THE DUAL OF THE BERGMAN SPACE $A^1$ IN SYMMETRIC SIEGEL DOMAINS OF TYPE II

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Abstract. An affirmative answer is given to the following conjecture of R. Coifman and R. Rochberg: in any symmetric Siegel domain of type II, the dual of the Bergman space $A^1$ coincides with the Bloch space of holomorphic functions and can be realized as the Bergman projection of $L^\infty$.

Let $D$ be a symmetric Siegel domain of type II; let $v$ denote the Lebesgue measure in $D$ and $H(D)$ the space of holomorphic functions in $D$. The Bergman space $A^p(D), 0 < p \leq \infty$, is defined by $A^p(D) = H(D) \cap L^p(dv)$.

In the one-dimensional case, $D = \Pi^+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}$, R. Coifman and R. Rochberg [4] gave a proof of the following fact: the dual of the Bergman space $A^1(\Pi^+)$ coincides with the Bloch space of holomorphic functions and can be realized as the Bergman projection of $L^\infty$. Then, these authors conjectured that this characterization of the dual of $A^1$ should hold in any symmetric Siegel domain of type II.

In [1] (resp. [2 and 3]), an affirmative answer to this conjecture was given in the particular case of the Cayley transform of the unit ball in $\mathbb{C}^n, n > 1$, defined by $\{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : \text{Im} \, z_1 > |z'|^2 \}$ (resp. in the tube $\mathbb{R}^{n+1} + i \Gamma, n \geq 1$, over the spherical cone $\Gamma$ defined in $\mathbb{R}^{n+1}$ by $\Gamma = \{(y_0, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{n+1} : y_0 - y_1^2 - \cdots - y_n^2 > 0, y_0 > 0 \}$).

The purpose of this paper is to prove the Coifman-Rochberg conjecture in the general case of any symmetric Siegel domain of type II.

We shall denote by $B(\xi, z)$ the Bergman kernel of such a domain $D$. When $r$ is a strictly positive integer, S. G. Gindikin [5] defined a differential operator $\mathcal{D}_r$ in $D$ that satisfies the property

$$(\mathcal{D}_r) B(\xi, z) = C_r B^{1+r}(\xi, z), \quad \xi, z \in D.$$ 

The Bloch space $\mathcal{B}$, corresponding to the integer $r$ is defined in terms of that operator $\mathcal{D}_r$. A holomorphic function $g$ in $D$ is said to be a Bloch function when it satisfies the estimate

$$\sup_{z \in D} \{|\mathcal{D}_r g(z)| B^{-r}(z, z)\} < +\infty.$$
Denote by \( \mathcal{N} \) the subspace of \( H(D) \) consisting of those functions satisfying \( \mathcal{D}_g = 0 \); the Bloch space \( \mathcal{B} \), is defined as the quotient space \( \{ \text{Bloch functions} \}/\mathcal{N} \).

In view of the proof of the Coifman-Rochberg conjecture, let us recall at once that, unlike the bounded domains, the Bergman kernel \( B(\xi, z) \) of \( D \) is not integrable with respect to \( z \) because of its bad behaviour when \( z \) tends to infinity.

In the first chapter of this paper, we use S. G. Gindikin’s general theory [5] to characterize those integers \( r \) for which the function \( z \mapsto B^{1+r}(\xi, z) \) belongs to \( L^1(D) \), \( \xi \in D \); also enclosed in this chapter are some results that will be useful later.

In the second chapter, we prove that when \( r \) is sufficiently large, the dual of \( A^1 \) coincides with the Bloch space \( \mathcal{B}_r \). More precisely, let \( L \) be a bounded linear functional on \( A^1(D) \); by the Hahn-Banach theorem, there exists a bounded function \( b \) in \( D \) such that for any \( f \) in \( A^1(D) \), the following equality holds: \( L(f) = \int_D b f \, dv \).

We associate with \( b \) the holomorphic function \( G \) defined in \( D \) by

\[
G(\xi) = C \int_D B^{1+r}(\xi, z) b(z) \, dv(z);
\]

this function \( G \) possesses the following two properties:

1° \[
\sup_{\xi \in D} \{ |G(\xi)| B^{-r}(\xi, \xi) \} < +\infty;
\]

2° \[
L(f) = \int_D G(\xi) \bar{f}(\xi) B^{-r}(\xi, \xi) \, dv(\xi)
\]

for any \( f \) in \( A^1(D) \).

Since it is well known (cf. [7]) that the differential equation \( \mathcal{D}_r \tilde{g} = G \) in \( D \) possesses holomorphic solutions, we then conclude that \( L \) can be represented by the element \( g \) of \( \mathcal{B}_r \), consisting of all the holomorphic solutions to this equation.

In the third chapter, we denote by \( P \) the operator that assigns to a bounded function \( b \) the above-defined element \( g \) of the Bloch space \( \mathcal{B}_r \), and we call it “Bergman projection” of \( L^\infty \) into \( \mathcal{B}_r \), for the following reason: although \( P \) is not the integral operator \( \mathcal{D} \) associated with the Bergman kernel \( B(\xi, z) \) of \( D \), which has no meaning on \( L^\infty \), we shall prove that for any function \( b \) in \( (L^2 \cap L^\infty)(D) \), the element \( Pb \) of \( \mathcal{B}_r \) can be represented by the holomorphic function \( \mathcal{D} b \). Moreover, we prove that \( P \) defines a bounded operator from \( L^\infty \) onto \( \mathcal{B}_r \), and, consequently, the dual of \( A^1(D) \) (which coincides with the Bloch space \( \mathcal{B}_r \)) can be realized as the Bergman projection of \( L^\infty \).

Finally, as usual, the same letter \( C \) will denote constants that may be different from each other.

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I. Preliminary results about the Bergman kernel

1. The canonical decomposition of the domain. Let \( V \) denote an affine-homogeneous cone in \( \mathbb{R}^n \), which contains no straight line; we first recall the canonical decomposition of \( V \) as settled in [5].
Notations. (i) At the jth step, \( j = 1, 2, \ldots \), the real line \( \mathbb{R} \) will be denoted by \( \mathbb{R}_j^1 \) and at the kth step, \( k = 2, 3, \ldots \), \( \mathbb{R}_k^k \) will denote the \( n_k \)-dimensional euclidean space \( \mathbb{R}^{n_k} \).

(ii) Let \( P \) denote the real Siegel domain

\[
P = \{ (y, t) \in \mathbb{R}^n \times \mathbb{R}^1 : y - \varphi(t, t) \in \Gamma \}
\]

where \( \Gamma \subseteq \mathbb{R}^n \) is an affine-homogeneous cone which contains no straight line and \( \varphi \) is a \( \Gamma \)-bilinear symmetric form defined in \( \mathbb{R}^n \); we shall denote by \( V(P) \) the cone

\[
V(P) = \{ (y, t, r) \in \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^1 : r > 0 \text{ and } (yr, t) \in P \}.
\]

In order to obtain the canonical decomposition of the cone \( V \), we consider at the first step the cone \( V^{(1)} = (0, \infty) \subseteq \mathbb{R}_1^1 \).

At the second step, we associate with the cone \( V^{(1)} \) and with a \( V^{(1)} \)-bilinear symmetric form \( F^{(2)} \) defined in \( \mathbb{R}^2_2 \) a real Siegel domain \( P^{(2)} \) contained in \( \mathbb{R}^1_{11} \times \mathbb{R}^2_2 \) and the cone

\[
V^{(2)} = V(P^{(2)}) \subseteq \mathbb{R}^1_{11} \times \mathbb{R}^2_2 \times \mathbb{R}^2_2.
\]

At the third step, we associate with the cone \( V^{(2)} \) and with a \( V^{(2)} \)-bilinear symmetric form \( F^{(3)} \) defined in \( \mathbb{R}^3_3 \) a real Siegel domain \( P^{(3)} \) contained in \( \mathbb{R}^1_{11} \times \mathbb{R}^2_2 \times \mathbb{R}^3_3 \) and the cone

\[
V^{(3)} = V(P^{(3)}) \subseteq \mathbb{R}^1_{11} \times \mathbb{R}^2_2 \times \mathbb{R}^2_2 \times \mathbb{R}^3_3 \times \mathbb{R}^2_3.
\]

And so on, at the \( i \)th step, we associate with the cone \( V^{(i-1)} \) and with a \( V^{(i-1)} \)-bilinear symmetric form \( F^{(i)} \) defined in \( \mathbb{R}^i_i \) a real Siegel domain \( P^{(i)} \) and the cone \( V^{(i)} = V(P^{(i)}) \).

It follows from results in [5] that every affine-homogeneous cone \( V \), which contains no straight line, can be decomposed in the form \( V = V^{(i)} \) (up to an affine isomorphism). The required number of steps to obtain \( V \) in this form is called the rank \( l \) of the cone \( V \) (\( V = V^{(l)} \)). This yields the following decomposition of the space \( \mathbb{R}^n \) containing \( V \):

\[
\mathbb{R}^n = \mathbb{R}_{11}^1 \times \cdots \times \mathbb{R}_{ij}^i \times \mathbb{R}_2^2 \times \cdots \times \mathbb{R}_j^j, \quad n = l + \sum_{j=2}^l n_j.
\]

Now, let \( F_{ij}^{(j)} \), \( 1 \leq i < j \leq l \), denote the projection of the \( V^{(j-1)} \)-bilinear symmetric form \( F^{(j)} \) on the real line \( \mathbb{R}^1_{ij} \), and let \( \mathbb{R}_{ij}^j \) denote the \( n_{ij} \)-dimensional subspace of \( \mathbb{R}_j^j \) where the form \( F_{ij}^{(j)} \) is positive definite; then the decomposition \( \mathbb{R}_j^j = \prod_{i=1}^{l-1} \mathbb{R}_{ij}^j \) holds.

We recall next that the cone \( V \) is self-conjugate if and only if the integers \( n_{ij} \), \( 1 \leq i < j \leq l \), are equal among themselves. In that case, when \( x_{ij} \) denotes the projection of \( x \in \mathbb{R}^n \) on \( \mathbb{R}_{ij}^j \), the cone \( V \) is self-conjugate with respect to the scalar product

\[
\langle x, x' \rangle = \sum_{j=1}^l \frac{(x_{jj}x'_{jj})}{2} + \sum_{1 \leq i < j \leq l} x_{ij}x'_{ij}, \quad x, x' \in \mathbb{R}^n.
\]
Thus, in order that the cone \( V \subset \mathbb{R}^n \) of rank \( l \) be self-conjugate, we shall assume that \( n - l \) is a multiple in \( \mathbb{N} \) of \( l(l - 1)/2 \) and that all the \( n_{ij} \)'s, \( 1 \leq i < j \leq l \), are equal to \( p = 2(n - l)/l(l - 1) \).

We shall also assume that the vectors in the subspaces \( \mathbb{R}^p_{ij} \) are represented by their coordinates with respect to bases satisfying the following property: for any \( \lambda_j \) in \( \mathbb{R}^n_j \), one has the equality \( F^{(i)}_{ij}(\lambda_j, \lambda_j) = \|\lambda_j\|^2 \), where \( \| \cdot \| \) denotes the euclidean norm in \( \mathbb{R}^p_{ij} \).

Furthermore, let \( F^{(k)}_{ij} \), \( 1 \leq i < j < k \leq l \), denote the projection of the \( V^{(k-1)} \)-bilinear symmetric form \( F^{(k)} \) on \( \mathbb{R}^p_{ij} \); the homogeneity assumption on the cone \( V \) implies that the form \( F^{(k)}_{ij} \) is concentrated on \( \mathbb{R}^p_{ik} \times \mathbb{R}^p_{jk} \).

Now, let \( D \subset \mathbb{C}^n \times \mathbb{C}^N \) denote a symmetric Siegel domain of type II, associated to the cone \( V \) and to the \( V \)-Hermitian form \( F \). We next describe the canonical decomposition of the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \).

First, the decomposition

\[
\mathbb{R}^n = \prod_{j=1}^{l} \mathbb{R}^1_{ij} \times \bigcap_{1 \leq j < k \leq l} \mathbb{R}^p_{jk}
\]

given above yields in a natural way the decomposition

\[
\mathbb{C}^n = \prod_{j=1}^{l} \mathbb{C}^1_{ij} \times \bigcap_{1 \leq j < k \leq l} \mathbb{C}^p_{jk}.
\]

Secondly, let \( F_{ij} \) denote the projection of the form \( F \) on the complex plane \( \mathbb{C}^1_{ij} \), and let \( \mathbb{C}^q_{ij} \) denote the \( q \)-dimensional subspace of \( \mathbb{C}^N \) where the form \( F_{ij} \) is positive definite; the decomposition \( \mathbb{C}^N = \prod_{j=1}^{l} \mathbb{C}^q_{ij} \) holds, and the canonical decomposition of the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \) is given by

\[
\mathbb{C}^n \times \mathbb{C}^N = \prod_{j=1}^{l} \mathbb{C}^1_{ij} \times \bigcap_{1 \leq j < k \leq l} \mathbb{C}^p_{jk} \times \prod_{j=1}^{l} \mathbb{C}^q_{ij}.
\]

Moreover, with respect to its canonical form, the cone \( V \) can be described in the following quantitative manner. Let \( \lambda \) be a point of \( V \) and let \( \lambda_j \), \( j = 2, \ldots, l \), denote the projection of \( \lambda \) on \( \mathbb{R}^p_{ij} \); there exists an automorphism (in the transitive group of \( V \) given in [5]) that assigns to \( \lambda \) a point \( \tilde{\lambda} \) of \( V \) satisfying \( \tilde{\lambda}_j = 0 \), for every \( j = 2, \ldots, l \). Such an automorphism can be built in \( l - 1 \) stages: a first automorphism maps \( \lambda \) to \( \lambda^{(1)} \in V \) satisfying \( \lambda^{(1)}_j = 0 \), a second one takes \( \lambda^{(1)} \) to \( \lambda^{(2)} \in V \) satisfying \( \lambda^{(2)}_{l-1} = \lambda^{(2)}_{l-2} = 0, \ldots, \) and finally, the \((l - 1)\)th automorphism assigns to \( \lambda^{(l-2)} \), satisfying \( \lambda^{(l-2)}_{l-1} = \cdots = \lambda^{(l-2)}_1 = \lambda^{(l-2)}_0 = 0 \) the point \( \lambda^{(l-1)} = \lambda \). More explicitly, the points \( \lambda^{(m)} \), \( m = 0, 1, \ldots, l - 1 \), of \( V \) are given by the following recurrence formulas (formulas (1.26), (1.27) in [5]):

\[
\lambda^{(0)} = \lambda,
\]

\[
\lambda^{(m+1)} = \lambda^{(m)} - \frac{F_{ij}(\lambda^{(m)}_{l-m}, \lambda^{(m)}_j)}{\lambda^{(m)}_{l-m, l-m}}, \quad 1 \leq i < j < l - m.
\]

Now, set \( \chi_j(\lambda) = \lambda^{(-j)}_{ij} \); the cone \( V \) is defined by the \( l \) inequalities \( \chi_j(\lambda) > 0 \), \( j = 1, 2, \ldots, l \) (formula (1.25) in [5]).
Furthermore, let us recall that the functions \( \chi_j(\lambda) \) can also be obtained in the following way. The transitive group \( G(V) \) of the cone \( V \), defined in [5], is represented by \( l \times l \) triangular matrices, and the automorphism \( T \in G(V) \) that assigns \( \lambda \) to \( \lambda \) is a triangular matrix of \( G(V) \) whose diagonal elements are all equal to 1. On the other hand, let \( e \) denote the point of \( V \subseteq \prod_{j=1}^{l} \mathbb{R}^{l_j} \times \prod_{j=2}^{l} \mathbb{R}^{l_j} \) whose coordinates are \( e_{jj} = 1 \) and \( e_k = 0 \) for \( j = 1, \ldots, l \) and \( k = 2, \ldots, l \); then for any \( \lambda \) in \( V \), there exists in \( G(V) \) a unique matrix \( g \) such that \( g(e) = \lambda \) and \( g^{-1} \) is the product of \( T \) by a diagonal matrix. Hence, the matrix \( g \) can be written as the product \( g = td \) of a diagonal matrix \( d \) of \( G(V) \) by a triangular matrix \( t \) of \( G(V) \) whose diagonal elements are all equal to 1; the point \( \lambda \) and the functions \( \chi_j(\lambda) \) are then given by the formulas \( \lambda = d(e) \) and \( \chi_j(\lambda) = d_j \), where \( d_j \) is the \( j \)-th diagonal element of \( d \).

Finally, let us recall that the self-conjugate cone \( V \) can also be defined by the \( l \) functions \( \chi_j^*(\lambda) \) that generally define the dual cone \( V^* \) of \( V \). Those functions \( \chi_j^*(\lambda) \) are given as follows.

Let \( G^*(V) \) denote the adjoint group of \( G(V) \) with respect to the scalar product \( \langle \cdot, \cdot \rangle \); the group \( G^*(V) \) acts transitively on \( V^* = \overline{V} \) and for any \( \lambda \) in \( V^* = \overline{V} \), there exists a unique automorphism \( g^* \) in \( G^*(V) \) satisfying \( g^*(e) = \lambda \). The group \( G^*(V) \) can also be represented by triangular matrices, and the matrix \( g^* \) can be uniquely written in \( G^*(V) \) as the product \( g^* = t^*d^* \) of a diagonal matrix \( d^* \) by a triangular matrix \( t^* \) whose diagonal elements are all equal to 1. The functions \( \chi_j^*(\lambda) \) are then given by \( \chi_j^*(\lambda) = d_j^* \), where \( d_j^* \) is the \( j \)-th diagonal element of \( d^* \).

We shall carry the following notations: for any vector \( \rho = (\rho_1, \ldots, \rho_l) \) in \( \mathbb{R}^l \), we set

\[
\lambda^\rho = \prod_{j=1}^{l} (\chi_j(\lambda))^\rho_j \quad \text{and} \quad (\lambda^*)^\rho = \prod_{j=1}^{l} (\chi_j^*(\lambda))^\rho_j.
\]

2. The Bergman kernel. Let \( n \) and \( l \) be two integers satisfying \( 1 \leq l \leq n \), \( n > 1 \), and such that \( l(l - 1)/2 \) divides \( n - l \) in \( \mathbb{N} \). By \( V \) we shall denote an affine-homogeneous, self-conjugate cone of rank \( l \) in \( \mathbb{R}^n \), and we shall assume that the cone \( V \) is irreducible: this implies that \( 1 < l < n \).

We therefore exclude the case \( n = l = 1 \), where the cone \( V \) is the positive real half-line and its associated Siegel domains, that are Cayley transforms of balls, were studied in [1].

Let \( D \) denote a symmetric Siegel domain of type II contained in \( \mathbb{C}^n \times \mathbb{C}^N \), associated to the cone \( V \) and to the \( V \)-Hermitian form \( F \) defined in \( \mathbb{C}^N \); the space \( \mathbb{C}^n \times \mathbb{C}^N \) containing \( D \) will be considered in its canonical form

\[
\mathbb{C}^n = \prod_{j=1}^{l} \mathbb{C}^l_j \times \prod_{k=2}^{n} \mathbb{C}^n_k,
\]

\[
\mathbb{C}^n_k = \prod_{j < k} \mathbb{C}^l_j, \quad p = \frac{2(n - l)}{l(l - 1)},
\]

\[
\mathbb{C}^N = \prod_{j=1}^{l} \mathbb{C}^q_j.
\]
We shall denote by \( q \) the vector of \( \mathbb{N}^l \) whose coordinates are \( q_j, j = 1, \ldots, l \), and we shall denote by \( d \) and \( m \) the vectors of \( \mathbb{R}^l \) whose respective coordinates are \( d_j = -(1 + p(j - 1)/2), m_j = p(j - 1) \).

It then follows from results in [5] that the Bergman kernel \( B(\xi, z) \) of \( D \) has the following two expressions:

**Proposition 1.2.1.** The Bergman kernel \( B(\xi, z) \) of \( D \) is given by the formulas

\[
B(\xi, z) = c\left(\frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\xi_2, z_2)\right)^{2d-q}
\]

(1)

\[
B^1 + \rho(\xi, z) = c\int_V \exp\left(-\langle\lambda, \frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\xi_2, z_2)\rangle\right)(\lambda^*)^{-d+q} d\lambda,
\]

where \( \xi = (\xi + i\eta, \xi_2) \) and \( z = (x + iy, z_2) \) are two points of \( D \subset \mathbb{C}^n \times \mathbb{C}^N \).

Subsequently, we shall carry the following notations:

(i) let \( \rho = (\rho_1, \ldots, \rho_l) \) and \( \rho' = (\rho'_1, \ldots, \rho'_l) \) be two vectors in \( \mathbb{R}^l \); we shall write \( \rho > \rho' \) whenever \( \rho_j > \rho'_j \) for every \( j = 1, \ldots, l \); we shall also set

\[
\rho \rho' = (\rho_1 \rho'_1, \ldots, \rho_l \rho'_l) \quad \text{and} \quad \rho / \rho' = (\rho_1 / \rho'_1, \ldots, \rho_l / \rho'_l),
\]

when none of the \( \rho'_j \)'s, \( j = 1, \ldots, l \), is zero;

(ii) for any two points \( \xi = (\xi + i\eta, \xi_2) \) and \( z = (x + iy, z_2) \) in \( D \subset \mathbb{C}^n \times \mathbb{C}^N \), we let \( B^\rho(z, z) \) and \( B^1 + \rho(\xi, z) \) denote the expressions

\[
B^\rho(z, z) = c\left(\frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\xi_2, z_2)\right)^{(2d-q)\rho}
\]

and

\[
B^1 + \rho(\xi, z) = c\left(\frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\xi_2, z_2)\right)^{2d-q+(2d-q)\rho}
\]

We will use the following lemma, due to S. G. Gindikin [5]:

**Lemma 1.2.1.** A holomorphic function \( G \) in \( D \) belongs to \( A^2(D) \) if and only if it can be expressed in the form

\[
G(z) = \int_V g(\lambda, z_2) \exp(i\langle\lambda, z_1\rangle) d\lambda,
\]

for any \( z = (z_1, z_2) \) in \( D \subset \mathbb{C}^n \times \mathbb{C}^N \), where the function \( g(\lambda, z_2) \) satisfies the following properties:

(i) the function \( z_2 \rightarrow g(\lambda, z_2) \) is entire in \( \mathbb{C}^N \);

(ii) the integral

\[
\int_V \int_{\mathbb{C}^N} |g(\lambda, z_2)|^2 \exp(-2\langle\lambda, F(z_2, z_2)\rangle)(\lambda^*)^d dv(z_2) d\lambda
\]

converges and is equal to \( c\int_D |G(z)|^2 dv(z) \).

We state another lemma of Gindikin [5]:

**Lemma 1.2.2.** For any vector \( \rho \) in \( \mathbb{R}^l \) such that \( \rho > m/2 \) and for any \( z \) in \( \mathbb{C}^n \) such that \( \Re z \in \mathcal{V} \), we have

\[
\int_V \exp(-\langle z, \lambda \rangle)(\lambda^*)^{-d+q} d\lambda = c_\rho(z)^{-\rho}.
\]
Our next purpose is to prove the following estimate for the Bergman kernel:

**Lemma 1.2.3.** For any \( \xi \) in \( D \), the kernel \( B^{1+\alpha}(\xi, z) \), \( \alpha \in \mathbb{R} \), belongs to \( L^1(dv(z)) \) if and only if \( (-2d + q)\alpha > m/2 \); in that case,

\[
\int_D |B^{1+\alpha}(\xi, z)| dv(z) = c_\alpha B^{\alpha}(\xi, \xi).
\]

**Proof.** Because of the homogeneity of the domain \( D \), it suffices to prove the lemma for \( f = (ie, 0) \in D \subset C^a \times C^N \), where \( e \) is the point of \( V \subset \prod_{j=1}^l \mathbb{R}_{ij} \times \prod_{j<k} \mathbb{R}_{jk} \) whose components are \( e_{jj} = 1 \) for \( j = 1, \ldots, l \) and \( e_{jk} = 0 \) when \( j < k \).

Take then \( f = (ie, 0) \). The assumption \((-2d + q)\alpha > m/2\) yields the inequality \([-2d - q + (2d - q)\alpha]/2 > m/2\); now, in view of Proposition 1.2.1 and Lemma 1.2.2, \( B^{(1+\alpha)/2}(\xi, z) \) can be written.

\[
B^{(1+\alpha)/2}(\xi, z) = c_\alpha \left( -\frac{x}{2i} + \frac{e + y}{2} \right)^{(2d - q + (2d - q)\alpha)/2}.
\]

Next, the Plancherel formula given in Lemma 1.2.1 yields

\[
\int_D |B^{1+\alpha}(\xi, z)| dv(z) = c_\alpha \int_V \exp\left( -\left( \lambda, -\frac{x}{2i} + \frac{e + y}{2} \right) \right) (\lambda^*)^{q + (-2d + q)\alpha}/2 d\lambda.
\]

Let us show that the right-hand side of this last equality converges if \((-2d + q)\alpha > m/2\). We first integrate with respect to \( z_2 \); the following lemma is proved in [5]:

**Lemma 1.2.4.** The integral

\[
\int_{C^N} \exp(-\langle \xi, F(z_2, z_2) \rangle) dv(z_2)
\]

is convergent.

In view of lemma 1.2.4 and of the homogeneity of the cone \( V \), we get

\[
\int_{C^N} \exp(-2\langle \lambda, F(z_2, z_2) \rangle) dv(z_2) = c(\lambda^*)^{-q};
\]

hence,

\[
(3) \quad \int_D |B^{1+\alpha}((ie, 0), z)| dv(z) = c_\alpha \int_V \exp(-\langle \xi, e \rangle)(\lambda^*)^{d + (-2d + q)\alpha} d\lambda.
\]

It then follows from Lemma 1.2.2 that \( B^{1+\alpha}((ie, 0), z) \) belongs to \( L^1(dv(z)) \) when \((-2d + q)\alpha > m/2\). Conversely, S. G. Gindikin [5] has proved that the right-hand side of (3) converges only if this condition on \( \alpha \) is satisfied. The proof of the lemma is complete.
Remark. The necessary and sufficient condition on $\alpha$ given in Lemma 1.2.3 contradicts a statement of R. Coifman and R. Rochberg (Lemma 2.2 of [4]). Even in the particular case of the tube over the spherical cone of $\mathbb{R}^n$, $n \geq 3$ (cone of rank two), the condition $\alpha > 0$ given by those authors was proved before to be insufficient (cf. [2 and 3]).

We shall also use Lemma 2.3 of [4]:

Lemma 1.2.5. $d$ denotes the Bergman distance in $D$. There exists a constant $C_D$ such that for any two points $\xi$ and $\xi_0$ in $D$ satisfying $d(\xi, \xi_0) < 10$ and for any $z$ in $D$, the following inequality holds:

$$|B(\xi, z)/B(\xi_0, z) - 1| \leq c_D d(\xi, \xi_0).$$

II. The dual of $A^1$ and Bloch functions

In the following, $\rho$ denotes a vector of $\mathbb{N}^l$ whose coordinates $\rho_j$, $j = 1, \ldots, l$, satisfy two conditions:

(i) $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{l-1} \geq \rho_l = 1$;

(ii) $(\lambda^*)^\rho = \prod_{j=1}^l \lambda_j^{\rho_j}$ is a polynomial of $\lambda \in \mathbb{R}^n$.

According to the terminology in [5], a vector $\rho$ satisfying property (ii) is said to be $V$-integral, and for any cone $V$ of rank $l$ an example of a $V$-integral vector is $\rho = (2^{l-2}, 2^{l-3}, \ldots, 2, 1, 1)$.

We shall denote by $\mathcal{R}$ the differential polynomial in $\mathbb{C}^n$ that possesses the property

$$\mathcal{R}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^\rho \exp(\langle \lambda, z_1 \rangle), \quad z_1 \in \mathbb{C}^n.$$  

Referring to [5], $\mathcal{R}$ is a Riemann-Liouville differential operator of $D \subset \mathbb{C}^n \times \mathbb{C}^n$.

Let $r$ be a vector of $\mathbb{N}^l$ satisfying the following two conditions (the notations are those introduced in section I.2):

(iii) $(\lambda^*)^r$ is a polynomial of $\lambda \in \mathbb{R}^n$; i.e., $r\rho$ is also a $V$-integral vector;

(iv) $r\rho > m/2$.

Let $\mathcal{D} = \mathcal{R}^r$ denote the differential polynomial in $\mathbb{C}^n$ that possesses the following property:

$$\mathcal{D}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^{r\rho} \exp(\langle \lambda, z_1 \rangle), \quad z_1 \in \mathbb{C}^n.$$  

In other words, $\mathcal{D}$ is the Riemann-Liouville differential operator of $D$ obtained by iterating $\mathcal{R}$ $r$ times.

Now, fix two such vectors $\rho$ and $r$; the corresponding Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is defined as follows. A holomorphic function $g$ in $D$ is said to be a Bloch function if it satisfies

$$\|g\|_* = \sup_{(z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^n} \left\{ |\mathcal{D}_{z_1} g(z)| B^{-(r\rho/(-2d+\alpha))(z, z)} \right\} < +\infty.$$  

Let $\mathcal{N} = \{ g \in H(D): \mathcal{D}_{z_1} g(z) = 0 \}$; the Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is the quotient space $\{ \text{Bloch functions} \}/\mathcal{N}$.

The Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ possesses the following property:

Proposition II.1. Under the quotient norm induced by $\| \|_*$, the Bloch space $\mathcal{B} = \mathcal{B}_{\rho, r}$ of $D$ is a Banach space.
The proof of this proposition essentially relies on the following lemma [7, p. 477]:

**Lemma II.1.** For any holomorphic function \( h(z_1, z_2) \) in \( D \subset \mathbb{C}^n \subset \mathbb{C}^N \), there exists a holomorphic function \( f(z_1, z_2) \) such that \( \partial_z f = h \).

We next prove the following theorem:

**Theorem II.1.** For any vectors \( p \) and \( r \) in \( \mathbb{N}^4 \) satisfying conditions (i)-(iv), the dual of the Bergman space \( A^1(D) \) coincides with the Bloch space \( B = B_{p,r} \) of \( D \).

**Proof.** It is easy to prove that the Bloch space \( B_{p,r} \) of \( D \) coincides with a subspace of the dual of \( A^1(D) \). This is done as follows: any element \( g \) of \( B_{p,r} \) defines an element \( L \) of the dual of \( A^1(D) \) by

\[
L(f) = \int_D \partial_z g(z) \bar{\partial}_z f(z) \frac{1}{B^{0/(-2d+q)}(z, z)} dv(z), \quad f \in A^1(D)
\]

and the conclusion follows from the easily proved inequality \( ||L|| \leq ||g||_* \).

Conversely, let us prove that any element \( L \) of the dual of \( A^1(D) \) can be represented by an element \( g \) of \( B_{p,r} \). First, by the Hahn-Banach theorem, there exists a bounded function \( b \) in \( D \) such that for any \( f \) in \( A^1(D) \), \( L(f) = \int_D b f dv \).

Secondly, in view of the condition \( rp > m/2 \) and of Lemma I.2.3, we can assign to this function \( b \) the holomorphic function \( G \) defined in \( D \) by

\[
G(\xi) = \int_D B^{1+rp/(-2d+q)}(\xi, z) b(z) dv(z);
\]

\( G \) satisfies the estimate

\[
\sup_{\xi \in D} \{ |G(\xi)| B^{rp/(-2d+q)}(\xi, \xi) \} \leq C\|b\|_\infty
\]

and yields an element \( L' \) of the dual of \( A^1(D) \) defined by

\[
L'(f) = C \int_D G(\xi) \bar{\partial}_z f(\xi) \frac{1}{B^{0/(-2d+q)}(\xi, \xi)} dv(\xi).
\]

Now, it follows from the Fubini theorem that

\[
L'(f) = C \int_D \left( \int_D B^{1+rp/(-2d+q)}(\xi, z) \bar{\partial}_z f(\xi) B^{rp/(-2d+q)}(\xi, \xi) dv(\xi) \right) b(z) dv(z).
\]

We then conclude, in view of the reproducing formula

\[
\int_D B^{1+rp/(-2d+q)}(\xi, z) \bar{\partial}_z f(\xi) B^{rp/(-2d+q)}(\xi, \xi) dv(\xi) = C^{-1} f(z),
\]

that the linear functionals \( L \) and \( L' \) coincide on \( A^1(D) \).

Finally, the linear functional \( L \) on \( A^1(D) \) can be represented by the element \( g \) of the Bloch space \( B = B_{p,r} \) defined as follows: by Lemma II.1 and using the estimate (\( \ast \)) for \( G \), take \( g \) to be the equivalence class of all holomorphic solutions to the equation

\[
\partial_z g(z) = CG(z), \quad z = (z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N.
\]

Also, \( \|g\|_* \leq C\|b\|_\infty \). The proof of Theorem II.1 is complete.
Remark. Bloch spaces can be defined exactly in the same way on homogeneous but nonsymmetric Siegel domains of type II associated with a self-conjugate cone. Furthermore, the argument in the proof of Theorem II.1 leads to the same conclusion in such a domain because all results we need from the first chapter are still valid. Concerning the existence of such domains, let us recall that the “historical” example of a homogeneous but nonsymmetric Siegel domain of type II, due to Pyateckii-Shapiro (in his solution to E. Cartan’s problem about the existence of bounded homogeneous but nonsymmetric domains; cf. [8, p. 269]) is associated with the (self-conjugate) spherical cone of \(\mathbb{R}^3\).

III. The Bergman projection of \(L^\infty\).

In this chapter, \(D\) again denotes a symmetric Siegel domain of type II contained in \(\mathbb{C}^n \times \mathbb{C}^N\). The Bergman projection \(\mathcal{P}\) of \(D\) is the orthogonal projection of \(L^2(dv)\) onto the subspace \(A^2(D)\) of \(L^2(dv)\) consisting of holomorphic functions; for any \(\varphi\) in \(L^2(dv)\), \(\mathcal{P}\varphi\) is given by the integral formula

\[
\mathcal{P}\varphi(\xi) = \int_D B(\xi, z) \varphi(z) \, dv(z), \quad \xi \in D,
\]

where \(B\) denotes the Bergman kernel of \(D\), explicitly given in (1) and (2). However, since by Lemma 1.2.3, the kernel \(B(\xi, z)\) does not belong to \(L^1(dv(z))\), \(\xi \in D\), expression (5) does not make sense when \(\varphi\) is any bounded function in \(D\).

In the following, we fix two vectors \(\rho\) and \(r\) in \(\mathbb{N}^l\), satisfying conditions (i)–(iv) introduced at the beginning of Chapter II and \(\mathcal{B} = \mathcal{B}_{\rho, r}\) denotes the corresponding Bloch space.

Our next concern is to define an operator \(P\) from \(L^\infty(D)\) into the Bloch space \(\mathcal{B}\) that satisfies the following two properties:

1° The image \(Pb\) in \(\mathcal{B}\) of a function \(\varphi \in (L^\infty \cap L^2)(D)\) can be represented by the function \(\mathcal{P}\varphi\) defined in (5): for this reason, we shall also call this new operator \(P\) the “Bergman projection”;

2° The “Bergman projection” \(P\) is a bounded operator from \(L^\infty\) onto \(\mathcal{B}\), and, consequently, the dual of the Bergman space \(A^1\) (the Bloch space \(\mathcal{B}\)) coincides with the “Bergman projection” \(PL^\infty\) of \(L^\infty\).

Let \(b\) be a bounded function in \(D\). In the proof of Theorem II.1, we assigned to \(b\) a holomorphic function \(G\) defined in \(D\) by

\[
G(\xi) = C \int_D B^{1+\rho/(-2d+q)}(\xi, z) b(z) \, dv(z), \quad \xi \in D,
\]

and the equivalence class \(g\) in the Bloch space \(\mathcal{B}\), consisting of all holomorphic solutions \(\tilde{g}\) of the differential equation

\[
\mathcal{D}_{\xi_1} \tilde{g}(\xi) = G(\xi), \quad \xi = (\xi_1, \xi_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N.
\]

The operator \(P\) from \(L^\infty\) into \(\mathcal{B}\), which assigns \(g\) to \(b\), will be called the “Bergman projection” in view of the following lemma:

Lemma III.1. For any \(\varphi\) in \((L^\infty \cap L^2)(D)\), the element \(P\varphi\) of the Bloch space \(\mathcal{B}\) can be represented by the function \(\mathcal{P}\varphi\) defined in (5).
Proof. This follows immediately from the equality

\[(6) \mathcal{D}_z B(\xi, z) = CB^{1+\rho/(2d+q)}(\xi, z), \quad \xi = (\xi_1, \xi_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N.\]

It was already shown that $P$ is a bounded operator from $L^\infty$ into $\mathcal{B}$; we now prove that $P$ is onto. More precisely, we show that any element $g$ of $\mathcal{B}$ is the Bergman projection $Pb$ of a bounded function $b$ defined by

\[b(z) = \mathcal{D}_z B(\xi, z) B^{-\rho/(2d+q)}(\xi, z), \quad z = (z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N.\]

This is an immediate consequence of the following proposition:

**Proposition III.1.** For any holomorphic function $G$ in $D$ satisfying the estimate

\[(*) \sup_{z \in D} \{|G(z)|B^{-\rho/(2d+q)}(z, z)|B^{\rho/(2d+q)}(\xi, z)|\} < +\infty, \]

the following equality holds:

\[(7) G(\xi) = C \int_D G(z) B^{-\rho/(2d+q)}(z, z) B^{1+\rho/(2d+q)}(\xi, z) \, dv(z). \]

Proof. This proposition will be proved in an equivalent form on a bounded realization of $D$. We need the following lemma:

**Lemma III.2.** Every symmetric Siegel domain of type II contained in $\mathbb{C}^M$ is biholomorphic to a bounded domain $\Omega$ which possesses the following property: if $z = (z_1, \ldots, z_M)$ is a point of $\Omega$, then for any complex numbers $\alpha_j$, satisfying $|\alpha_j| \leq 1$, $j = 1, \ldots, M$, $(\alpha_1 z_1, \ldots, \alpha_M z_M)$ is also a point of $\Omega$ ($\Omega$ is said to be a bounded "circular" domain).

Furthermore, let $B_\Omega(\xi, z)$ denote the Bergman kernel of $\Omega$; there exists a nonzero constant $c_\Omega$ such that $B_\Omega(0, z) = C_\Omega$.

The first part of the lemma was proved by A. Koranyi and J. A. Wolff in [6]. The second part is contained in the proof of Lemma 2.3 of [4] (Lemma I.2.5 above); that proof, due to Koranyi, uses all of Lemma III.2.

We can now begin the proof of Proposition III.1. Let $\Phi$ denote a biholomorphic transformation of a bounded circular domain $\Omega$, associated with $D$ by Lemma III.2, onto $D$, which assigns to the origin 0 the point $e = (e_1, e_2)$ of $D \subset \mathbb{C}^n \times \mathbb{C}^N$, whose components are $(e_1)_{jj} = 1$, $(e_1)_{jk} = 0$ when $j < k$ and $e_2 = 0$.

This transformation $\Phi$ yields the following relation between the respective Bergman kernels $B_D$ and $B_\Omega$ of the domains $D$ and $\Omega$:

\[B_\Omega(\xi', z') = B_D(\Phi(\xi'), \Phi(z')) [J\Phi(\xi')]^a \left[ \overline{J\Phi(z')} \right], \]

where $J\Phi(w)$ denotes the jacobian of $\Phi$ at the point $w$ of $\Omega$. When one puts $z' = 0$ in this relation, it follows from the second part of Lemma III.2 that $J\Phi(\xi') = CB_D^{-1}(\Phi(\xi'), e)$; furthermore, according to the notations introduced in section 1.2, we set

\[ [J\Phi(\xi')]^a = C_a B_D^a(\Phi(\xi'), e) \]

and

\[ B_\Omega^a(\xi', z') = B_D^a(\Phi(\xi'), \Phi(z')) [J\Phi(\xi')]^a \left[ \overline{J\Phi(z')} \right]^a. \]
By means of the biholomorphic transformation $\Psi = \Phi^{-1}$, it is then easy to check that Proposition III.1 is equivalent to the following result:

**Proposition III.2.** For any holomorphic function $\tilde{G}$ in $\Omega$ satisfying the estimate

$$\sup_{z' \in \Omega} \left\{ \left| \tilde{G}(z') |B_{\Omega}^{-rp/(-2d+q)}(z', z') | J\Phi(z') |^{2rp/(-2d+q)} \right\} < +\infty,$$

the following equality holds:

$$\tilde{G}(z')[J\Phi(z')]^{1+rp/(-2d+q)}$$

$$= C \int_{\Omega} \tilde{G}(z')[J\Phi(z')]^{1+rp/(-2d+q)}$$

$$\cdot B_{\Omega}^{1+rp/(-2d+q)}(\xi, z') B_{\Omega}^{-rp/(-2d+q)}(z', z') \, dv(z'), \quad \xi' \in \Omega.$$

**Proof of Proposition III.2.** With respect to the measure $B_{\Omega}^-{rp/(-2d+q)}(z, z) \, dv(z)$ the Bergman kernel of $\Omega$ is $CB_{\Omega}^{1+rp/(-2d+q)}(\xi, z)$. Carrying ourselves onto $\Omega$ by using the biholomorphic transformation $\Psi = \Phi^{-1}$, we easily deduce that with respect to the measure $B_{\Omega}^{-rp/(-2d+q)}(z', z') \, dv(z')$, the Bergman kernel of $\Omega$ is $CB_{\Omega}^{1+rp/(-2d+q)}(\xi', z')$.

It now suffices to prove that the function $\tilde{G}(z')[J\Phi(z')]^{1+rp/(-2d+q)}$ is integrable in $\Omega$ with respect to the measure $B_{\Omega}^{-rp/(-2d+q)}(z', z') \, dv(z')$.

Since we assume that the function $\tilde{G}(z')[J\Phi(z')]^{2rp/(-2d+q)}B_{\Omega}^{-rp/(-2d+q)}(z', z')$ is bounded in $\Omega$, it is enough to prove the following lemma:

**Lemma III.3.** $\int_{\Omega} |J\Phi(z')|^{1-rp/(-2d+q)} \, dv(z') < +\infty$.

**Proof of Lemma III.3.** Put $\Psi = \Phi^{-1}$; the assertion in the lemma is equivalent in $D$ to the estimate

$$\int_{D} |J\Psi(z)|^{1+rp/(-2d+q)} \, dv(z) < +\infty.$$

The desired conclusion then follows from Lemma I.2.3 because $J\Psi(z) = CB_{D}(e, z)$ and $rp > m/2$. The proof of Lemma III.3 is complete, and Propositions III.2 and III.1 are thus entirely proved.

**Remark.** In view of Proposition III.1, under the measure $B^{-rp/(-2d+q)}(z, z) \, dv(z)$ the Bergman kernel of $D$ does not only reproduce functions in Bergman spaces $A^{p}$, but also those holomorphic functions in $D$, satisfying the uniform estimate ($\ast$). This property is trivial on bounded domains and seems to be new concerning unbounded domains; however, in this case, unlike the bounded domains, the meaning of the right-hand side of (7) requires the presence of a weight.

We have just proved that the Bergman projection $P$ of $D$ is bounded from $L^\infty$ onto the Bloch space $B$. Furthermore, if $R$ denotes the operator defined in $B$ with values in $L^\infty$ that assigns to an element $g$ of $B$ the bounded function

$$Rg(z) = B_{z, g}(z) B^{-rp/(-2d+q)}(z, z), \quad z = (z_1, z_2) \in D \subset \mathbb{C}^n \times \mathbb{C}^N,$$

$R$ is a "continuous right inverse" of $P$, i.e., $PR = \text{Id}_B$. 

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The following theorem comes as a summary of this chapter:

**Theorem III.1.** Let $D$ be a symmetric Siegel domain of type II.

1° The Bergman projection $P$ of $D$ is a bounded operator from $L^\infty$ onto the Bloch space $\mathcal{B}$; furthermore, $P$ possesses a continuous right inverse $R: \mathcal{B} \to L^\infty$;

2° Consequently, the dual of the Bergman space $A^1$ (the Bloch space $\mathcal{B}$) can be realized as the Bergman projection of $L^\infty$.

**Remarks.** 1° At the end of the second chapter, we noticed that the dual of $A^1$ still coincides with the Bloch space in any homogeneous, but nonsymmetric Siegel domain of type II, associated with a self-conjugate cone. In such a domain, the Bergman projection can be defined exactly in the same way as an operator from $L^\infty$ into $\mathcal{B}$. The question therefore arises whether Theorem III.1 can be extended to this type of domain; a relevant problem would be to find a substitute for Lemma III.2.

2° Unlike the particular cases of the Cayley transform of the unit ball and of the tube over the spherical cone, respectively studied in [1 and 3], we have not defined here the Bergman projection $P$ of $L^\infty$ as an integral operator associated with a kernel. The problem of determining a defining kernel for $P$ will be considered in a forthcoming paper.

**References**


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