

## DESINGULARIZATIONS OF PLANE VECTOR FIELDS

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**ABSTRACT.** The singularities of a plane vector field can be reduced under quadratic blowing ups. We describe a control method for the singularities of the vector field which works for ground fields of any characteristic and which has no essential obstruction for generalizing to higher dimensional cases.

**0. Introduction.** In his paper [6], Seidenberg proves the following result: "Let  $k$  be an algebraically closed field of characteristic zero and let  $D = A\partial/\partial X + B\partial/\partial Y$  be a vector field with  $A, B \in k[[X, Y]]$  relatively prime elements. If  $r = \min(\nu(A), \nu(B))$ , where  $\nu$  denotes the order of the power series, then the solutions of  $D$  can be made to go over the solutions of a finite number of vector fields with  $r \leq 1$ , by a finite number of transformations of the form  $X' = X, Y' = Y/X$ , translations tacitly included."

After this, A. van den Essen [7] gave another simpler proof of the same result based on the use of intersection multiplicities.

The above result may be seen as a "punctual" reduction of the singularities of the vector field  $D$ . In his works [2, 3], Giraud proves implicitly a result (which may be stated as Corollary 2 below) on "global" reduction of the singularities of the vector field  $D$ , also valid for positive characteristic. Giraud considers vector fields that are tangents to a normal-crossing divisor and he uses

$$c = \text{length}(k[[X, Y]]/(A, B))$$

as a main invariant for the control of the singularity,  $A, B$  being the coefficients.

Giraud's invariant and Seidenberg's proof depend essentially on the fact that  $A = B = 0$  defines a scheme of dimension zero. So they are not generalizable to higher dimension.

In this paper we describe a control method for the singularity of the vector field based upon Hironaka's techniques in [4], which has no essential obstruction for generalizing to higher dimension (we have shown punctually in dimension 3 in [1]). We obtain a global result (Corollary 2 below) which is valid for all the characteristics. In §4 we apply this technique to the case for which the minimum of the order of the coefficients would be 1 (see also [2]); in this case one cannot make a global reduction, but it is always possible to make a punctual reduction.

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**1. Preliminaries.**

(1.1) Let  $X$  be an excellent regular scheme of dimension two over an algebraically closed field  $k$  of arbitrary characteristic. Let  $E$  be a reduced effective divisor of  $X$  whose support has only normal crossings as singularities and let us denote by  $\Theta_X[E]$  the sub- $\mathcal{O}_X$ -module of the tangent sheaf  $\Theta_X$  locally given by the germs of vector fields which are germs of tangent vector fields to  $E$ .

(1.2) For any invertible sub- $\mathcal{O}_X$ -module  $\mathcal{D}$  of  $\Theta_X$ , let us denote by  $\alpha(\mathcal{D})$  the invertible sub- $\mathcal{O}_X$ -module of  $\Theta_X$  which is locally the double orthogonal (i.e. the  $D^{\perp\perp}$ ) of  $\mathcal{D}$  relative to the natural pairing  $\Omega_X \times \Theta_X \rightarrow \mathcal{O}_X$ . Let us denote by  $(\mathcal{D}, E)$  the invertible sub- $\mathcal{O}_X$ -module of  $\Theta_X[E]$  given by  $(\mathcal{D}, E) = \mathcal{D} \cap \Theta_X[E]$ . If  $\mathcal{D} = (\alpha(\mathcal{D}), E)$ , the sub- $\mathcal{O}_X$ -module  $\mathcal{D}$  of  $\Theta_X[E]$  will be said to be *multiplicatively irreducible and adapted to  $E$* .

(1.3) Let  $(x, y)$  be a regular set of parameters for the local ring  $\mathcal{O}_{X,P}$  of  $X$  at  $P$ . Let us suppose that either  $P \notin E$  or the local equations for  $E$  at  $P$  are  $x = 0, y = 0$  or  $xy = 0$ . Such a regular set of parameters will be called an *adapted to  $E$  regular set of parameters*.

Let  $\{\partial/\partial x, \partial/\partial y\}$  be the dual basis of  $\{dx, dy\}$ . Then the fiber  $\Theta_{X,P}[E]$  of  $\Theta_X[E]$  at  $P$  is a free module over  $\mathcal{O}_{X,P}$  which is freely generated by  $\{\partial_x, \partial_y\}$ , where  $\partial_x = \partial/\partial x, \partial_y = \partial/\partial y$  if  $P \notin E$ ;  $\partial_x = x \cdot \partial/\partial x, \partial_y = \partial/\partial y$  if  $E$  has the local equation  $x = 0$  at  $P$ ;  $\partial_x = \partial/\partial x, \partial_y = y \cdot \partial/\partial y$  if  $E$  has the local equation  $y = 0$  at  $P$ , and finally  $\partial_x = x \cdot \partial/\partial x, \partial_y = y \cdot \partial/\partial y$  if  $E$  has the local equation  $xy = 0$  at  $P$ .

A necessary and sufficient condition for  $\mathcal{D}$  to be multiplicatively irreducible and adapted to  $E$  is that for each closed point  $P$ , the fiber  $\mathcal{D}_P$  would be generated by a germ of a vector field  $D = a\partial_x + b\partial_y$ ,  $a$  and  $b$  being relatively prime elements in  $\mathcal{O}_{X,P}$ .

Throughout the paper we will suppose that  $\mathcal{D}$  is multiplicatively irreducible and adapted to  $E$ .

(1.4) Consider  $\pi: X^\sim \rightarrow X$ , the blowing-up of  $X$  with center in a closed point  $P$ . Let  $\mathcal{D}'$  denote the image of  $\mathcal{D}$  by the composition of the two natural morphisms

$$(1.4.1) \quad \Theta_X \rightarrow \text{Hom}_{\mathcal{O}_X}(\Omega_X, \pi_*\mathcal{O}_{X^\sim}) \rightarrow \pi_* \text{Hom}_{\mathcal{O}_{X^\sim}}(\pi^*\Omega_X, \mathcal{O}_{X^\sim})$$

(the second one being an isomorphism) and let  $\mathcal{D}''$  denote the image of the composition of the two natural morphisms

$$(1.4.2) \quad \pi^*\mathcal{D}' \rightarrow \pi^*\pi_* \text{Hom}_{\mathcal{O}_{X^\sim}}(\pi^*\Omega_X, \mathcal{O}_{X^\sim}) \rightarrow \text{Hom}_{\mathcal{O}_{X^\sim}}(\pi^*\Omega_X, \mathcal{O}_{X^\sim}).$$

Finally, denote by  $\mathcal{D}^\pi$  the inverse image of  $\mathcal{D}''$  by the natural morphism

$$(1.4.3) \quad \Theta_{X^\sim} \rightarrow \text{Hom}_{\mathcal{O}_{X^\sim}}(\pi^*\Omega_X, \mathcal{O}_{X^\sim}).$$

$\mathcal{D}^\pi$  is an invertible sub- $\mathcal{O}_{X^\sim}$ -module of  $\Theta_{X^\sim}$ . Let  $E^\pi$  be the divisor of  $X^\sim$  (with normal crossings) obtained by considering on  $\pi^{-1}(E \cup \{P\})$  its reduced structure. Define the *adapted to  $E$  strict transform  $\mathcal{D}^\sim$  of  $\mathcal{D}$  by  $\pi$*  to be  $\mathcal{D}^\sim = (\alpha(\mathcal{D}^\pi), E^\pi)$ .

(1.5) In terms of coordinates,  $\mathcal{D}^\sim$  can be expressed as follows: Let  $(x, y)$  be an adapted to  $E$  regular set of parameters at the point  $P$ . For every closed point  $P'$  of  $X^\sim$  with  $\pi(P') = P$ , an adapted to  $E^\pi$  regular set of parameters  $(x', y')$  can be

obtained by exactly one of the two following transformations

$$(1.5.1) \quad x' = x; \quad x'(y' + \zeta) = y, \quad \zeta \in k,$$

$$(1.5.2) \quad x'y' = x; \quad y' = y.$$

According to the notation in [4], the first one will be called a  $(T - 1, \zeta)$  transformation and the second one a  $(T - 2)$  transformation.

Suppose that we have a  $(T - 1, \zeta)$  transformation. If  $P \notin E$  or if  $x = 0$  is a local equation for  $E$  at  $P$  or if  $\zeta \neq 0$ , then  $x' = 0$  is a local equation for  $E^\pi$  at  $P'$ . If  $\zeta = 0$  and  $y = 0$  or  $xy = 0$  are local equations for  $E$  at  $P$ , then  $x'y' = 0$  is a local equation for  $E^\pi$  at  $P'$ . Let  $\mathcal{D}_P$  be generated by  $D = a\partial_x + b\partial_y$ . Then  $\mathcal{D}_{P'}^\sim$  is generated by the germ of the vector field  $D'$  given by

$$(1.5.3) \quad D' = x''(ax'^{-1}\partial_{x'} + (b - (y' + \zeta)a)x'^{-1}\partial_{y'})$$

if  $P \notin E$ ;

$$(1.5.4) \quad D' = x''(a\partial_{x'} + (bx'^{-1} - (y' + \zeta)a)\partial_{y'})$$

if  $x = 0$  is a local equation for  $E$  at  $P$ ;

$$(1.5.5) \quad D' = x''(ax'^{-1}\partial_{x'} + (b - ax'^{-1})\partial_{y'})$$

if  $y = 0$  is a local equation for  $E$  at  $P$  and  $\zeta = 0$ ;

$$(1.5.6) \quad D' = x''(ax'^{-1}\partial_{x'} + (b - ax'^{-1})(y' + \zeta)\partial_{y'})$$

if  $y = 0$  is a local equation for  $E$  at  $P$  and  $\zeta \neq 0$ ;

$$(1.5.7) \quad D' = x''(a\partial_{x'} + (b - a)\partial_{y'})$$

if  $xy = 0$  is a local equation for  $E$  at  $P$  and  $\zeta = 0$ ; and

$$(1.5.8) \quad D' = x''(a\partial_{x'} + (b - a)(y' + \zeta)\partial_{y'})$$

if  $xy = 0$  is a local equation for  $E$  at  $P$  and  $\zeta \neq 0$ . In the above expressions  $t$  is the integer such that the coefficients of  $\partial_{x'}$  and  $\partial_{y'}$  in  $D'$  are relatively prime elements in the regular local ring  $\mathcal{O}_{X', P'}$ .

For a  $T - 2$  transformation, one can obtain analogous expressions for  $D'$ . Namely, these expressions are those obtained by interchanging  $x$  and  $y$ ,  $x'$  and  $y'$  in the above cases for  $\zeta = 0$ .

(1.6) The adapted order  $\nu_P(\mathcal{D}, E)$  of  $\mathcal{D}$  at a closed point  $P$  of  $X$  is defined to be the maximum integer  $n$  such that

$$(1.6.1) \quad \mathcal{D}_P \subseteq \mathfrak{m}_P^n \cdot \Theta_{X,P}[E],$$

where  $\mathfrak{m}_P$  is the maximal ideal of  $\mathcal{O}_{X,P}$ . The adapted order  $\nu_P(\mathcal{D}, E)$  is also the minimum of the orders, relative to the filtration given by  $\mathfrak{m}_P$  in  $\mathcal{O}_{X,P}$ , of the coefficients of any generator of  $\mathcal{D}_P$  in any basis of  $\Theta_{X,P}[E]$ . If  $\nu_P(\mathcal{D}, E) = 0$ ,  $P$  will be said to be a regular point for  $\mathcal{D}$  adapted to  $E$ .

Looking at the expressions (1.5.3)–(1.5.8) it can be easily deduced that

$$(1.6.2) \quad \nu_{P'}(\mathcal{D}^\sim, E^\pi) \leq \nu_P(\mathcal{D}, E).$$

## 2. Construction of the invariant.

(2.1) Let us denote by  $s_p(E)$  the number of irreducible components of  $E$  through  $P$ . Let  $(x, y)$  be an adapted to  $E$  regular set of parameters at  $P$  and let us suppose that  $\mathcal{D}_p$  is generated by  $D = a\partial_x + b\partial_y$ . The point  $P$  will be said to be of *type zero* for  $\mathcal{D}$  if:

(i)  $s_p(E) = 1$ .

(ii) The order  $\nu(a)$  (resp.  $\nu(b)$ ) of  $a$  (resp.  $b$ ) relative to  $\mathfrak{m}_p$  is equal to  $\nu_p(\mathcal{D}, E)$  in the case that  $y = 0$  (resp.  $x = 0$ ) is a local equation for  $E$  at  $P$ .

Otherwise,  $P$  will be said to be of *type one*.

For an element  $f \in \mathcal{O}_{X,P}$  with order  $\nu(f) \geq r$ ,  $r \in \mathbf{N}$ , denote by  $\mathcal{J}^r(f)$  the affine plane  $\mathbf{A}^2(k)$  if  $r < \nu(f)$  and the Hironaka tangent space (or directrix, see [4]) if  $r = \nu(f)$  (i.e., in this case  $\mathcal{J}^r(f)$  is the maximum linear subvariety of the tangent cone of  $f = 0$  leaving it invariant by translations).

Set  $r = \nu_p(\mathcal{D}, E)$ . We will define the *directrix*  $\mathcal{J}_p(\mathcal{D}, E)$  of  $\mathcal{D}$  at  $P$  adapted to  $E$  as either

$$(2.1.1) \quad \mathcal{J}_p(\mathcal{D}, E) = \mathcal{J}^r(a) \cap \mathcal{J}^r(b)$$

if  $s_p(E) = 0$  or 2, or

$$(2.1.2) \quad \mathcal{J}_p(\mathcal{D}, E) = \mathcal{J}^r(a)$$

if  $P$  is of type zero (resp. one) and  $y = 0$  (resp.  $x = 0$ ) is a local equation for  $E$  at  $P$ , or finally

$$(2.1.3) \quad \mathcal{J}_p(\mathcal{D}, E) = \mathcal{J}^r(b)$$

if  $P$  is of type zero (resp. one) and  $x = 0$  (resp.  $y = 0$ ) is a local equation for  $E$  at  $P$ .

We always have that  $\dim \mathcal{J}_p(\mathcal{D}, E) \leq 1$ . If  $\dim \mathcal{J}_p(\mathcal{D}, E) = 1$ , the directrix defines a closed point  $\mathbf{P}(\mathcal{J}_p(\mathcal{D}, E))$  in the exceptional divisor of the blowing-up  $\pi: X^- \rightarrow X$  of  $X$  with center  $P$ .

(2.2) **LEMMA 1.** *If  $P'$  is a closed point of the exceptional divisor of  $\pi$  such that  $\nu_{P'}(\mathcal{D}^-, E^\pi) = \nu_p(\mathcal{D}, E)$ , then  $\dim \mathcal{J}_{P'}(\mathcal{D}, E) = 1$  and  $P' = \mathbf{P}(\mathcal{J}_p(\mathcal{D}, E))$ .*

**PROOF.** Let us suppose that  $P$  is of the type zero and that  $x = 0$  is a local equation for  $E$  at  $P$ . Set  $r = \nu_p(\mathcal{D}, E)$  and let us suppose that we have a  $(T - 1, \zeta)$  transformation and that  $\mathcal{D}_{P'}^-$  is generated by  $D' = a'\partial_{x'} + b'\partial_{y'}$ . Looking at (1.5.4) one realizes that  $t = -1(r - 1)$  and that

$$(2.2.1) \quad a' = ax'^{-r+1}; \quad b' = bx'^{-r} - (y' + \zeta)ax'^{-r+1}.$$

If  $\nu'$  denotes the order relative to  $\mathfrak{m}_{P'}$ ,  $\nu_{P'}(\mathcal{D}^-, E^\pi) = r$  implies that  $\nu'(bx'^{-r}) = r$  since the order  $\nu(b)$  of  $b$  relative to  $\mathfrak{m}_p$  is  $r$ . Let  $\phi$  denote the initial form of  $b$ . A necessary condition in order to have  $\nu'(bx'^{-r}) = r$  is that one has  $\phi = \lambda(y - \zeta\underline{x})$ ,  $\lambda \in k^*$ , where  $\underline{x}$  (resp.  $\underline{y}$ ) is the initial form of  $x$  (resp.  $y$ ), and in this case  $\mathcal{J}^r(b) = (\underline{y} - \zeta\underline{x} = 0) = \mathcal{J}_p(\mathcal{D}, E)$ . If  $\dim \mathcal{J}_p(\mathcal{D}, E) = 0$ , then  $\phi$  is not a power of a linear form and  $\nu_{P'}(\mathcal{D}^-, E^\pi) < r$ .

A similar argument can be considered for  $T - 2$  and in the other cases.

(2.3) Let us suppose that  $s_p(E) \geq 1$  and  $\dim \mathcal{J}_p(\mathcal{D}, E) = 1$ . By making, if necessary, changes of type  $x_1 = y$  or  $y_1 = y + \lambda x$ ,  $\lambda \in k$ , in an adapted to  $E$  regular set of parameters  $(x, y)$  one can assume that  $(x, y)$  can be chosen such that a generator  $D = a\partial_x + b\partial_y$ , of  $\mathcal{D}_p$  takes exactly one of the following forms:

- I.  $E$  given by  $x$ , type zero,  $\mathcal{J}^r(b) = (y = 0)$ .
- II.  $E$  given by  $y$ , type zero,  $\mathcal{J}^r(a) = (y = 0)$ .
- (2.3.1) III.  $E$  given by  $x$ , type one,  $\mathcal{J}^r(a) = (y = 0)$ .
- IV.  $E$  given by  $y$ , type one,  $\mathcal{J}^r(b) = (y = 0)$ .
- V.  $E$  given by  $xy$ ,  $\mathcal{J}_p(\mathcal{D}, E) = (y + \zeta x = 0)$ ,  $\zeta \in k$ .

If the forms are I or III, we define the *stability invariant*  $st_p(\mathcal{D}, E)$  to be 0 and if otherwise  $st_p(\mathcal{D}, E) = 1$ .

Following notations as in [4], for a power series  $f = \sum_{i,j} f_{ij}x^i y^j \in k[[x, y]]$  and  $r \in \mathbb{N}$ , set

$$(2.3.2) \quad \gamma^r(f; x, y) = \min(i(r - j)^{-1}; j < r, f_{ij} \neq 0).$$

If  $r =$  order of  $f$ ,  $(x', y')$  are given by a  $(T - 1, 0)$  transformation and  $f' = f \cdot x'^{-r}$ , one has that

$$(2.3.3) \quad \gamma^r(f'; x', y') = \gamma^r(f; x, y) - 1.$$

Moreover, the necessary and sufficient condition for  $f$  to have order  $< r$  is that  $\gamma^r(f; x, y) < 1$ , and if  $r =$  order of  $f$ , then  $\mathcal{J}^r(f) = (y = 0)$  if and only if  $\gamma^r(f; x, y) > 1$ .

Set  $r = \nu_p(\mathcal{D}, E)$ . We define  $\gamma(\mathcal{D}, E; x, y)$  to be equal to

$$(2.3.4) \quad \begin{aligned} &\min(\gamma^r(ya; x, y), \gamma^r(b; x, y)), \quad \text{if I,} \\ &\min(\gamma^r(a; x, y), \gamma^r(yb; x, y)), \quad \text{if II,} \\ &\min(\gamma^{r+1}(ya; x, y), \gamma^{r+1}(b; x, y)), \quad \text{if III,} \\ &\min(\gamma^{r+1}(a; x, y), \gamma^{r+1}(yb; x, y)), \quad \text{if IV,} \\ &\min(\gamma^r(a; x, y), \gamma^r(b; x, y)), \quad \text{if V.} \end{aligned}$$

Since  $a$  and  $b$  are relatively prime, one has that  $\gamma_p(\mathcal{D}, E; x, y) < \infty$  if  $r \geq 2$ .

$\mathcal{D}$  will be said to be *prepared with respect to*  $(x, y)$  if one of the following situations holds:

- (i)  $\gamma = \gamma(\mathcal{D}, E; x, y) \notin \mathbb{N}$ .
- (ii)  $\gamma \in \mathbb{N}$  and  $\mathcal{D}_p$  takes one of the forms II, IV, or V.
- (iii)  $\gamma \in \mathbb{N}$  and  $\gamma$  does not increase after any change of the type  $y_1 = y + \lambda x^\gamma$ ,  $\lambda \in k$ .

If  $r \geq 2$  and  $\mathcal{D}$  is not prepared with respect to  $(x, y)$ , then from  $(x, y)$  one can obtain a new adapted to  $E$  regular set of parameters  $(x, y^\sim)$  such that  $\mathcal{D}$  is prepared with respect to  $(x, y^\sim)$  by making successive changes of the type  $y_1 = y + \lambda x^\gamma$  increasing  $\gamma = \gamma(\mathcal{D}, E; x, y)$ . The number of these changes is always finite since otherwise  $a$  and  $b$  would not be relatively prime elements in  $\mathcal{O}_{X,P}$  ( $a$  and  $b$  are

relatively prime in  $\mathcal{O}_{X,P}$  if and only if they are relatively prime in the completion  $\widehat{\mathcal{O}}_{X,P}$ . If  $r = 1$  it may happen that one has to make infinitely many changes  $y \mapsto y_1$  (see Example (4.5) below).

Now, we define  $\gamma_P(\mathcal{D}, E)$  to be the minimum  $\gamma_P(\mathcal{D}, E; x, y)$ , where  $(x, y)$  runs over all prepared situations. If  $r = 1$  and if the number of changes  $y \mapsto y_1$  needed from any adapted to  $E$  regular set of parameters  $(x, y)$  to reach a prepared situation is infinite or if for any adapted to  $E$  regular set of parameters  $(x, y)$  with respect to which  $\mathcal{D}$  is prepared we have that  $\gamma(\mathcal{D}, E; x, y) = \infty$ , then we define  $\gamma_P(\mathcal{D}, E) = \infty$ . For the sake of completeness, we make the convention  $\gamma_P(\mathcal{D}, E) = 0$  if  $s_P(E) = 0$  or  $\dim \mathcal{I}_P(\mathcal{D}, E) = 0$ .

(2.4) By Lemma 1, we can obtain by successive blowing-ups only one sequence

$$(2.4.1) \quad P = P_0 \xleftarrow{\pi_0} P_1 \xleftarrow{\pi_1} \dots,$$

each  $P_i$  being the center of  $\pi_i$  and lying in the exceptional divisor of  $\pi_{i-1}$ , in such a way that the adapted order of the successive adapted to  $E_{i-1}$  strict transforms  $\mathcal{D}_i$  of  $\mathcal{D}_{i-1}$  does not change ( $\mathcal{D}_0 = \mathcal{D}, E_0 = E$ ). This sequence will be called the *adapted to  $E$  stationary way of  $P$  for  $\mathcal{D}$*  and it will be denoted by  $S_P(\mathcal{D}, E)$ . The length  $l_P(\mathcal{D}, E)$  of  $S_P(\mathcal{D}, E)$  may be “a priori” finite or infinite.

Let  $i \leq l_P(\mathcal{D}, E)$ . We define the number  $\delta_P(\mathcal{D}, E; P_i)$  to be equal to 1 if  $i = 0$  and equal to 0 if  $i > 0$ .

Now, we define the (lexicographic) *invariant*  $\text{Inv}_P(\mathcal{D}, E; P_i)$  as

$$(2.4.2) \quad \text{Inv}_P(\mathcal{D}, E; P_i) = (\dim \mathcal{I}_{P_i}(\mathcal{D}_i, E_i), \delta_P(\mathcal{D}, E; P_i), \text{type of } P_i, \text{st}_{P_i}(\mathcal{D}_i, E_i), \gamma_{P_i}(\mathcal{D}_i, E_i)).$$

### 3. Desingularization results (order $\geq 2$ ).

(3.1) LEMMA 2. *Let  $0 \leq i \leq l_P(\mathcal{D}, E)$  and let us suppose that  $\mathcal{D}_i$  takes at  $P_i$  the form I (resp. III, resp. IV, resp. V) and that  $\mathcal{D}_{i+1}$  takes at  $P_{i+1}$  the form I (resp. III, resp. V, resp. V). Then one has that*

$$(3.1.1) \quad \gamma_{P_{i+1}}(\mathcal{D}_{i+1}, E_{i+1}) \leq \gamma_{P_i}(\mathcal{D}_i, E_i) - 1.$$

PROOF. If  $\gamma_{P_i}(\mathcal{D}_i, E_i) = \infty$ , there is nothing to see. Suppose that  $\gamma_{P_i}(\mathcal{D}_i, E_i) < \infty$ . If  $(x, y)$  is an adapted to  $E_i$  regular set of parameters with respect to which  $\mathcal{D}_i$  is prepared at  $P_i$  and for which  $\mathcal{D}_i$  takes the form I, III, IV or V, then the only transformation to make is  $(T - 1, 0)$  (note that in case V one necessarily has  $\zeta = 0$ , since otherwise one would not have the form V in  $\mathcal{D}_{i+1}$ ). Because of (2.3.3) one can see that

$$(3.1.2) \quad \gamma(\mathcal{D}_{i+1}, E_{i+1}; x', y') = \gamma(\mathcal{D}_i, E_i; x, y) - 1.$$

Now, in view of the commutativity of making the change  $y'_1 = y' + \lambda x'^{\gamma'}$  after  $(T - 1, 0)$  or making  $(T - 1, 0)$  after the change  $y_1 = y + \lambda x^\gamma$ , with  $\gamma = \gamma' + 1$ , one has that  $\mathcal{D}_{i+1}$  is prepared with respect to  $(x', y')$  and (3.1.1) follows from this fact.

(3.2) LEMMA 3. Let  $0 < i \leq l_p(\mathcal{D}, E)$  and let us suppose that  $\mathcal{D}_i$  takes the form II at  $P_i$ . Then we have that  $i = l_p(\mathcal{D}, E)$  if  $r = v_p(\mathcal{D}, E) \geq 2$ . If  $r = 1$  and  $i < l_p(\mathcal{D}, E)$ , then we have that

$$(3.2.1) \quad \dim \mathcal{J}_{P_{i+1}}(\mathcal{D}_{i+1}, E_{i+1}) = 0.$$

REMARK. If we have (3.2.1), then  $i + 1 = l_p(\mathcal{D}, E)$  by Lemma 1.

PROOF OF LEMMA 3. By the above remark,  $\dim \mathcal{J}_{P_{i-1}}(\mathcal{D}_{i-1}, E_{i-1}) = 1$ . If  $s_{P_{i-1}}(E_{i-1}) \geq 1$ , then  $\mathcal{D}_{i-1}$  takes at  $P_{i-1}$  one of the forms III or V (the other forms cannot be transformed into II).

Let us suppose that  $\mathcal{D}_{i-1}$  has the form III and that it is generated at  $P_{i-1}$  by

$$(3.2.2) \quad a(x\partial/\partial x) + b\partial/\partial y$$

with order of  $b \geq r + 1$  and  $\mathcal{J}^r(a) = (\underline{y} = 0)$ . If we apply  $(T - 1, 0)$ , then  $\mathcal{D}_i$  is generated at  $P_i$  by

$$(3.2.3) \quad a'(x'\partial/\partial x') + b'\partial/\partial y',$$

where  $a' = a/x''$ ,  $b' = b/x''^{r+1} - y'a/x''$ . Because of our hypotheses on the form II of  $\mathcal{D}_i$  at  $P_i$ , we have that

$$(3.2.4) \quad a' = \lambda y'' + x'(\dots), \quad b' = \mu x'' + (\text{terms of higher order})$$

with  $\lambda, \mu \in k^* = k - \{0\}$ . Interchanging  $x'$  and  $y'$ , we can suppose that  $\mathcal{D}_i$  is generated at  $P_i$  by

$$(3.2.5) \quad (\mu y' + (\dots))\partial/\partial x + (\lambda x' + y(\dots))(\underline{y}\partial/\partial y).$$

By applying  $(T - 1, 0)$  to (3.2.5) one can see that the adapted order drops if  $r \geq 2$ , so  $i = l_p(\mathcal{D}, E)$ , and that the dimension of the directrix drops if  $r = 1$  and if the adapted order does not drop, so we have (3.2.1).

Let us suppose that  $\mathcal{D}_{i-1}$  has the form V and that it is generated at  $P_{i-1}$  by

$$(3.2.6) \quad a(x\partial/\partial x) + b(y\partial/\partial y),$$

where  $\mathcal{J}^r(a) \cap \mathcal{J}^r(b) = (\underline{y} - \zeta \underline{x} = 0)$  (necessarily  $\zeta \neq 0$ , otherwise  $s_{P_i}(E_i) = 2$ ). If we apply  $(T - 1, 0)$ , then  $\mathcal{D}_i$  is generated at  $P_i$  by (3.2.3) where  $a' = a/x''$  and  $b' = (b - a)(y' + \zeta)/x''$ . By the hypotheses, one has necessarily  $\text{In}(b') = \mu \underline{x}''$ ,  $\mu \neq 0$ , so  $\text{In}(a) = \text{In}(b) = \lambda(\underline{y} - \zeta \underline{x})^r$ ,  $\lambda \neq 0$ , since otherwise  $\underline{y}'$  will occur in  $\text{In}(b')$ . Thus we will have (3.2.4) and the same conclusions as in the preceding case.

Now suppose that  $s_{P_{i-1}}(E_{i-1}) = 0$  (so  $i - 1 = 0$ ). We can take an adapted to  $E$  regular set of parameters  $(x, y)$  such that  $\mathcal{J}_P(\mathcal{D}, E) = (\underline{y} = 0)$  and we can suppose that  $\mathcal{D}$  is generated at  $P$  by

$$(3.2.7) \quad a\partial/\partial x + b\partial/\partial y$$

with order  $(b) \geq r + 1$  (otherwise, after  $(T - 1, 0)$  we obtain the form I, or  $\dim \mathcal{J}_{P_1}(\mathcal{D}_1, E_1) = 0$ , or the adapted order drops), so  $\mathcal{J}^r(a) = (\underline{y} = 0)$  and after  $(T - 1, 0)$  one comes to (3.2.4).

(3.3) THEOREM 1. Let  $0 < i \leq l_p(\mathcal{D}, E)$  and assume  $r = v_p(\mathcal{D}, E) \geq 2$ . Then we have that

$$(3.3.1) \quad \text{Inv}(S_p(\mathcal{D}, E); P_i) < \text{Inv}(S_p(\mathcal{D}, E); P_{i-1})$$

for the lexicographic order.

PROOF. Because of Lemma 1,  $\dim \mathcal{I}_{P_{i-1}}(\mathcal{D}_{i-1}, E_{i-1}) = 1$ , otherwise  $i - 1 = l_p(\mathcal{D}, E)$ . If  $i = 1$  or if  $\dim \mathcal{I}_P(\mathcal{D}, E) = 0$ , the result is trivial. Let us suppose that  $i > 1$  and that  $\dim \mathcal{I}_P(\mathcal{D}, E) = 1$ . Then  $s_{P_{i-1}}(E_{i-1}) \geq 1$ ,  $s_{P_i}(E_i) \geq 1$ , and  $\mathcal{D}_{i-1}$  and  $\mathcal{D}_i$  have one of the forms I–V.

As a notation let us write for instance  $\text{III} \rightarrow \text{I}$  if  $\mathcal{D}_{i-1}$  has the form III at  $P_{i-1}$  and  $\mathcal{D}_i$  has the form I at  $P_i$ . If we have  $\text{I} \rightarrow \text{I}$ ,  $\text{III} \rightarrow \text{III}$ ,  $\text{IV} \rightarrow \text{V}$ ,  $\text{V} \rightarrow \text{V}$ , Lemma 2 gives us the result. The only transition beginning at I is  $\text{I} \rightarrow \text{I}$ . Because of Lemma 3, there is no transition beginning at II. If  $\text{III} \rightarrow \text{I}$  or  $\text{III} \rightarrow \text{II}$ , the type drops and the transitions  $\text{III} \rightarrow \text{IV}$  and  $\text{III} \rightarrow \text{V}$  are not possible. The only transition beginning at IV is  $\text{IV} \rightarrow \text{V}$ . If  $\text{V} \rightarrow \text{I}$  or  $\text{V} \rightarrow \text{II}$  the type drops, if  $\text{V} \rightarrow \text{III}$  the stability invariant drops and  $\text{V} \rightarrow \text{IV}$  is not possible.

(3.4) Note that if  $\gamma_p(\mathcal{D}, E) < \infty$ , then  $\gamma_p(\mathcal{D}, E) \in (1/r!) \mathbf{N}$ . As the other invariants are nonnegative integers, we have the following:

COROLLARY 1. *If  $\nu_p(\mathcal{D}, E) \geq 2$ , then  $S_p(\mathcal{D}, E)$  is always finite.*

(3.5) As the adapted order is semicontinuous and  $\mathcal{D}$  is multiplicatively irreducible and adapted to  $E$ , the points of adapted order greater or equal to 1 form a finite set of  $X$ , thus we can state:

COROLLARY 2 (desingularization theorem). *All the possible sequences*

$$(3.5.1) \quad X = X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \dots$$

obtained by blowing-up successive points with adapted order for the strict transform of  $\mathcal{D}$  greater than or equal to 2 are finite sequences. Thus the strict transform  $(\mathcal{D}^-, E^-)$  of  $\mathcal{D}$  by the composition of all the blowing-ups of any one of these sequences has only points whose adapted order is less than or equal to 1.

**4. The case of adapted order equal to 1.**

(4.1) LEMMA 4. *Let us suppose that  $\mathcal{D}$  has the form I at  $P$  in the adapted to  $E$  regular set of parameters  $(x, y)$  and that*

$$(4.1.1) \quad \gamma(\mathcal{D}, E; x, y) > \gamma_p(\mathcal{D}, E).$$

Then  $\gamma_p(\mathcal{D}, E) \in \mathbf{N}$  and there is a coordinate change  $y_1 = y + \xi x^\gamma$ ,  $\gamma = \gamma_p(\mathcal{D}, E)$ , such that  $\mathcal{D}$  is prepared at  $P$  with respect to  $(x, y_1)$  and such that

$$(4.1.2) \quad \gamma(\mathcal{D}, E; x, y_1) = \gamma_p(\mathcal{D}, E).$$

PROOF. Following notations as in [4], for a power series  $f = \sum_{i,j} f_{i,j} x^i y^j$ ,  $r \in \mathbf{N}$ , and  $\gamma \in \mathbf{Q}$ , denote

$$(4.1.3) \quad \{f\}_{(x,y)}^{\gamma,r} = \sum_{j=0}^r f_{\gamma(r-j),j} x^{\gamma(r-j)} y^j$$

(if  $\gamma(r - j) \notin \mathbf{N}$ , set  $f_{\gamma(r-j),j} = 0$ ). Let us suppose that  $\mathcal{D}$  is generated at  $P$  by

$$(4.1.4) \quad D = a \cdot x \partial / \partial x + b \partial / \partial y,$$

where  $\mathcal{I}^r(b) = (y = 0)$ ,  $r = \nu_p(\mathcal{D}, E)$ . Let  $\gamma = \gamma(\mathcal{D}, E; x, y)$  and denote

$$(4.1.5) \quad \{D\}_{(x,y)} = \left( (1/y) \{ya\}_{(x,y)}^{\gamma,r}, \{b\}_{(x,y)}^{\gamma,r} \right)$$

which is an element of  $(k[x, y])^2$ . The fact of  $\mathcal{D}$  being prepared with respect to  $(x, y)$  only depends on  $\{D\}_{(x,y)}$  (modulo  $k^* = k - \{0\}$ ).

Let  $(x_2, y_2)$  be an adapted to  $E$  regular set of parameters such that  $\mathcal{D}$  is prepared with respect to  $(x_2, y_2)$  and such that

$$(4.1.6) \quad \gamma(\mathcal{D}, E; x_2, y_2) = \gamma_P(\mathcal{D}, E).$$

We have that

$$(4.1.7) \quad x_2 = \lambda \cdot x \cdot u, \quad y_2 = \mu \cdot y \cdot v + \rho x^n \cdot w$$

where  $u, v, w$  are units in  $k[[x, y]] = \hat{\mathcal{O}}_{X,P}$  which are equal to 1 modulo the maximal ideal  $\hat{m}_P$  and  $\lambda, \mu \in k^*, \rho \in k$ . Because of (4.1.1) one has  $\rho \neq 0$  and  $n = \gamma_P(\mathcal{D}, E)$ ; in particular  $\gamma_P(\mathcal{D}, E) \in \mathbb{N}$ . In fact, looking at the monomial  $\sigma \cdot y^r, \sigma \in k^*$ , which occurs in the coefficient  $b$ , one can see that if  $n < \gamma(\mathcal{D}, E; x, y)$  and  $\rho \neq 0$  then

$$(4.1.8) \quad \gamma(\mathcal{D}, E; x_2, y_2) = n$$

and if  $n > \gamma(\mathcal{D}, E; x, y)$  then

$$(4.1.9) \quad \gamma(\mathcal{D}, E; x_2, y_2) \geq \gamma(\mathcal{D}, E; x, y).$$

Consider the coordinate change

$$(4.1.10) \quad x_1 = x; \quad y_1 = y + (\rho/\lambda^n)x^n.$$

Then  $\gamma(\mathcal{D}, E; x_1, y_1) = n$  and by an easy computation one has that

$$(4.1.11) \quad \{D\}_{(x_1,y_1)} = \mu^r \cdot \{D\}_{(x_2,y_2)};$$

thus  $\mathcal{D}$  is prepared with respect to  $(x, y_1)$ .

(4.2) THEOREM 2. *If  $\nu_P(\mathcal{D}, E) = 1$ , then the stationary way  $S_P(\mathcal{D}, E)$  is infinite if and only if there is a step  $P_l, l > 1$ , such that  $\mathcal{D}_l$  takes the form I at  $P_l$  and  $\gamma_{P_l}(\mathcal{D}_l, E_l) = \infty$ . Moreover, if  $i \geq l$  then  $\mathcal{D}_i$  takes the form I at  $P_i$  and  $\gamma_{P_i}(\mathcal{D}_i, E_i) = \infty$ .*

PROOF. If  $\gamma_{P_i}(\mathcal{D}_i, E_i) < \infty$  for all  $i = 1, \dots$ , then the same techniques as in Theorem 1 show that  $S_P(\mathcal{D}, E)$  is finite. Thus, for some  $l \geq 1$  one has that  $\gamma_{P_l}(\mathcal{D}_l, E_l) = \infty$ . Because of the multiplicative irreducibility of  $\mathcal{D}_l$ ,  $\mathcal{D}_l$  takes the form I or II at  $P_l$ . If  $\mathcal{D}_l$  takes the form II, then Lemma 3 leads us to a contradiction with the fact that  $S_P(\mathcal{D}, E)$  is infinite, so  $\mathcal{D}_l$  takes the form I. As the only way to lose the form I with the transformation  $(T - 1, 0)$  is to reach  $\gamma_{P_l}(\mathcal{D}_l, E_l) = 1$  and in this case  $\dim \mathcal{J}_{P_l}(\mathcal{D}_l, E_l) = 0$  (so  $S_P(\mathcal{D}, E)$  is finite), then the last statement holds.

For the converse, let us suppose that  $\mathcal{D}_l$  takes the form I at  $\mathcal{D}_l$  and  $\gamma_{P_l}(\mathcal{D}_l, E_l) = \infty$ . It is enough to prove that  $l < l_P(\mathcal{D}, E)$  and that  $\mathcal{D}_{l+1}$  takes the form I at  $P_{l+1}$  and  $\gamma_{P_{l+1}}(\mathcal{D}_{l+1}, E_{l+1}) = \infty$ . As  $\gamma_{P_l}(\mathcal{D}_l, E_l) = \infty$ , we can choose an adapted to  $E_l$  regular set of parameters  $(x, y)$  at  $P_l$  such that

$$(4.2.1) \quad \gamma(\mathcal{D}_l, E_l; x, y) > 2;$$

by applying  $(T - 1, 0)$  we have that

$$(4.2.2) \quad \gamma(\mathcal{D}_{l+1}, E_{l+1}; x', y') > 1$$

and then  $l < l_P(\mathcal{D}, E)$  and  $\mathcal{D}_{l+1}$  has the form I. Now, suppose that

$$(4.2.3) \quad \gamma_{P_{l+1}}(\mathcal{D}_{l+1}, E_{l+1}) = \gamma < \infty.$$

Then we can choose  $(x, y)$  such that

$$(4.2.4) \quad \gamma(\mathcal{D}_l, E_l; x, y) > \gamma + 1,$$

and by applying  $(T - 1, 0)$  we have that

$$(4.2.5) \quad \gamma(\mathcal{D}_{l+1}, E_{l+1}; x', y') > \gamma.$$

By Lemma 4, there is a coordinate change  $x'_1 = x', y'_1 = y' + \xi x'^\gamma$  such that  $\mathcal{D}_{l+1}$  is prepared with respect to  $(x'_1, y'_1)$  and

$$(4.2.6) \quad \gamma(\mathcal{D}_{l+1}, E_{l+1}; x'_1, y'_1) = \gamma.$$

Consider the change  $x_1 = x, y_1 = y + \xi x^{\gamma+1}$ . Then by (3.1.2) and the argument following (3.1.2) we have that

$$(4.2.7) \quad \gamma(\mathcal{D}_l, E_l; x_1, y_1) = \gamma + 1 < \infty$$

and  $\mathcal{D}_l$  is prepared with respect to  $(x_1, y_1)$ , which is a contradiction with the fact that  $\gamma_P(\mathcal{D}_l, E_l) = \infty$ . Then

$$(4.2.8) \quad \gamma_{P_{l-1}}(\mathcal{D}_{l+1}, E_{l+1}) = \infty.$$

(4.3) **REMARK.** As a corollary of Theorem 2 one can easily obtain the result of Seidenberg [6, Theorem 12].

(4.4) Let us suppose that  $\mathcal{D}$  takes the form I at  $P$  and that  $\gamma_P(\mathcal{D}, E) = \infty$ . Then there are two possibilities:

A. There is an adapted to  $E$  regular set of parameters  $(x, y)$  such that  $\mathcal{D}$  is prepared with respect to  $(x, y)$ .

B. There is not such an adapted regular set of parameters.

In case A,  $\gamma(\mathcal{D}, E; x, y) = \infty$  and  $\mathcal{D}$  is generated at  $P$  by

$$(4.4.1) \quad D = ax\partial/\partial x + (y \cdot u)\partial/\partial y,$$

where  $u$  is a unit in  $\mathcal{O}_{X,P}$ . Take the irreducible closed curve  $Y$  of  $X$  given at  $P$  by  $y = 0$ . One can desingularize  $Y$  by a finite number of blowing-ups in closed points different from  $P$ , and making more blowing-ups, if necessary, one can suppose that  $Y$  has normal crossings with  $E$ . Now, if one makes the blowing-up of  $X$  with center at  $Y$ , it has the only effect of adding  $Y$  to  $E$  and thus in view of (4.4.1) the adapted order at  $P$  comes to 0. In this case we have an “*algebraic resolution*” at  $P$ .

If case B holds, let  $X^\wedge$  be the formal scheme of  $X$  at  $P$  and let  $\mathcal{D}^\wedge$  be obtained from the morphism  $X^\wedge \rightarrow X$  in a way similar to  $\mathcal{D}^\pi$ . If  $(x, y)$  is an adapted to  $E$  regular set of parameters in which  $\mathcal{D}$  takes the form I, then the infinite sequence of preparations coordinate changes  $y \mapsto y_1$  is always convergent in  $\mathcal{O}_{X^\wedge, P} = \mathcal{O}_{\hat{X}, P}$  and leads us to a regular set of parameters  $(x, y^\wedge)$  of the maximal ideal of  $\mathcal{O}_{X^\wedge, P}$  which is adapted to  $E^\wedge$  ( $=$  inverse image in  $X^\wedge$  of  $E$ ) and such that

$$(4.4.2) \quad \gamma(\mathcal{D}^\wedge, E^\wedge; x, y^\wedge) = \infty.$$

If  $Y^\wedge$  is given by  $y^\wedge = 0$ , then the blowing-up of  $X^\wedge$  with center at  $Y^\wedge$  adds  $Y^\wedge$  to  $E^\wedge$  and then the adapted to  $E^\wedge \cup Y^\wedge$  order of  $\mathcal{D}^\wedge$  at  $P$  is 0. One has an “*algebroid resolution*”.

(4.5) EXAMPLE. Case B effectively occurs as the following example shows:  $X = \mathbf{A}^2(k)$ ,  $P = \text{origin}$  and  $\mathcal{D}$  is globally generated by

$$(4.5.1) \quad D = x \cdot x\partial/\partial x + (y + x)\partial/\partial y,$$

where  $E$  is given by  $x = 0$ .

In fact, let us suppose that there exists a change  $y \mapsto y_1$ ,  $y_1 \in \mathcal{O}_{X,P}$  such that, locally at  $P$ ,

$$(4.5.2) \quad D = x \cdot x\partial/\partial x + (u \cdot y_1)\partial/\partial y,$$

where  $u$  is a unit in  $\mathcal{O}_{X,P}$ . Then the sequence of the infinitely near points obtained by blowing-up  $y_1 = 0$  at the origin is necessarily equal to  $S_P(\mathcal{D}, E)$ . By computing  $S_P(\mathcal{D}, E)$  from (4.5.1) one obtains that  $y_1 = 0$  defines the algebroid curve

$$(4.5.3) \quad y = - \sum_{i>1} (i-1)!x^i,$$

which is not an algebraic curve.

#### REFERENCES

1. F. Cano, *Teoría de distribuciones sobre variedades algebraicas*, Colecc. de Mon. del Instituto Jorge Juan, C.S.I.C., Madrid, 1983.
2. J. Giraud, *Forme normale d'une fonction sur une surface de caractéristique positive*, Bull. Soc. Math. France **111** (1983), 109–124.
3. ———, *Condition de Jung pour les revêtements radiciels de hauteur 1*, Proc. Algebraic Geometry, Tokyo/Kyoto 1982, Lecture Notes in Math., vol. 1016, Springer-Verlag, 1983, pp. 313–333.
4. H. Hironaka, *Desingularization of excellent surfaces*, Advanced Science Seminar in Algebraic Geometry (Summer 1967), Mimeographed notes by B. Bennett, Bowdoin College.
5. ———, *Characteristic polyhedra of singularities*, J. Math. Kyoto Univ. **10** (1967), 251–293.
6. A. Seidenberg, *Reduction of singularities of the differential equation  $Ady = Bdx$* , Amer. J. Math. **90** (1968), 248–269.
7. A. van den Essen, *Reduction of singularities of the differential equation  $Ady = Bdx$* , Équations Différentielles et Systèmes de Pfaff dans le Champ Complexe, Lecture Notes in Math., vol. 712, Springer-Verlag, 1979, pp. 44–49.

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