ISOMETRIES ON $L_{p,1}$

N. L. CAROTHERS AND B. TURETT

Abstract. The extreme points of the sphere of the Lorentz function space $L_{p,1}[0,1]$ are computed. As an application, the linear isometries from $L_{p,1}$ into itself are completely described.

1. Introduction. Since its introduction in 1950 by G. G. Lorentz, the Lorentz function space $L_{p,1} = L_{p,1}[0,1]$ has found frequent application to problems in interpolation theory and weighted-norm inequalities. In recent years the isomorphic structure of $L_{p,1}$ (as a Banach space) has received increasing attention and is now reasonably well understood while very little has been written on the isometric structure of $L_{p,1}$. In §2, we develop several interesting isometric tools; in particular, we compute the extreme points of the closed unit ball of $L_{p,1}$ (Theorem 1). As an application of these results, we give, in §3, a complete description of the linear isometries from $L_{p,1}$ into itself (Theorem 2).

Recall that Lamperti's proof of Banach's classical theorem on the linear isometries $T: L_p \to L_p$ [18, p. 333] proceeds in two major steps. In the first it is shown that $T$ must preserve disjointness (this via an observation concerning the $L_p$-norm). The second step is quite general: $T$ now induces a homomorphism of the measure algebra, and this homomorphism is necessarily induced by an automorphism, $\tau$, of the underlying measure space. It now follows easily that $T$ may be written: $Tf = h \cdot (f \circ \tau)$, where $h = T1$. Moreover, the converse is also true; that is, if $h$ is norm-one in $L_p$ and $\tau$ is an automorphism of $[0,1]$ with $\tau^{-1}[0,1] = \text{support of } h$, then $Tf = h \cdot (f \circ \tau)$ defines an isometry on $L_p$.

By modifying the first step of this argument, Bru and Heinich [2] are able to show that the positive (onto) isometries on a large class of Banach lattices (which includes $L_{p,1}$) are likewise induced by automorphisms of the underlying measure space and so may also be written: $Tf = h \cdot (f \circ \tau)$, where $h = T1$. However, even in the case of $L_{p,1}$, it is not clear whether the converse holds. Rather than deduce information for $L_{p,1}$ from this lattice result, we opt for a somewhat more analytic approach to the question. We treat the isometries on $L_{p,1}$ as an application of specific geometric and "distributional" tools intrinsic to $L_{p,1}$. In particular, by using a stronger version of...
the first step outlined above (see Lemma 5, below), we shall see that there are actually fewer isometries on \( L_{p,1} \) than might be anticipated from the results in [2]. Specifically, not every norm-one \( h \) in \( L_{p,1} \) can be written as \( T1 \) for some isometry \( T \).

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Our notation is, for the most part, standard and follows that of Lindenstrauss and Tzafriri [15]. We write \( \mu(A) \) for the Lebesgue measure of subset of \( A \) of \( \mathbb{R} \) and, given a measurable function \( f : [0,1] \to \mathbb{R} \) and, given a measurable function \( f : [0,1] \to \mathbb{R} \), we define

\[
\text{dist}(f; t) = d_f(t) = \mu(\{s : |f(s)| > t\}), \quad f^*(t) = \inf\{s : d_f(s) \leq t\}, \\
F(t) = \int_0^t f^*(s) \, ds, \quad \text{and} \quad \text{supp}f = \{s : f(s) \neq 0\}.
\]

Notice that \( d_f \) is actually the (probability) distribution of \( |f| \). Also we apologize in advance for all the usual abuses (and omissions) of the phrase “almost everywhere.” For example, we shall sometimes write \( f \geq 0 \) when we mean \( f \geq 0 \) a.e., and \( A \subset B \) instead of \( \mu(B \setminus A) = 0 \), etc.

Now, for \( 1 < p < \infty \), the Lorentz function space \( L_{p,1} = L_{p,1}(0,1) \) is defined to be the collection of all (equivalence classes of) measurable functions \( f : [0,1] \to \mathbb{R} \) for which \( \|f\|_{p,1} < \infty \) where

\[
(1) \quad \|f\|_{p,1} = \int_0^1 f^*(t) \, d(t^{1/p}).
\]

Simple change-of-variable and integration-by-parts arguments show that (1) can be written in a variety of guises:

\[
(2) \quad \|f\|_{p,1} = \int_0^\infty d_f(t)^{1/p} \, dt = \int_0^1 \left( \frac{1}{p} \right) t^{1/p-1} \, dF(t) \\
= \int_0^1 F(t) \, d\left( \left( \frac{1}{p} \right) t^{1/p-1} \right) + \|f\|_1.
\]

(Notice that if \( f \in L_{p,1} \), then \( t^{1/p-1}F(t) \to 0 \) as \( t \to 0^+ \).) We shall also use the usual \( L_p \)-spaces with norm

\[
\|f\|_p = \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p},
\]

and also the well-known duality: \( (L_{p,1})^* = L_{p',\infty} \), \( 1/p + 1/p' = 1 \), where

\[
(3) \quad \|f\|_{p',\infty} = \sup_{0 < t < 1} t^{-1/p} \int_0^t f^*(s) \, ds.
\]

(See [16 or 13].)

It is also well known that \( L_{p,1} \) is a separable dual space [10] with an unconditional basis [15, p. 156]. Moreover, \( L_{p,1} \) is known to satisfy a lower \( p \)-estimate for disjoint elements [7, 1]; that is, if \( f_1, \ldots, f_n \in L_{p,1} \) are disjointly supported, then

\[
(4) \quad \left\| \sum_{i=1}^n f_i \right\|_{p,1}^{p} \geq \sum_{i=1}^n \|f_i\|_{p,1}^p.
\]

In particular, (4) implies that \( \|f\|_p \leq \|f\|_{p,1} \). (Also see [13 or 15, Proposition 2.6.9].)
2. Extreme points. Our first two lemmas (which are essentially known) examine the triangle inequality in $L_{p,1}$.

**Lemma 1.** If $f, g \in L_{p,1}$ with $\|f + g\|_{p,1} = \|f\|_{p,1} + \|g\|_{p,1}$, then $(f + g)^* = f^* + g^*$.

**Proof.** First notice that since
\[
\|
f + g\|_{p,1} = \|
f\|_{p,1} + \|
g\|_{p,1} \leq \|
f\|_{p,1} + \|
g\|_{p,1} + \|
f + g\|_{p,1}.
\]
we must have $|f + g| = |f| + |g|$ a.e. Thus $f \cdot g \geq 0$ a.e. and $\|f + g\|_{1} = \|f\|_{1} + \|g\|_{1}$. Now set
\[
F_1(x) = \int_0^x (f + g)^*(t) \, dt \quad \text{and} \quad F_2(x) = \int_0^x (f^*(t) + g^*(t)) \, dt
\]
for $0 \leq x \leq 1$. Then $F_1 \leq F_2$ and we need to show that $F_1 = F_2$. (See [15, p. 125].)

But
\[
0 = \|f\|_{p,1} + \|g\|_{p,1} - \|f + g\|_{p,1} = \int_0^1 \left[ F_2(t) - F_1(t) \right] \left( \left( \frac{1}{p} \right)^{1/p - 1} \right),
\]
and $-(1/p)t^{1/p-1}$ is increasing; thus $F_2 - F_1 = 0$. Consequently, $(f + g)^* = f^* + g^*$. □

**Remark.** Note that $(f + g)^* = f^* + g^*$ implies that $f \cdot g \geq 0$ and that
\[
\text{supp}(f + g)^* = \text{supp}f^* \cup \text{supp}g^*.
\]
Thus
\[
\mu(\text{supp}f \cup \text{supp}g) = \max\{\mu(\text{supp}f), \mu(\text{supp}g)\};
\]
that is, we either have supp$f \subset$ supp$g$ or else supp$g \subset$ supp$f$.

**Lemma 2.** For $f \in L_{p,1}$ and $a > 0$, let $f^a = |f| \lor a - a$ and $f_a = |f| \land a$. Then $\|f\|_{p,1} = \|f^a\|_{p,1} + \|f_a\|_{p,1}$. In particular, if $\|f\|_{p,1} = 1$ and $0 < a < 1$, then $\|f^a\|_{p,1} > 0$ and $\|f_a\|_{p,1} > 0$, and hence $|f|$ is a convex combination of $f^a/\|f^a\|_{p,1}$ and $f_a/\|f_a\|_{p,1}$.

**Proof.** A straightforward computation shows that $f^* = (f^*)^a + (f_a)^*$ and so $\|f\|_{p,1} = \|f^a\|_{p,1} + \|f_a\|_{p,1}$. Now, for $0 < a < 1$, it is easy to see that $0 < a \cdot (d_f(a))^{1/p} \leq \|f\|_{p,1} \leq a < 1$, and hence also $\|f^a\|_{p,1} \geq 1 - a > 0$. □

We are now in a position to give a simple description of the extreme points of the closed unit ball of $L_{p,1}$.

**Theorem 1.** Let $1 < p < \infty$ and let $f \in L_{p,1}$ with $\|f\|_{p,1} = 1$. Then the following are equivalent:

(i) $\|f\|_{p} = 1$.

(ii) $f$ is an extreme point of the closed unit ball of $L_{p,1}$.

(iii) $|f| = \mu(E)^{-1/p} \chi_E$ for some $E \subset [0,1]$.

(iv) $\|f|^{p-1}\|_{p,\infty} = 1$, where $1/p + 1/p' = 1$.

**Proof.** Suppose (i) holds. If $f = (g + h)/2$ with $\|g\|_{p,1} = \|h\|_{p,1} = 1$, then $\|g\|_{p} = \|h\|_{p} = 1$. The strict convexity of $L_p$ then implies $f = g = h$. 

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Now suppose (ii) holds. Then, for any $0 < a < 1$, Lemma 2 implies that $f_a = \lambda |f|$ and $f^a = (1 - \lambda) |f|$ where $\lambda = \|f_a\|_{p,1}$. But this easily implies that $|f| = \mu(E)^{-1/p} \chi_E$ for some $E \subset [0,1]$.

That (iii) implies (iv) is obvious; so finally suppose (iv) holds. Then $\|f\|_{p}^{p-1} = \|\|f\|_{p-1}^{p-1}\|_{p'} \geq \|\|f\|_{p-1}^{p-1}\|_{p',\infty} = 1$ and so $\|f\|_{p} = 1$. □

**Remark.** The analogue of Theorem 1 for the Lorentz sequence space $l_{p,1}$ is well known and is due to W. J. Davis (cf. e.g. [3]).

Since $L_{p,1}$ is a separable dual space, its closed unit ball should have a wealth of strongly exposed points. (See [8 or 9].) As it happens, each extreme point of the closed unit ball of $L_{p,1}$ is also strongly exposed. To see this, suppose $f \in L_{p,1}$ is an extreme point of the closed unit ball and let $g = (\text{sgn} f) \cdot |f|^{-1}$. Then $\|f\|_p = \|g\|_{p'} = 1 = \langle f, g \rangle$ and $f$ (considered as an element of $L_p$) is strongly exposed by $g$ (considered as an element of $L_{p'}$). Thus, if $\|f_n\|_{p,1} \leq 1$ and $\langle f_n, g \rangle \to \langle f, g \rangle$, then $f_n \to f$ in $L_p$ and hence $\|f_n\|_p \to \|f\|_p = 1$. But this implies that $\|f_n\|_{p,1} \to \|f\|_{p,1} = 1$. That these conditions are sufficient to imply the convergence of the sequence $(f_n)$ to $f$ in $L_{p,1}$ is given as

**Lemma 3.** Let $(f_n)$ be a sequence in $L_{p,1}$ such that $(f_n)$ converges to $f$ in $L_p$ and $(\|f_n\|_{p,1})$ converges to $\|f\|_{p,1}$. Then $(f_n)$ converges to $f$ in $L_{p,1}$.

**Proof.** First notice that it is enough to show that every subsequence of $(f_n)$ has a further subsequence converging to $f$ in $L_{p,1}$. Consequently, we may assume that $(f_n)$ converges to $f$ a.e. Also, for convenience, we shall take $\|f\|_{p,1} = 1$. Now let $\varepsilon > 0$ and choose $\delta > 0$ so that $\|f \chi_A\|_{p,1} < \varepsilon$ whenever $\mu(A) < \delta$. Next, use Egorov’s theorem to choose $A$ so that $\mu(A) < \delta$ and so that $f_n$ converges uniformly to $f$ on $A^c$. Finally, choose $n$ sufficiently large so that the following hold:

(i) $\|(f - f_n) \chi_{A^c}\|_{p,1} < \varepsilon$,
(ii) $\|f_n\|_{p,1} < 1 + \varepsilon,$ and
(iii) $\|f_n\|_{p,1} - \|f_n \chi_{A^c}\|_{p,1} < \varepsilon$.

Then

$$\|f - f_n\|_{p,1} \leq \|f \chi_A\|_{p,1} + \|f_n \chi_A\|_{p,1} + \|(f - f_n) \chi_{A^c}\|_{p,1} \leq 2\varepsilon + \left(\|f_n\|_{p,1}^{p} - \|f_n \chi_{A^c}\|_{p,1}^{p}\right)^{1/p} \leq 2\varepsilon + \left(p(1 + \varepsilon)^{p-1}\varepsilon\right)^{1/p} < 2p\varepsilon^{1/p}.$$ 

Thus $(f_n)$ converges to $f$ in $L_{p,1}$. □

Before we can entertain any discussion of isometries on $L_{p,1}$, we shall need some condition stated in terms of the norm in $L_{p,1}$ which will guarantee that two functions are disjointly supported. The next two lemmas provide such conditions; the first is quite general (and really just a minor variation of Lemma 7.2 in [14]), while the second examines the case of equality in (4).

**Lemma 4.** Given $f, g \in L_{p,1}$, let $f \oplus g$ denote the sum of disjoint copies of $f$ and $g$; that is, $d_{f \oplus g} = d_f + d_g$. (Of course, we may need to take $f \oplus g \in L_{p,1}[0,2]$.) If $f \cdot g \geq 0$, then $\|f + g\|_{p,1} \geq \|f \oplus g\|_{p,1}$ and equality occurs only when $f \cdot g = 0$. 

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Proof. The first conclusion is a general fact in any rearrangement invariant space. Indeed, just as in the proof of Lemma 1, we need only to observe that if
\[ H_1(x) = \int_0^x (f \oplus g)^* (t) \, dt \quad \text{and} \quad H_2(x) = \int_0^x (f + g)^* (t) \, dt, \]
then \( H_1 \leq H_2 \). But, since \( f \cdot g \geq 0 \),
\[ H_1(x) = \sup_{\mu \in X} \int_E \left| f \oplus g \right|(s) \, ds \leq \sup_{\mu \in X} \left\{ \int_E \left| f(s) \right| \, ds + \int_E \left| g(s) \right| \, ds \right\} \]
\[ = \sup_{\mu \in X} \int_E \left| f + g \right|(s) \, ds = H_2(x). \]
Again, as in Lemma 1, the case \( \|f + g\|_{p,1} = \|f \oplus g\|_{p,1} \) would imply that \( H_1 = H_2 \); that is, \( d_{f+g} = d_f \oplus g = d_f + d_g \). Then \( f \cdot g \geq 0 \) implies
\[ \mu(\text{supp}\ f \cup \text{supp}\ g) = d_{f+g}(0) = d_f(0) + d_g(0) = \mu(\text{supp}\ f) + \mu(\text{supp}\ g). \]
Thus \( f \cdot g = 0 \). □

Lemma 5. Let \( f, g \in L_{p,1} \) with \( f \cdot g \geq 0 \). If \( \|f + g\|_{p,1} = \|f \oplus g\|_{p,1} \), then \( f \cdot g = 0 \) and, moreover, \( d_f \) and \( d_g \) are proportional.

Proof. The first conclusion is immediate from (4) and Lemma 4. Indeed,
\[ \|f\|_{p,1}^p + \|g\|_{p,1}^p = \|f + g\|_{p,1}^p \geq \|f \oplus g\|_{p,1}^p \geq \|f\|_{p,1}^p + \|g\|_{p,1}^p \]
and thus \( f \cdot g = 0 \). But now
\[ \left\{ \int_0^\infty \left( d_f(t) + d_g(t) \right)^{1/p} \, dt \right\}^p = \|f + g\|_{p,1}^p \geq \|f\|_{p,1}^p + \|g\|_{p,1}^p \]
\[ = \left\{ \int_0^\infty d_f(t)^{1/p} \, dt \right\}^p + \left\{ \int_0^\infty d_g(t)^{1/p} \, dt \right\}^p. \]
That is, we have equality in the triangle inequality in \( L_{1/p}[0, \infty) \). Hence \( d_f \) and \( d_g \) are proportional. (In particular, \( \|f\|_{p,1}^p = \|g\|_{p,1}^p \) would imply that \( d_f = d_g \)). □

Remark. The observation made in [2] is that \( L_{p,1} \) is “order convex”; that is, if \( f \cdot g \geq 0 \) and if \( \|f - g\|_{p,1} = \|f + g\|_{p,1} \), then \( f \cdot g = 0 \). Notice that \( f \cdot g \geq 0 \) and \( \|f + g\|_{p,1}^p = \|f\|_{p,1}^p + \|g\|_{p,1}^p \) imply that \( \|f + g\|_{p,1} = \|f - g\|_{p,1} \).

3. Isometries. Finally we are ready to describe the linear isometries from \( L_{p,1} \) into itself. What might not be expected here is that the only linear isometries are the obvious ones: changes of sign, rearrangements, and dilations. That is, if \( T : L_{p,1} \to L_{p,1} \) is an isometry and \( \lambda = \mu(\text{supp}\ T1) \), then
\[ (Tf)^*(t) = \lambda^{-1/p} f^*(t/\lambda) \]
for every \( f \in L_{p,1} \) and \( 0 \leq t \leq 1 \). (We take \( f^*(s) = 0 \) for \( s > 1 \).)

Before we set a plan of attack for proving (5), let us first reduce to the case of positive isometries. In what follows, \( T : L_{p,1} \to L_{p,1} \) is any linear isometry (not necessarily onto or positive) and, for each \( n = 1, 2, \ldots \) and \( i = 1, 2, \ldots, n \), \( z_{n,i} \) will denote the characteristic function of the interval \( [(i - 1)/n, i/n) \).
Lemma 6. Let $T: L_{p,1} \to L_{p,1}$ be a linear isometry. For every $f \in L_{p,1}$,

$$\text{supp } Tf \subset \text{supp } T1.$$  

**Proof.** It suffices to show that (6) holds in the case $f = z_{n,i}$ for any $n > 2^{p-1}$ and $i = 1, 2, \ldots, n$. To do this, it suffices to show that $Tz_{n,i}$ and $T(1 - z_{n,i})$ are disjointly supported for all $n > 2^{p-1}$ and all $i = 1, 2, \ldots, n$. Fix $n > 2^{p-1}$ and $1 \leq i \leq n$, and set $f = Tz_{n,i}$, $g = T(1 - z_{n,i})$, $h = f + g = T1$, $k = f - g = T(2z_{n,i} - 1)$. Then, since $T$ is an isometry, it is easy to see that

$$\text{supp } h \subset \text{supp } k \text{ or } \text{supp } k \subset \text{supp } h \text{ or } \text{supp } g \subset \text{supp } h \text{ or } \text{supp } h \subset \text{supp } g.$$  

The equations in (8) follow from the fact that $\|1 + x\|_{p,1} = 1 + \|x\|_{p,1}$ for any $x \in L_{p,1}$, $x \geq 0$.

The equations in (8) combine with the remark following Lemma 1 to yield several consequences:

$$f \cdot h \geq 0, \quad f \cdot k \geq 0, \quad g \cdot h \geq 0, \quad (-g) \cdot k \geq 0$$

and

$$\text{supp } f \subset \text{supp } h \text{ or } \text{supp } h \subset \text{supp } f,$$

$$\text{supp } f \subset \text{supp } k \text{ or } \text{supp } k \subset \text{supp } f,$$

$$\text{supp } g \subset \text{supp } h \text{ or } \text{supp } h \subset \text{supp } g,$$

$$\text{supp } g \subset \text{supp } k \text{ or } \text{supp } k \subset \text{supp } g.$$  

The inequalities in (9) imply that at any point for which $f \cdot g \neq 0$ we have $h \cdot k = 0$ and, moreover, that $f \cdot g \geq 0$ on supp $h$ and $f \cdot g \leq 0$ on supp $k$. Consequently, if we set:

$$A = \text{supp } f \setminus \text{supp } g,$$

$$B = \text{supp } g \setminus \text{supp } f,$$

$$C = \text{supp } h \setminus (A \cup B) = \{\text{sgn } f = \text{sgn } g\},$$

$$D = \text{supp } k \setminus (A \cup B) = \{\text{sgn } f = -\text{sgn } g\},$$

then $A$, $B$, $C$, $D$ are pairwise disjoint and

$$f = fA + \frac{1}{2}hA, \quad g = gB + \frac{1}{2}hC, \quad h = \frac{1}{2}hC + \frac{1}{2}kC, \quad k = \frac{1}{2}hA - \frac{1}{2}kC.$$  

But now (10) implies several conditions on $A$, $B$, $C$, and $D$. In fact, it is not hard to see that either $A = B = \emptyset$ or else $C = D = \emptyset$ (that is, either $h \cdot k = 0$ or $f \cdot g = 0$). Indeed, suppose for instance that $A \neq \emptyset$ and $C \neq \emptyset$. Then supp $g = B \cup C \cup D$ and supp $k = A \cup B \cup D$ cannot satisfy either of the containments supp $g \subset$ supp $k$ or supp $k \subset$ supp $g$. Thus we need only point out that $A = B = \emptyset$ (i.e., $h \cdot k = 0$) is impossible. But $\|h + k\|_{p,1} = 2\|f\|_{p,1} = 2n^{-1/p}$ while from (4), $h \cdot k = 0$ would imply $\|h + k\|_{p,1} \leq 2^{1/p}$; our choice of $n > 2^{p-1}$ makes this impossible. Thus $f \cdot g = 0$ as desired. □
Lemma 7. Let $T: L_{p,1} \to L_{p,1}$ be a linear isometry. The map $S: L_{p,1} \to L_{p,1}$ defined by $Sf = (\text{sgn } T1) \cdot Tf$ is a positive isometry. In particular, if $f \cdot g \geq 0$, then $Tf \cdot Tg \geq 0$.

Proof. As mentioned above, for any $f \geq 0$, we have $\|Tf\|_{p,1} = \|T1\|_{p,1} + \|Tf\|_{p,1}$ and so, from Lemma 1, $T1 \cdot Tf \geq 0$. Since Lemma 6 states that $\text{supp } Tf \subset \text{supp } T1$, we can conclude that $(\text{sgn } T1) \cdot Tf = |Tf|$. □

Our next goal will be to provide a more tractable (i.e., linear) replacement for (5). To this end, it may be helpful to think of $T$ as an isometry from $L_{p,1}$ into $L_{p,1}([0,1]^2)$. The reasons for this are essentially cosmetic: if we define $f \otimes g$ by $(f \otimes g)(s,t) = f(s)g(t)$, then it is easy to see that the map $f \to f \otimes g$ defines an isometry satisfying (5) whenever $|g| = \mu(E)^{-1/p}$. Indeed, the distribution of $f \otimes \chi_E$ is $\mu(E) \cdot d_f$ and so, in this case,

$$(f \otimes g)^*(t) = \mu(E)^{-1/p} f^*(t/\mu(E)).$$

Now it is also known that

$$(14) \quad \|f\|_{p,1} \cdot \|g\|_{p,1} \leq \|f \otimes g\|_{p,1} \leq \|f\|_{p,1} \cdot \|g\|_{p,1}$$

for any $f, g \in L_{p,1}$ [4, 5 and 17, Theorem 7.4] and so our program is easy to outline. We shall then show that $Tf$ must have same distribution as $f \otimes g$ where $g = T1$. We shall then show that $g = T1$ forces equality in (14). This implies that $\|g\|_{p,1} = \|g\|_{p,1} = 1$ or equivalently, by Theorem 1, that $g = \mu(E)^{-1/p} \chi_E$ for some measurable set $E$.

Lemma 8. Let $T: L_{p,1} \to L_{p,1}$ be a linear isometry and let $g = T1$. Then $Tf$ has the same distribution as $f \otimes g$ for every $f$ in $L_{p,1}$.

Proof. Since $T$ is linear and continuous and since step functions of the form $\sum_{i=1}^n a_i z_{n,i}$ are dense in $L_{p,1}$, it is enough to show that, for each $n$, the functions $Tz_{n,i}$, $i = 1, 2, \ldots, n$, are disjointly supported and have the same distribution. Fix $n$ and $1 \leq i \neq j \leq n$. Since $T$ is a linear isometry, we have $\|Tz_{n,i}\|_{p,1} = \|Tz_{n,j}\|_{p,1}$ and $\|Tz_{n,i} + Tz_{n,j}\|_{p,1} = \|Tz_{n,i}\|_{p,1} + \|Tz_{n,j}\|_{p,1}$.

But now Lemma 5 and Lemma 7 give us that $Tz_{n,i}$ and $Tz_{n,j}$ are disjointly supported and have the same distribution. Necessarily $Tz_{n,i}$ has the same distribution as $z_{n,i} \otimes g$ and linearity implies that $T(\sum_{i=1}^n a_i z_{n,i})$ has the same distribution as $(\sum_{i=1}^n a_i z_{n,i}) \otimes g$. □

Next we consider the case of (near) equality on the left-hand side of inequality (14). The following lemma is suggested by the proof of Lemma 8.8 of [14]. (Also see [15, Theorem 7.2.7 and 6].)

Lemma 9. Given a positive integer $k$ and $\alpha > 1$, define $f_{k,a} \in L_{p,1}$ by $f_{k,a}(t) = k^{\alpha/p} \land t^{-1/p}$. Then, for any $h = \sum_{i=1}^k a_i z_{k,i}$,

$$(15) \quad \|f_{k,a} \otimes h\|_{p,1} \leq (1 + \alpha^{-1}) \cdot \|h\|_{p} \cdot \|f_{k,a}\|_{p,1}.$$ 

Proof. We shall show that if $\|h\|_{p} \leq 1$, then $(f_{k,a} \otimes h)^* \leq f_{k,a}^{(1)}$; (15) then follows from a simple calculation:

$$\|f_{k,a} + 1\|_{p,1} = \left(1 + \frac{\alpha + 1}{p} \log k \leq (1 + \alpha^{-1}) \cdot \left(1 + \frac{\alpha}{p} \log k\right) = (1 + \alpha^{-1}) \cdot \|f_{k,a}\|_{p,1}. $$
Assume that $h = \sum_{i=1}^{k} a_i z_{k,i}$ satisfies $h \geq 0$ and $\|h\|_p = k^{-1} \sum_{i=1}^{k} a_i^p \leq 1$; we wish to estimate $(f_{k,a} \otimes h)^*$. But $f_{k,a} \otimes h = \sum_{i=1}^{k} a_i (f_{k,a} \otimes z_{k,i})$ and the functions $f_{k,a} \otimes z_{k,i}$ are disjointly supported and have distribution $k^{-1} \text{dist}(f_{k,a}; t)$. Thus

$$\text{dist}(f_{k,a} \otimes h; t) = k^{-1} \sum_{i=1}^{k} \text{dist}(f_{k,a}; \frac{t}{a_i}).$$

Now $\text{dist}(f_{k,a}; t) = t^{-p} \wedge \chi_{[0, k^{a/p}]}(t)$, and it is easy to check that, for $0 < a \leq k^{1/p}$, we have

$$\text{dist}(f_{k,a}; t/a) = a^p t^{-p} \wedge \chi_{[0, k^{a/p}]}(t/a) \leq \chi_{[0,1]}(t) + a^p t^{-p} \chi_{[1, k^{(a+1)/p}]}(t).$$

Consequently,

$$\text{dist}(f_{k,a} \otimes A; t/a) = a^p t^{-p} \wedge \chi_{[0, k^{a/p}]}(t/a) \leq t^{-p} \wedge \chi_{[0, k^{(a+1)/p}]}(t) = \text{dist}(f_{k,a+1}; t).$$

Thus, $(f_{k,a} \otimes h)^* \leq f_{k,a+1}$. □

We are finally ready to combine all of the preceding observations to give a simple proof of our main result.

**Theorem 2.** Let $T: \ell^p \to \ell^p$ be a linear isometry and let $\lambda = \mu(\text{supp} 11)$. Then $T$ satisfies

$$\lambda^{1/p} f^*(t) = \lambda^{1/p} f^*(t_{\mu})$$

for every $f$ in $\ell^p$ and $0 \leq t \leq 1$.

**Proof.** Let $g = T1$. Lemma 8 shows that $(Tf)^* = (f \otimes g)^*$ for every $f \in \ell^p$. Thus, by (13) and Theorem 1, we need only show that $\|g\|_p = 1$.

Let $0 < \varepsilon < 1$ and let $h = \sum_{i=1}^{k} a_i z_{k,i}$ be a step function such that $\|g - h\|_p \leq \varepsilon \|g\|_p$. Next let $f = f_{k,1/\varepsilon}$ be the function given in Lemma 9 (for $\alpha = 1/\varepsilon$). Then

$$\|f\|_p = \|Tf\|_p = \|f \otimes g\|_p \leq \|f \otimes h\|_p + \|f \otimes (g - h)\|_p \leq (1 + \varepsilon) \|h\|_p \cdot \|f\|_p + \|(g - h)\|_p \cdot \|f\|_p \leq \left[1 + \varepsilon^2 + \varepsilon \|g\|_p \cdot \|f\|_p \right] (where the second inequality follows from (14) and (15)). Letting $\varepsilon$ tend to 0 yields $\|g\|_p = 1$ as promised. □

**Remark.** Notice that every linear isometry on $\ell^p$ turns out to be an isometry on $\ell^p$ and, in fact, a multiple of an isometry on every $L_r$. Thus a linear isometry $T$ on $\ell^p$ maps extreme points to extreme points and is onto exactly when $|T1| = 1$. 

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078

DEPARTMENT OF MATHEMATICAL SCIENCES, OAKLAND UNIVERSITY, ROCHESTER, MICHIGAN 48063 (Current address of B. Turett)

Current address (N. L. Carothers): Department of Mathematics, Texas A & M University, College Station, Texas 77843

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