_i(z_{i,j})|^2 - |f_{i+s}(z_{i,j})|^2 + O(\varepsilon).
\end{aligned}$$

From (6.2) it follows that the above equals

$$D_i(z) = (|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2)|\phi_{i,j}(z)|^2 + O(\varepsilon). \tag{6.5}$$

Thus $D_i(z) \geq O(\varepsilon)$ for each $1 \leq i \leq s$. Summing yields $\|F(z)+G(z)\|^2 - \|F(z)\|^2 \geq O(\varepsilon)$ and therefore

$$\|F(z) + G(z)\| - \|F(z)\| \geq O(\varepsilon). \tag{6.6}$$

Suppose that $\|F(z) + G(z)\| < a - \varepsilon^{1/7}$ for some $z \in S$. Choose a ball $B(z_{i,j}, r)$ containing z. Then (6.1) and (6.6) imply

$$\|F(z_{i,j})\| \leq \|F(z)\| + \varepsilon \leq (a - \varepsilon^{1/7}) + O(\varepsilon) < a - \tfrac{1}{2}\varepsilon^{1/7}$$

if ε_0 is sufficiently small. Thus $a - \|F(z_{i,j})\| \geq \frac{1}{2}\varepsilon^{1/7}$, and by (6.3)

$$|\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 = \frac{1}{2s}(a^2 - \|F(z_{i,j})\|^2) > \frac{1}{8s}\varepsilon^{1/7}.$$

Since $z \in B(z_{i,j}, r)$, we have $|\phi_{i,j}(z)|^2 \geq C\varepsilon^{1/2}$ by Lemma 5.3(c). Thus (6.5) implies

$$\|F(z) + G(z)\|^2 - \|F(z)\|^2 = \sum_{i=1}^{s} D_i(z) \geq \frac{C}{8s}\varepsilon^{1/7+1/2} + O(\varepsilon) > 2\varepsilon^{2/3}.$$

Therefore

$$\|F(z) + G(z)\| - \|F(z)\| = \frac{\|F(z) + G(z)\|^2 - \|F(z)\|^2}{\|F(z) + G(z)\| + \|F(z)\|} > \varepsilon^{2/3},$$

and (b) is proved.

PROOF OF (c). If K is a compact subset of D, let

$$\delta = \min\{\|z - w\| : z \in K,\ w \in S\} > 0.$$

Then $|\phi_{i,j}(z)| \leq e^{-m\delta}$ for all $z \in K$. Since each $\mathcal{F}_i$ contains no more than C_3/r^{2n} balls and $3\alpha_2 m r^2 \geq 1$, we have

$$|g_i(z)| \leq \frac{C_3}{r^{2n}} e^{-m\delta} \leq C_4 m^n e^{-m\delta}.$$

If m is sufficiently large, the right-hand side is less than $\varepsilon/\sqrt{2s}$ which implies $\|G(z)\| < \varepsilon$. This proves (c).

PROOF OF (d). Fix a point $z \in S$, and let $I(z)$ be as in the proof of (a) above. If i is not in $I(z)$, then $|g_i(z)|^2 < \eta^2 < \eta$ by Lemma 5.3(a). If on the other hand $i \in I(z)$, we assume $1 \leq i \leq s$, and we group together the terms with indices i and $i + s$. By Lemma 5.3(b), (6.1), and (6.3) we have

$$\begin{aligned} |g_i(z)|^2 + |g_{i+s}(z)|^2 &< |\beta_{i,j}|^2 + |\beta_{i+s,j}|^2 + 6\eta = \frac{1}{2s}(a^2 - \|F(z_{i,j})\|^2) + 6\eta \\ &< \frac{1}{s}(a - \|F(z_{i,j})\|) + 6\eta \leq \frac{1}{s}(a - \|F(z)\|) + 7\eta. \end{aligned}$$

Summing yields

$$\|G(z)\|^2 < a - \|F(z)\| + 7s\eta < a - \|F(z)\| + \varepsilon < 1 - \|F(z)\|.$$

This proves (d), and Lemma 6.1 is proved.

7. Embedding into balls. In this section we shall conclude the proof of Theorem 1.3. Let $h = (h_1, \dots, h_p)$ be as in the statement of Theorem 1.3. Choose a sequence $\{a_k\}$ of positive numbers strictly increasing to 1, $\frac{1}{2} < a_k < 1$, and a sequence $\{\varepsilon_k\}$ decreasing to 0 such that $\sum \varepsilon_k$ is finite, $\sum \varepsilon_k^{2/3} = \infty$, $a_{k-1} + \varepsilon_k < a_k$, and $\sup_S \|h\| < a_0$. Using Lemma 6.1 we shall inductively construct a sequence $\{G_k\}$ of entire mappings

$$G_k = (g_1, \dots, g_{2s}, 0, \dots, 0) \colon \mathbb{C}^n \to \mathbb{C}^{2s+p}$$

and an increasing sequence $\{K_k\}$ of compact subsets of D, $\bigcup_k K_k = D$, such that, if we put $F_0 = (0, \dots, 0, h)$ and $F_k = F_0 + \sum_{j=1}^k G_j$, the following properties hold:

(a) $\|F_{k-1}(z)\| \geq \min_{w \in S} \|F_{k-1}(w)\| - 2^{-k}$ if $z \in \overline{D} \backslash K_k$,

(b) $\|F_k(z)\| < a_k$ if $z \in \overline{D}$,

(c) if $\|F_k(z)\| < a_{k-1} - \varepsilon_k^{1/7}$ for some $z \in S$, then $\|F_k(z)\| \geq \|F_{k-1}(z)\| + \varepsilon_k^{2/3}$,

(d) $\|G_k(z)\| < \varepsilon_k$ if $z \in K_k$, and

(e) $\|G_k(z)\|^2 \leq 1 - \min_S \|F_{k-1}\|$ if $z \in \overline{D}$.

We start with $k = 1$, $F_0 = (0, \dots, 0, h)$, $\|F_0\| < a_0$ on $\overline{D}$. Since $\|F_0\|$ is continuous on $\overline{D}$, we can find a compact subset K_1 of D such that (a) holds with $k = 1$. We then apply Lemma 6.1 to the data F_0, a_0, ε_1 and K_1 to find an entire function G_1 such that the properties (b)–(e) hold for $k = 1$. Now we pick a compact subset K_2 of D containing K_1 such that (a) holds for $k = 2$, and we apply Lemma 6.1 to find a G_2 satisfying (b)–(e) with $k = 2$. Clearly we can proceed this way to construct sequences $\{G_k\}$ and $\{K_k\}$ such that $\bigcup_k K_k = D$.

Property (d) implies that the series $F = \lim F_k = F_0 + \sum_{j=1}^\infty G_j$ converges uniformly on every compact subset of D, and hence the limit F is holomorphic on D. Property (b) implies that $\|F\| \leq 1$ on $\overline{D}$. If h is not constant, then F is not constant, and the maximum modulus principle implies $\|F\| < 1$ on D. Hence F maps D into $\mathbb{B}^{2s+p}$. If h is constant, we can apply the same proof with h replaced by $(\delta z_1, h)$ for a small $\delta > 0$.

It remains to show that F maps D properly into $\mathbb{B}^{2s+p}$. Let

$$u_k(z) = \min\{\|F_k(z)\|, a_{k-1} - \varepsilon_k^{1/7}\}, \qquad z \in S.$$

Clearly u_k is a continuous function on S. We claim that $\lim u_k = 1$ uniformly on S. It suffices to show that for each $z \in S$ the sequence $u_k(z)$ is increasing towards one. If $\|F_k(z)\| \leq a_{k-1} - \varepsilon_k^{1/7}$, then the property (c) implies

$$u_k(z) = \|F_k(z)\| \geq \|F_{k-1}(z)\| + \varepsilon_k^{2/3} \geq u_{k-1}(z) + \varepsilon_k^{2/3}. \tag{7.1}$$

On the other hand, if $\|F_k(z)\| > a_{k-1} - \varepsilon_k^{1/7}$, then

$$u_k(z) = a_{k-1} - \varepsilon_k^{1/7} > a_{k-2} - \varepsilon_{k-1}^{1/7} \geq u_{k-1}(z).$$

This proves that $\{u_k(z)\}$ is strictly increasing.

Suppose that $\lim u_k(z) < 1$ for some $z \in S$. Then

$$a_{k-1} - \varepsilon_k^{1/7} > \|F_k(z)\| \quad \text{for all } k > k_0,$$

and hence by (7.1)

$$u_k(z) \geq u_{k-1}(z) + \varepsilon_k^{2/3}, \qquad k > k_0.$$

Summing we have for all $k > k_0$

$$u_k(z) \geq u_{k_0}(z) + \sum_{j=k_0+1}^{k} \varepsilon_j^{2/3}.$$

Since the series $\sum \varepsilon_j^{2/3}$ diverges, it follows that $\lim_{k\to\infty} u_k(z) = \infty$ which is a contradiction. This proves that the functions u_k converge to one uniformly on S and consequently $\lim \|F_k\| = 1$ uniformly on S. If we define $b_k = \inf_S \|F_k\|$ and $r_k = (1 - b_k)^{1/2}$, then

$$\lim b_k = 1 \quad \text{and} \quad \lim r_k = 0. \tag{7.2}$$

Let $z \in K_{k+1} \backslash K_k$. The triangle inequality gives

$$\|F(z)\| \geq \|F_{k-1}\| - \|G_k(z)\| - \sum_{j=k+1}^{\infty} \|G_j(z)\|.$$

We can estimate the first term on the right using the property (a), the second term using (e) and the third term using (d) to obtain

$$\|F(z)\| \geq (b_{k-1} - 2^{-k}) - r_k - \sum_{j=k+1}^{\infty} \varepsilon_j, \qquad z \in K_{k+1} \backslash K_k. \tag{7.3}$$

Since $\sum \varepsilon_j$ is convergent, (7.2) implies

$$\lim_{k\to\infty} b_{k-1} - 2^{-k} - r_k - \sum_{j=k+1}^{\infty} \varepsilon_j = 1. \tag{7.4}$$

From (7.3) and (7.4) it follows that $\|F(z)\|$ tends to one as the point z tends towards the boundary of D. This means that the map $F: D \to \mathbb{B}^{2s+p}$ is proper. Theorem 1.3 is proved.

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