EQUIVARIANT MINIMAL IMMERSIONS OF $S^2$ INTO $S^{2m}(1)$

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Abstract. We classify the directrix curves associated with equivariant minimal immersions of $S^2$ into $S^{2m}(1)$ and obtain some applications.

0. Introduction. Minimal immersions of the 2-sphere $S^2$ into the standard $n$-dimensional unit sphere $S^n(1)$ in the euclidean space $\mathbb{R}^{n+1}$ were studied by O. Boruvka [1], E. Calabi [6], S. S. Chern [7], J. L. M. Barbosa [2], and R. L. Bryant [5]. On the other hand, K. Uhlenbeck [16] handled equivariant harmonic maps of $S^2$ into $S^n(1)$ as completely integrable systems.

In this paper, we study equivariant minimal immersions of $S^2$ into $S^n(1)$ of type $(m(1), \ldots, m(m))$ (see §3) by using Chern and Barbosa's method [7, 2]. That is, we classify directrix curves associated with equivariant (generalized) minimal immersions of $S^2$ into $S^{2m}(1)$ of type $(m(1), \ldots, m(m))$. We see that the volume of the generalized minimal immersions is equal to $4\pi(m(1) + \cdots + m(m))$ and the regularity of the generalized minimal immersions is equivalent to $m(1) = 1$, which gives another proof of [16]. In particular, examples constructed by Barbosa [2] are equivariant minimal immersions of type $(1, \ldots, m-1, k)$. Furthermore, in §4, we investigate minimal immersions of the real projective 2-space $P^2$ into the standard $2m$-dimensional real projective space $P^{2m}(1)$ and show that there is no full minimal immersion of $P^2$ into $S^{2(2m-1)}(1)$. We classify equivariant minimal immersions of $P^2$ into $P^{2m}(1)$ of type $(m(1), \ldots, m(m))$ and prove that an equivariant minimal immersion of $P^2$ into $P^{2m}(1)$ of type $(m(1), \ldots, m(m))$ is unique. Hence we note that a minimal immersion with volume $m(m+1)\pi$ is the standard minimal immersion $P^2(2/m(m+1)) \to P^{2m}(1)$. Using this fact, we obtain an application to P. Li and S. T. Yau's inequality [12]. In §5, we show that the minimal cone of a full minimal immersion of $S^2$ into $S^{2m}(1)$ is stable. The minimal cone of the holomorphic immersion of $S^2$ into $S^{6}(1)$ with almost complex structure defined by Cayley numbers has the parallel calibration $\omega$ [11] and hence is homologically volume minimizing. Conversely we prove that the full minimal immersion of $S^2$ into $S^{2m}(1)$ whose minimal cone has a parallel calibration is holomorphic in $S^{6}(1)$. Using this equivalence, we classify equivariant holomorphic immersion of $S^2$ into $S^{6}(1)$. On the other hand, it is known that 3-dimensional totally real submanifolds in $S^{6}(1)$ are minimal [8] and their minimal cones have the parallel calibration $\ast\omega$ and hence are
homologically volume minimizing [13]. In §7, we prove that some tubes in the direction of the first and second normal bundle of holomorphic curves give 3-dimensional totally real submanifolds in $S^6(l)$. Using this fact, we see that circle bundles of $S^2$ of positive even Chern number ($\geq 4$) are minimally immersed in $S^6(l)$. In particular, the minimal immersion of $S^2(\frac{1}{l})$ into $S^6(l)$ is constructed by the above method as well as the holomorphic immersion of $S^2(\frac{1}{l})$ into $S^6(l)$.

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1. Higher fundamental forms. Let $\bar{M}^n(c)$ be an $n$-dimensional Riemannian manifold of constant curvature $c$. We denote by $\langle \cdot, \cdot \rangle$ and $\nabla$ the metric and the covariant differentiation of $\bar{M}^n(c)$, respectively. Let $M$ be an $m$-dimensional manifold immersed in $\bar{M}^n(c)$, $\chi$ the immersion and $\nabla$ the covariant differentiation of $M$ with respect to the induced metric. Then the second fundamental form $\sigma_2$ of $M$ is given by

$$\sigma_2(X, Y) = \nabla_X Y - \nabla_Y X$$

and satisfies

$$\sigma_2(X, Y) = \sigma_2(Y, X).$$

Let $N_x(M)$ be the normal space at $x$. We call the subspace $N_1(x)$ of $N_x(M)$ spanned by $\sigma_2(X, Y)$ for all $X, Y \in T_x(M)$ the first normal space at $x$ and we denote $\bigcup_{x \in M} N_1(x)$ by $N_1(M)$. Let $M_1$ be the subset of $M$ defined by

$$\{ x \in M : \text{dim } N_1(x) = \max_{x \in M} \text{dim } N_1(x) \}.$$ 

Then, by the definition of $M_1$, $M_1$ is open in $M$. Since the restriction $N_1(M_1)$ of $N_1(M)$ to $M_1$ is a subbundle of $N(M_1)$, we can define the third fundamental form $\sigma_3$ by

$$\sigma_3(X_1, X_2, X_3) = \text{the component of } \nabla^{N}_{X_1} \sigma_2(X_2, X_3),$$

which is orthogonal to $N_1(M_1)$, where $\nabla^{N}$ is the normal connection of $N(M)$.

It is easy to see that $\sigma_3$ is a 3-symmetric tensor. Continuing this process, we can define the $(s + 1)$st fundamental form $\sigma_{s+1}$, the $s$th normal bundle $N_s(M_s) (M_0 = M)$ and the open set $M_s$ for $s \geq 1$. Furthermore we have the fact that $\sigma_{s+1}$ is an $(s + 1)$-symmetric tensor. We set $r_s = \text{rank } N_s(M_s)$. If there is an $s_0$ such that $r_{s_0} = 0$, then by [10], $N(M_{s_0})$ has the Whitney sum decomposition:

$$N_1(M_{s_0}) + \cdots + N_{s_0-1}(M_{s_0}) + P,$$

where $N_1(M_{s_0})$ is the restriction of $N_1(M)$ to $M_{s_0}$ and $P$ is the bundle which is parallel with respect to $\nabla^{N}$. By J. Erbacher [10], we obtain

$$\chi(M_{s_0}) \subset \text{a totally geodesic submanifold of codimension dim } P.$$

2. Minimal immersions of $S^2$ into $S^n(1)$. In this section, we review necessary results on minimal immersions of $S^2$ into $S^n(1) \subset R^{n+1}$. 

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If $S^2$ is fully immersed in $S^n(1)$, then $n$ is an even integer ($= 2m$). Moreover the higher fundamental forms $\sigma_s$ for $s = 2, \ldots, m$ satisfy

$$\sum_{i=1}^{2} \sigma_s(e_i, e_i, X_1, X_2, \ldots, X_{s-2}) = 0,$$

$\sigma_s(X_1, \ldots, X) = \sigma_s(X_1, \ldots, X, Y)$ is orthogonal to $\sigma_s(X_1, \ldots, X)$, $\|\sigma_s(X_1, \ldots, X, Y)\| = l_{s-1}$, where $\{e_1, e_2\}$ is an orthonormal basis and $X, Y$ are orthonormal vectors of $T(S^2_{s-2})$. Since the immersion is full and analytic, we obtain $l_1, \ldots, l_{m-1} \neq 0$ on any open subset. For an orthonormal local cross section $e_3, \ldots, e_{2m}$ of $N(M_{m-1})$ defined by

$$e_{2s-1} = \frac{1}{l_{s-1}} \sigma_s(e_1, \ldots, e_1), \quad e_{2s} = \frac{1}{l_{s-1}} \sigma_s(e_1, \ldots, e_1, e_2),$$

we set $E_s = e_{2s-1} + ie_{2s}$ for $2 \leq s \leq m$. Then we have

$$(2.1) \quad \nabla E_s = -\kappa_{s-1}^2 \phi E_{s-1} - i\omega_{2s-1,2s} E_s + \kappa_s \phi E_{s+1},$$

$\omega_{2s-1,2s} = s\omega_{1,2} + \theta_{s-1}, \quad \theta_s = d^c \log(\kappa_1, \ldots, \kappa_s),$

where $\kappa_s = l_s/l_{s-1}$ ($l_0 = 1$), $\phi = \omega_1 + i\omega_2$ such that $\omega_1, \omega_2$ are the dual frames of $\{e_1, e_2\}$, $\kappa_0 = 0$, $d^c = i(\overline{\partial} - \partial)$, and

$$\omega_{1,2}(X) = \langle \nabla_X e_1, e_2 \rangle, \quad \omega_{2s-1,2s}(X) = \langle \nabla_X e_{2s-1}, e_{2s} \rangle.$$ 

We have the following relations among $\kappa_1, \ldots, \kappa_m$:

$$(2.2) \quad \kappa_1^2 = \frac{1}{2}(1 - K), \quad \kappa_m = 0,$$

$$\frac{1}{2} \Delta \log(\kappa_1, \ldots, \kappa_s) + \kappa_s^2 - \kappa_{s+1}^2 - \frac{1}{2} (s + 1) K = 0,$$

where $K$ is the Gauss curvature of $M$. These results are given in [7]. Moreover we note the following [3, 7]:

$M - M_{m-1}$ consists of isolated points and the $s$th normal bundle is defined over isolated points.

Next we review Barbosa’s result [2].

Let $z$ be an isothermal coordinate of $S^2$ and $(, )$ the symmetrical product of $C^{2m+1}$, i.e., the complex linear extension of the euclidean product of $R^{2m+1}$. Then we construct vector valued functions $G_0, G_1, \ldots, G_m$ as follows:

$$G_0 = \chi, \quad G_1 = \overline{\partial} \chi,$$

$G_k = \overline{\partial}^k \chi - \sum_{j=1}^{k-1} a^j_k G_j$, $G_m = \overline{\partial}^m \chi - \sum_{j=1}^{m-1} a^j_m G_j,$

where the $a^j_k$ are chosen in such a way that $(G_k, G_j) = 0$ for $j < k$.

Barbosa obtains the following

**LEMMA 2.1 (BARBOSA [2]).** (1) $\overline{\partial} G_k = G_{k+1} + (\overline{\partial} \log|G_k|^2) G_k$,

(2) $\partial G_k = -|G_k|^2 G_{k-1}/|G_{k-1}|^2$ for $k > 0$,

(3) $\overline{\partial} G_m = (\overline{\partial} \log|G_m|^2) G_m.$
Note the fact that $\xi = G_m/|G_m|^2$ is holomorphic and
\begin{equation}
(\xi, \xi) = \cdots = (\xi^{m-1}, \xi^{m-1}) = 0,
\end{equation}
where $\xi^k = \partial^k \xi$. We call $\xi$ the associated holomorphic map of $\chi$. Furthermore

**Lemma 2.2 (Barbosa [2]).** $\xi$ has only isolated singularities with poles and $\xi$ gives a holomorphic map $\Xi$ of $S^2$ into a 2m-dimensional complex projective space $P_{2m}$.

We call the above holomorphic map $\Xi$ the directrix curve of the immersion $\chi$. We define $\psi$ by
\[
\psi = \xi \land \xi^1 \land \cdots \land \xi^{m-1} \land \xi^1 \land \cdots \land \xi^{m-1},
\]
which is a map into $\Lambda^{2m}C^{2m+1}$ and define $\tilde{\psi}$ by
\[
\tilde{\psi} = \begin{cases} 
\psi & \text{if } m \text{ is even}, \\
-i\psi & \text{if } m \text{ is odd}.
\end{cases}
\]

Regarding $\Lambda^{2m}C^{2m+1}$ as $C^{2m+1}$, we note that $\tilde{\psi}$ is parallel to $\chi$. Conversely let $\Xi$ be a holomorphic curve of $S^2$ into $P_{2m}$ which is not contained in any hyperplane of $P_{2m}$. Using an isothermal coordinate $z$ and the inhomogeneous coordinates of $P_{2m}$, we have a local expression $\xi(z)$ of $\Xi(z)$ into $C^{2m+1}$. Assume that $\xi$ satisfies (2.2). Then we can construct $\tilde{\psi}$ as above and we have the following

**Proposition 2.1 (Barbosa [2]).** The function $\tilde{\psi}/|\tilde{\psi}|$ is independent of the particular local coordinates used, and so it defines a global map $\chi$ from $S^2$ into $S^{2m}(1)$. Furthermore, we have, relative to a local coordinate $z$, that $(\partial \chi, \partial \chi) = 0$, $\partial \tilde{\partial} \chi$ is parallel to $\chi$ and
\[
(\partial \chi, \tilde{\partial} \chi) = |\xi_{m-1} \land \xi_{m-1}'|^2 / |\xi_{m-1}|^4,
\]
where $\xi_{m-1} = \xi \land \xi' \land \cdots \land \xi^{m-1}$.

Proposition 2.1 implies that $\chi$ is a generalized minimal immersion (see, for example, [2]). Let $\Xi$ be a holomorphic map of $S^2$ into $P_{2m}$ which is not contained in a hyperplane and whose local expression $\xi$ satisfies (2.2). Then we call $\Xi$ a totally isotropic curve. Consequently we obtain

**Theorem 2.1 (Barbosa [2]).** There exists a canonical 1-1 correspondence between the set of generalized minimal immersions $\chi$: $S^2 \to S^{2m}(1)$ which are not contained in any lower dimensional subspace of $R^{2m+1}$ and the set of totally isotropic holomorphic curves $\Xi$: $S^2 \to P_{2m}$ which are not contained in any complex hyperplane of $P_{2m}$. The correspondence is the one that associates with minimal immersion $\chi$ its directrix curve.

By the definition of $G_j$ and $E_j$, we obtain

**Lemma 2.3.** $G_j = \lambda^2/2k_1 \cdots k_{j-1} E_j$, where $\lambda^2 dz d\bar{z}$ is the metric tensor.

3. **Equivariant minimal immersions of $S^2$ into $S^{2m}(1)$.** Let $\rho$ and $\rho'$ be a circle action of $S^2$ and a one-parameter subgroup of isometries of $S^{2m}(1)$, respectively. Let $\chi$ be an equivariant minimal immersion of $S^2$ into $S^{2m}(1)$ which is not contained in any hyperplane of $R^{2m+1}$ and satisfies
\begin{equation}
\chi(\rho(\theta) x) = \rho'(\theta) \chi(x).
\end{equation}
Since \( \rho(\theta) \) is a circle action and gives a conformal transformation of \( S^2(1) \), there exists an isothermal coordinate \( z \) defined by the stereographic projection of \( S^2(1) \) onto \( \mathbb{R}^2 \) such that

\[
\rho(\theta) : z \to e^{i\theta}z.
\]

Choosing orthogonal coordinates \((x^1, y^1, \ldots, x^m, y^m, u)\) of \( \mathbb{R}^{2m+1} \), we have positive integers \( 0 \leq m(1) \leq m(2) \leq \cdots \leq m(m) \) such that

\[
\rho(\theta)(x^1, y^1, \ldots, x^m, y^m, u) = (\ldots, x^k \cos m(k)\theta - y^k \sin m(k)\theta, x^k \sin m(k)\theta + y^k \cos m(k)\theta, \ldots, u).
\]

The equivariant minimal immersion is said to be of type \((m(1), \ldots, m(m))\).

\( \chi \) gives the same vector valued functions \( G_j \) as \((2.1)\). Let \( D_j \) and \( F_j \) be the vector valued functions defined by \( \chi \cdot \rho, \bar{\rho} \cdot \chi \), respectively. Then we have

**Lemma 3.1.** \( D_j = e^{-i(\theta)}G_j \cdot \rho \) and \( F_j = \bar{\rho} \cdot G_j \).

**Proof.** From the definition of \( D_j \), we have

\[
D_1 = \bar{\delta}(\chi \cdot \rho) = e^{-i\theta}G_1 \cdot \rho(z).
\]

Assume \( D_j = e^{-i(\theta)}G_j \cdot \rho \) for \( j \leq k \). Then

\[
D_{k+1} = \frac{\partial^{k+1}}{\partial z^{k+1}}(\chi \cdot \rho) - \sum_{l=1}^{k} \left( \frac{\partial^{k+1}}{\partial z^{k+1}}(\chi \cdot \rho), \bar{D}_l \right) \frac{D_l}{\|D_l\|^2}
\]

\[
= \frac{\partial^{k+1}}{\partial z^{k+1}}(\chi \cdot \rho) - \sum_{l=1}^{k} \left( e^{-(k+1)\theta}\left( \frac{\partial^{k+1}}{\partial z^{k+1}}(e^{i\theta}z), \frac{e^{-i(\theta)}G_l \cdot \rho}{\|G_l \cdot \rho\|^2} \right) \bar{G}_l \right)
\]

\[
= e^{-(k+1)\theta}G_{k+1} \cdot \rho(z). \quad Q.E.D.
\]

Since \( \bar{\rho} \cdot \chi = \chi \cdot \rho \), we obtain

\[
\frac{D_m}{\|D_m\|^2} = \frac{F_m}{\|F_m\|^2},
\]

which implies

\[
e^{-im\theta}z(\rho(z)) = \bar{\rho}(\theta)\xi(z).
\]

Conversely, we have the following

**Lemma 3.2.** Let \( \chi \) be a full minimal immersion of \( S^2 \) into \( S^{2m}(1) \) and \( \Xi \) the directrix curve. Let \( z \) be an isothermal coordinate of \( S^2 \) defined by the stereographic projection of \( S^2(1) \) onto \( \mathbb{R}^2 \) and \( \xi(z) \) the expression of \( \Xi \). If \( \xi(\rho(\theta)z) \) is parallel to \( \bar{\rho}(\theta)\xi(z) \), then \( \chi \) is an equivariant minimal immersion.

**Proof.** From the definition of \( \psi \), we get

\[
\psi(\rho(\theta)z) = \xi(\rho(\theta)z) \wedge \cdots \wedge \xi^{m-1}(\rho(\theta)z)
\]

\[
\wedge \bar{\xi}(\rho(\theta)z) \wedge \cdots \wedge \bar{\xi}^{m-1}(\rho(\theta)z).
\]

It follows from \((3.2)\) that

\[
\psi(\rho(\theta)z) = \bar{\rho}(\theta)\xi(z) \wedge \cdots \wedge \bar{\rho}(\theta)\xi^{m-1}(z)
\]

\[
\wedge \bar{\rho}(\theta)\xi(z) \wedge \cdots \wedge \bar{\rho}(\theta)\xi^{m-1}(z).
\]
Since \( \tilde{p} \) acts on \( \Lambda^{2m}C^{2m+1} \), we have \( \psi(\rho(\theta)z) = \tilde{p}(\theta)\psi(z) \). This, together with \( \chi = \frac{\psi}{\|\psi\|} \), implies that \( \chi \) is an equivariant minimal immersion of \( S^2 \) into \( S^{2m}(1) \). Q.E.D.

Hence, by Theorem 2.1, the study of equivariant minimal immersions of type \( (m_{(1)}, \ldots, m_{(m)}) \) reduces to that of totally isotropic curves whose expression \( \xi \) satisfies (3.2). Then, since \( \xi \) has no essential singularity at \( z = 0 \), it can be written in some neighborhood of 0 as

\[
\xi(z) = \sum_{a=k}^{l} a_{a}z^{a},
\]

where \( a_{a} \in C^{2m+1} \) and \( k \) is the degree of poles at \( z = 0 \). Setting \( \xi'(z) = \sum_{a} A_{a}^{j}z^{a} \), we obtain

\[
e^{i(a-m)A_{a}^{j}} = A_{a}^{j}\cos m_{(j)}\theta - A_{a}^{j}\sin m_{(j)}\theta,
\]

\[
e^{i(a+m)A_{a}^{j}} = A_{a}^{j}\sin m_{(j)}\theta + A_{a}^{j}\cos m_{(j)}\theta.
\]

We note that \( A_{a}^{j-1}, A_{a}^{j} \neq 0 \) holds if and only if

\[
\left( \cos m_{(j)}\theta - e^{i(a-m)\theta} \right)^{2} + \sin^{2} m_{(j)}\theta = 0.
\]

Then \( \alpha = m - m_{(j)} \) or \( \alpha = m + m_{(j)} \) and \( A_{a}^{j} = A_{a}^{j-1}, A_{a}^{j} = -iA_{a}^{j-1} \). We denote \( A_{a}^{j-1} \) and \( A_{a}^{j+1} \) by \( A_{j} \) and \( B_{j} \), respectively. By \( (\xi, \xi) = 0 \), we obtain

\[
\xi^{2m+1}(z)^{2} + \left( 4 \sum_{j=1}^{m} A_{j}B_{j} \right)z^{2m} = 0
\]

and hence

\[
\xi^{2m+1}(z) = i\sqrt{4 \sum_{j=1}^{m} C_{j}z^{m}},
\]

where \( C_{j} = A_{j}B_{j} \). Setting \( \kappa = \sqrt{4 \sum_{j=1}^{m} C_{j}} \), we have

\[
(3.3) \quad \xi(z) = (\ldots, A_{j}z^{m_{(j)}}, B_{j}z^{m_{(j)}}, iA_{j}z^{m_{(j)}}, iB_{j}z^{m_{(j)}}, \ldots, \kappa z^{m}).
\]

By (3.3), \( m_{(1)} < \cdots < m_{(m)} \) holds, because \( \xi(z) \) is not contained in any subspace of \( C^{2m+1} \). Let \( a_{j}, b_{j} \) be the vectors of \( C^{2m+1} \) defined by

\[
a_{j} = A_{j}(e^{2j} + ie^{2j}) \quad \text{and} \quad b_{j} = B_{j}(e^{2j} - ie^{2j}),
\]

where \( e_{k} = (0, \ldots, 0, 1, 0, \ldots, 0) \) (one in the \( k \)th position). Then \( (a_{j}, b_{j}) = 2C_{j} \) for \( 1 \leq j \leq m \) clearly holds, and \( \xi \) can be written as

\[
\xi(z) = z^{-m+m_{(m)}}a_{m} + z^{-m+m_{(m-1)}}a_{m-1} + \sum_{j=1}^{m-1} a_{j}z^{m_{(j)}} + \sum_{j=1}^{m-1} b_{j}z^{m_{(j)}+m_{(j)}} + i\kappa z^{m_{(m)}}.
\]
Let $\eta(z)$ be the terms in \{\cdots\}. Then $\xi(z)$ is totally isotropic if and only if $\eta(z)$ is. $\eta'(z)$ is given by

$$\begin{align*}
z^{m(m) - m(m-1) - 1} \left( 2m^{(m)}_m b_m z^{m(m) + m(m) - 1} + (m^{(m)}_m - m^{(m-1)}_{m-1}) a_{m-1} \right. \\
+ (m^{(m)}_m + m^{(m-1)}_{m-1}) b_{m-1} z^{m(m-1)} + (m^{(m)}_m - m^{(m)}_{m-1}) a_{z^{m(m) - m(m-1)}} \\
+ (m^{(m)}_m + m^{(m)}_{m+1}) b_z z^{m(m) + m(m+1)} + (m^{(m+1)}_{m+1} - m^{(m+1)}_{m}) a_z z^{m(m+1) - m(m)} \\
+ (m^{(m)}_m + m^{(m)}_{m+1}) b_{z^{m(m) + m(m+1)}} + i k m^{(m)}_m z^{m(m-1) + m(m+1)} + i k m^{(m)}_m e z^{m(m-1)}. \end{align*}$$

We denote the terms in \{\cdots\} by $\eta_i$. Then

$$(\eta_1, \eta_1) = \cdots = (\eta^{m-2}_1, \eta^{m-2}_1) = 0$$

holds. Continuing this process, we obtain holomorphic curves $\eta(z), \eta(1)(z), \cdots, \eta(m-1)(z)$ such that

$$(\eta, \eta) = (\eta(1), \eta(1)) = \cdots = (\eta(m-1), \eta(m-1)) = 0,$$

which is equivalent to the fact that $\xi$ is totally isotropic. Thus we get

**Lemma 3.3.** $\xi$ is totally isotropic if and only if

1. $C_1 + \cdots + C_m = \frac{1}{4} k^2$,

2. $\left( m^{2}_{(m)} - m^{2}_{(m+j)} \right) \cdots \left( m^{2}_{(j+1)} - m^{2}_{(j)} \right) C_j + \sum_{k<j} \left( m^{2}_{(m)} - m^{2}_{(k)} \right) \cdots \left( m^{2}_{(j+1)} - m^{2}_{(j)} \right) C_k = \frac{1}{4} k^2 m^{2}_{(m)} \cdots m^{2}_{(j+1)}$ for each $j \leq m - 1$.

We can solve the equations (1) and (2), that is, we get

**Lemma 3.4.** The unique solutions $C_j$ of (1) and (2) are given by

$$C_j = (-1)^{j-1} \frac{\kappa^2 m^{2}_{(m)} \cdots m^{2}_{(j+1)} m^{2}_{(j-1)} \cdots m^{2}_{(1)}}{4 \left( m^{2}_{(m)} - m^{2}_{(j)} \right) \cdots \left( m^{2}_{(j+1)} - m^{2}_{(j)} \right) \left( m^{2}_{(j)} - m^{2}_{(j-1)} \right) \cdots \left( m^{2}_{(j)} - m^{2}_{(1)} \right)}.$$

**Proof.** It is easy to see that the solutions $C_1, \ldots, C_m$ are unique. We prove that the above $C_j$ satisfy (1) and (2). (2) holds if and only if

$$\sum_{k=1}^{j} \left( m^{2}_{(m)} - m^{2}_{(j)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(k+1)} \right) m^{2}_{(k)} \left( m^{2}_{(k)} - m^{2}_{(k-1)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(1)} \right) = \frac{1}{m^{2}_{(j)} \cdots m^{2}_{(1)}}.$$

For each $k > l$,

$$\frac{1}{m^{2}_{(k)} \cdots m^{2}_{(1)}} \left( m^{2}_{(k)} - m^{2}_{(l)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(l+1)} \right) m^{2}_{(k)} \left( m^{2}_{(k)} - m^{2}_{(k-1)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(1)} \right) \left( m^{2}_{(k)} - m^{2}_{(l)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(l+1)} \right) m^{2}_{(k)} \left( m^{2}_{(k)} - m^{2}_{(k-1)} \right) \cdots \left( m^{2}_{(k)} - m^{2}_{(1)} \right)$$

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converges to some value if \( m_{(k)} \to m_{(j)} \). Therefore the left-hand side of (3.4) converges to some value even if \( m_{(k)} \to m_{(j)} \). Choosing the common denominator, we note that the numerator has the divisor:

\[
(m_{(j)} - m_{(j-1)}) \cdots (m_{(j)} - m_{(1)}) (m_{(j-1)} - m_{(j-2)}) \cdots (m_{(j-1)} - m_{(1)}) \cdots (m_{(2)} - m_{(1)}) .
\]

Thus the left-hand side of (3.3) is given by

\[
\frac{L}{m_{(j)}^2 \cdots m_{(1)}^2}
\]

up to a real number \( L \). We can easily prove \( L = (-1)^{j-1} \) by induction and \( m_{(1)} \to \infty \). Since (3.3) holds for \( j = m \), we have (1).

**Q.E.D.**

**Lemma 3.5.** Let \( \chi \) be an equivariant minimal immersion of \( S^2 \) fully into \( S^{2m}(1) \) of type \( (m_{(1)}, \ldots, m_{(m)}) \). Then \( m_{(1)}, \ldots, m_{(m)} \) and the associated holomorphic map \( \xi \) of \( \chi \) is given by

\[
\xi(z) = \left( \ldots, A_j z^{m_{(j)}-m_{(j-1)}} + B_j z^{m_{(j)}+m_{(j)}}, iA_j z^{m_{(j)}-m_{(j-1)}}, -iB_j z^{m_{(j)}+m_{(j)}}, \ldots, i\kappa z^m \right),
\]

where \( A_j B_j = C_j \) are given by Lemma 3.4.

Choose an arbitrary pair of antipodal points over \( S^2 \), say \( p_1 \) and \( p_2 \), and take isothermal coordinates \( z \) and \( w \) defined by the stereographic projections at these points. Consider the holomorphic curve \( \Xi: S^2 \to P_{2m} \) defined by \( \xi(z) \) and \( \xi(w) \), where \( \xi(w) = w^{2m} \xi(1/w) \) and each of the local functions is supposed to represent \( \Xi \) in the corresponding coordinate neighborhood. Then Theorem 2.1 and Lemma 3.2 imply that \( \Xi \) is the directrix curve for an equivariant minimal immersion of certain type \( (m_{(1)}, \ldots, m_{(m)}) \). We remark that the example constructed in [2, p. 101] is an equivariant minimal immersion of type \( (1, 2, \ldots, m - 1, k) \), because the directrix curve is given by \( \eta(z) \).

Next we study the volume and regularity of the minimal surface \( \chi \) defined by \( \xi \) in Lemma 3.5.

Let \( S \) be a unitary matrix of degree \( 2m+1 \) given by

\[
S = \frac{2j-1}{2j} \begin{pmatrix}
1 - i\sqrt{2} & i\sqrt{2} \\
-1 + i\sqrt{2} & -1 - i\sqrt{2}
\end{pmatrix}.
\]

Then \( \phi = S \cdot \xi \) is given by

\[
\phi(z) = \frac{2j-1}{2j} \begin{pmatrix}
\sqrt{2} B_j z^{m_{(j)}+m_{(j)}} \\
-\sqrt{2} iA_j z^{m_{(j)}-m_{(j)}} \\
\vdots \\
i\kappa z^m
\end{pmatrix}
\]

and hence \( \xi_{m-1}(z) = S^{-1} \phi_{m-1}(z) \). Considering \( \phi_{m-1} \) a holomorphic curve in \( P_{(2m+1)} \) with holomorphic sectional curvature 2, by Proposition 2.1, we see that

\[
\text{volume}(\phi_{m-1}) = \text{volume}(\chi)
\]
and that \( \phi_{m-1} \) is regular if and only if \( \chi \) is. We need the following lemma to decide the regularity of \( \phi_{m-1} \).

**Lemma 3.6.** For real numbers \( l, l_1, \ldots, l_m \), we have

\[
\det \begin{pmatrix}
(l - l_1) \cdots (l - l - (k - 1)) \\
(l - l_1) \cdots (l - l - ((m - 1)) - 1)
\end{pmatrix}
= (l_1 - l_2) \cdots (l_1 - l_m) \cdots (l_{m-1} - l_m).
\]

**Proof.** The result follows from the fact that the left-hand side of (3.5) has common divisors \( (l_j - l_k) \). Q.E.D.

Let \( \{ e_{j_1} \wedge \cdots \wedge e_{j_m}, 1 \leq j_1 < j_2 < \cdots < j_m \leq 2m + 1 \} \) be the basis of \( \wedge^m \mathbb{C}^{2m+1} \). Then there are polynomial functions \( A_{j_1}, \ldots, A_{j_m} \) such that

\[
(3.6) \quad \phi_{m-1}(z) = \sum A_{j_1} \cdots A_{j_m} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_m}.
\]

It is clear that

\[
\min_{j_1 < \cdots < j_m} \{ \deg A_{j_1} \cdots A_{j_m}(z) \} \geq m^2 - m^2_{(m)} - \cdots - m_{(1)} - \frac{1}{2} m (m - 1),
\]

\[
\max_{j_1 < \cdots < j_m} \{ \deg A_{j_1} \cdots A_{j_m}(z) \} \leq m^2 + m_{(m)} + \cdots + m_{(1)} - \frac{1}{2} m (m - 1).
\]

By Lemma 3.6, the equalities hold. Thus we see that

\[
\text{volume}(\chi) = 4\pi (m_{(1)} + \cdots + m_{(m)}).
\]

It is easy to see that the regularity of \( \phi_{m-1} \) is equivalent to

\[
(3.7) \quad \left| \frac{\phi_{m-1} \wedge \phi_{m-1}'}{\phi_{m-1}^4} \right| \neq 0
\]

(see, for example, [2]). By Lemma 3.6,

\[
\phi_{m-1}(z) = (-\sqrt{2} i)^m A_1 \cdots A_m (m_{(1)} - m_{(2)}) \cdots m_{(m)}
\]

\[
\times z^{m^2 - m_{(1)} - \cdots - m_{(m)} - m(m-1)/2} e_2 \wedge e_4 \wedge \cdots \wedge e_{2m}
\]

\[
+ (-\sqrt{2} i)^{m-1} i^k A_2 \cdots A_m (m_{(2)} - m_{(3)}) \cdots (m_{(2)} - m_{(m)}) m_2
\]

\[
(m_{(3)} - m_{(4)}) \cdots (m_{(3)} - m_{(m)}) m_{(3)} \cdots m_{(m)}
\]

\[
\times z^{m^2 - m_{(2)} - \cdots - m_{(m)} - m(m-1)/2} e_4 \wedge e_6 \wedge \cdots \wedge e_{2k} \wedge \cdots \wedge e_{2m} \wedge e_{2m+1} + \cdots.
\]

Since we note that

\[
\left| z^{-(m^2 - m_{(1)} - \cdots - m_{(m)} - m(m-1)/2)} \phi_{m-1}(z) \right| \neq 0.
\]

(3.7) is equivalent to

\[
(3.8) \quad z^{-2(m^2 - m_{(1)} - \cdots - m_{(m)} - m(m-1)/2)} \phi_{m-1}(z) \wedge \phi_{m-1}'(z) \neq 0.
\]
By the calculation of $φ_{m-1}(z) \wedge φ_{m-1}(z)$, we see that $φ_{m-1}$ is regular if and only if $m_{(1)} = 1$. That is, $φ_{m-1}$ has two poles at 0 and $∞$ of degree $m_{(1)}$.

**Theorem 3.1.** Let $χ$ be an equivariant generalized minimal immersion of $S^2$ fully into $S^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$. Then

(i) the directrix curve for $χ$ is given by

$$ξ(z) = (\ldots, A_j z^{m_{(j)}-m_{(1)}} + B_j z^{m_{(j)}+m_{(1)}} + iA_j z^{m_{(j)}-m_{(1)}} - iB_j z^{m_{(j)}+m_{(1)}} , \ldots, iz^m),$$

where

$$A_j B_j = (-1)^{j-1} \frac{m_{(m)}^2 \cdots m_{(j+1)}^2 m_{(j-1)}^2 \cdots m_{(1)}^2}{4(m_{(m)}^2 - m_{(j+1)}^2) \cdots (m_{(j+1)}^2 - m_{(j)}^2)(m_{(j)}^2 - m_{(j-1)}^2) \cdots (m_{(1)}^2 - m_{(1)}^2)},$$

(ii) its volume is $4\pi(m_{(1)} + \cdots + m_{(m)})$,

(iii) $χ$ is an immersion if and only if $m_{(1)} = 1$.

**Remark.** (1) In the case that $m_{(1)} = 1, \ldots, m_{(m-1)} = m - 1$, $m_{(m)} = k$, Barbosa [2] shows that $\text{volume}(χ) = 2\pi(2k + m(m - 1))$ and $χ$ is an immersion.

(2) The regularity condition $m_{(1)} = 1$ is proved in [16].

Let $A$ be the element of $SO(2m + 1, C)$ given by

$$\begin{pmatrix}
\cdots & a_j & b_j \\
\cdots & -b_j & a_j
\end{pmatrix},$$

where $a_j^2 + b_j^2 = 1$. Then $Aξ(z)$ also gives a directrix curve of a certain minimal immersion of $S^2$ into $S^{2m}(1)$ [2]. Hence the coefficients $A'_j, B'_j$ of $Aξ(z)$ are given by

$$A'_j = (a_j + ib_j) A_j, \quad B'_j = (a_j - ib_j) B_j.$$

This implies that this action on equivariant minimal immersions of type $(m_{(1)}, \ldots, m_{(m)})$ is transitive and hence the class of equivariant minimal immersions of type $(m_{(1)}, \ldots, m_{(m)})$ is equal to $(R^+)^m$.

**4. Minimal immersions of $P^2$ into $P^{2m}(1)$.** The deck transformation of $S^2$ which gives $P^2$ is given by $ω$,

$$ω: z → -1/\bar{z}.$$ 

Let $\bar{χ}$ be a minimal immersion of $P^2$ fully into $P^{2m}(1)$. Then there exists a minimal immersion $χ$ of $S^2$ fully into $S^{2m}(1)$ such that

$$\begin{align*}
S^2 & \xrightarrow{χ} S^{2m}(1) \\
\downarrow π & \quad \downarrow π \\
P^2 & \xrightarrow{\bar{χ}} P^{2m}(1)
\end{align*}$$

is commutative and $χ(ω(z)) = χ(z)$ or $-χ(z)$.
Case 1: $\chi(\omega(z)) = \chi(z)$. This case implies that there exists a minimal immersion of $P^2$ into $S^{2m}(1)$.

By the same method as in (2.1), we construct vector-valued functions $G_j$ and $F_j$ from $\chi$ and $\chi \cdot \omega$, respectively. It is easy to show that

$$F_k(z) = \frac{G_k(-1/z)}{z^{2k}}.$$  

It follows that $\xi = G_m/|G_m|^2$ satisfies

(4.1) $\xi(z) = z^{2m}\xi(-1/z).$

Case 2: $\chi(\omega(z)) = -\chi(z)$. Similarly we obtain

(4.2) $\xi(z) = -z^{2m}\xi(-1/z).$

In both cases, we get

$$\psi(z) = \xi(z) \wedge \cdots \wedge \xi^{m-1}(z) \wedge \frac{1}{z^{2}} \xi'(z) \wedge \cdots \wedge \frac{1}{z^{2}} \xi^{m-1}(z) \wedge \xi(z)$$

$$= |z|^{4m} \xi'(z) \wedge \cdots \wedge \frac{1}{z^{2}} \xi^{m-1}(z) \wedge \xi(z)$$

Using Proposition 2.1, we obtain $\chi(z) = -\chi(-1/z)$ if $m$ is odd, $\chi(z) = \chi(-1/z)$ if $m$ is even, which implies

**PROPOSITION 4.1.** Let $\chi$ be a minimal immersion of $P^2$ fully into $P^{2m}(1)$. Then Case 2 occurs if $m$ is odd and Case 1 occurs if $m$ is even.

Next we study equivariant minimal immersions of $P^2$ into $P^{2m}(1)$ of type $(m_1, \ldots, m_m)$.

Case 1. By Theorem 3.1, we have $B_j = (-1)^{m-m(j)}A_j$ and hence $C_j = (-1)^{m+m(j)}|A_j|^2$. Furthermore we see that if $j$ is even, then so is $m + m(j)$ and if $j$ is odd, then so is $m + m(j)$. Let $\tilde{\chi}$ be another equivariant minimal immersion of type $(m_1, \ldots, m_m)$ with the directrix curve given by $\tilde{\xi}$ whose coefficients are $\tilde{A}_j$ and $\tilde{B}_j$. By Theorem 3.1, there exist nonzero complex numbers $\alpha_j$ for $1 \leq j \leq m$ such that

$$\tilde{A}_j = \alpha_j A_j \quad \text{and} \quad \tilde{B}_j = \frac{1}{\alpha_j} B_j.$$

Since $\tilde{B}_j = (-1)^{m-m(j)}A_j$, we have $\alpha_j \overline{\alpha_j} = 1$, which together with Theorem 3.1 implies that $\tilde{\chi}$ is congruent to $\chi$.

Case 2. Similarly, we see that if $j$ is even, then $m + m(j)$ is odd, and if $j$ is odd, then $m + m(j)$ is even, and the same result holds as for Case 1.

**PROPOSITION 4.2.** Let $\chi$ be an equivariant minimal immersion of $P^2$ fully into $P^{2m}(1)$ of type $(m_1, \ldots, m_m)$ with the directrix curve given by $\xi$ as in Theorem 3.1.
If \( m \) is even, then
\[
j: \text{even} \to m + m_{(j)}: \text{even},
\]
\[
j: \text{odd} \to m + m_{(j)}: \text{odd}.
\]

Conversely, for \((m_{(1)}, \ldots, m_{(m)})\) as above, there exists a unique equivariant full minimal immersion of \( P^2 \) into \( S^{2m}(1) \) and hence into \( P^{2m}(1) \) of type \((m_{(1)}, \ldots, m_{(m)})\).

If \( m \) is odd, then
\[
j: \text{even} \to m + m_{(j)}: \text{odd},
\]
\[
j: \text{odd} \to m + m_{(j)}: \text{even}.
\]

Conversely, for \((m_{(1)}, \ldots, m_{(m)})\) as above, there exists a unique equivariant full minimal immersion of \( P^2 \) into \( P^{2m}(1) \) of type \((m_{(1)}, \ldots, m_{(m)})\).

By Calabi [6], the volume of \( P^2 \) minimally and fully immersed in \( P^{2m}(1) \) exceeds \( m(m+1)\pi \). Next we study a minimal immersion \( \chi \) of \( P^2 \) into \( P^{2m} \) such that the volume is equal to \( m(m+1)\pi \).

The directrix curve \( \Xi \) of \( \chi \) is given by the associated holomorphic map \( \xi \):
\[
\xi(z) = \begin{cases} 
z^{2m}\xi(-1/z) & \text{if } m \text{ is even}, 
-z^{2m}\xi(-1/z) & \text{if } m \text{ is odd}. 
\end{cases}
\]

\( \xi \) is one expression of the directrix curve \( \Xi \) and it is a meromorphic function in \( C^{2m+1} \). Following Barbosa [2], we have another expression \( \eta \) of \( \Xi \) such that
\[
\eta(z) = a_0 + a_1z + \cdots + a_{2m}z^{2m} \neq 0,
\]
because the volume is equal to \( m(m+1)\pi \). Then we note that \( \eta(z) \) is proportional to \( \eta(-1/z) \) and hence there exists a nonzero constant \( \delta \) such that
\[
\delta(a_0 + a_1z + \cdots + a_{2m}z^{2m}) = (-1)^{2m}a_{2m} + \cdots + a_0 z^{2m}.
\]
Since \( \eta \) is totally isotropic, we get \((a_j, a_k) = (a_j, \overline{a_k})\) for \( j < k \) and \( j + k = 2m \).

Put
\[
b_k = \frac{a_k + \overline{a_k}}{2}, \quad c_k = \frac{a_k - \overline{a_k}}{2} \quad \text{and} \quad d_m = \begin{cases} 
a_m & \text{if } m \text{ is even}, 
-ia_m & \text{if } m \text{ is odd}.
\end{cases}
\]

Then \( \{b_1, \ldots, b_m, c_1, \ldots, c_m, d_m\} \) is a basis of \( R^{2m+1} \) and the planes spanned by \( \{b_k, c_k\} \) and \( d_m \) are orthogonal to each other. Let \( e_1, \ldots, e_{2m+1} \) be an orthonormal basis of \( R^{2m+1} \) such that
\[
b_k = a_ke_{2k-1} + \beta_k e_{2k}, \quad c_k = \gamma_k e_{2k-1} + \delta_k e_{2k} \quad \text{and} \quad e_{2m+1} = d_m/|d_m|.
\]

Therefore we get
\[
\eta(z) = \sum_{k=1}^{m} \left\{(a_k + i\gamma_k)z^{k-1} + (-1)^{k-1}(a_k - i\gamma_k)z^{2m-k-1} \right\}e_{2k-1}
\]
\[
+ \sum_{k=1}^{m} \left\{(\beta_k + i\delta_k)z^{k-1} + (-1)^{k-1}(\beta_k - i\delta_k)z^{2m-k} \right\}e_{2k} + \lambda z^m e_{2m+1},
\]
where \( \lambda = |d_m| \) if \( m \) is even and \( \lambda = i|d_m| \) if \( m \) is odd. Since \((\eta, \eta) = 0\), we get
\[
(a_k + i\delta_k)^2 + (\beta_k + i\gamma_k)^2 = 0.
\]
We may assume $\beta_k + i\delta_k = i(\alpha_k + i\gamma_k)$ so that $\eta$ gives an equivariant minimal immersion of $S^2$ into $S^{2m}(1)$ of type $(1, 2, \ldots, m)$ by Theorem 3.1. It follows from Proposition 4.1 that $\chi$ is unique. It is clear that the standard minimal immersion of $P^2(2/m(m + 1))$ into $P^{2m}(1)$ has volume $m(m + 1)\pi$.

**Corollary 4.1.** Let $\chi$ be a full minimal immersion of $P^2$ into $P^{2m}(1)$ with volume $m(m + 1)\pi$. Then $\chi$ is the standard minimal immersion.

P. Li and S. T. Yau prove the following

**Proposition A [12].** For any metric $ds^2$ on $P^2$, $\lambda_1 \cdot \text{Vol} \leq 12\pi$, where $\lambda_1$ is the first eigenvalue of the Laplacian of $ds^2$. Equality implies there exists a subspace of the first eigenspace of $ds^2$ which gives an isometric minimal immersion of $P^2$ into $S^4(1)$ if $\lambda_1 = 2$.

**Proposition B [12].** If $M$ is a compact surface in $R^n$ homeomorphic to $P^2$, then $\int |H|^2 \geq 6\pi$, where $H$ is the mean curvature vector of $M$. The equality holds only when $M$ is the image of a stereographic projection of some minimal surface in $S^4(1)$ such that the first eigenvalue of the Laplacian of $M$ is equal to 2.

Normalizing $\lambda_1 = 2$, we know that the volume $\leq 6\pi$. If the equality holds, then the metric is standard by Corollary 4.1, because the real projective space of volume $= 6\pi$ is minimally immersed in $S^4(1)$. Thus we get the following

**Corollary 4.2.** For $P^2$, if $\lambda_1 \cdot \text{volume} = 12\pi$, then the metric is standard.

**Corollary 4.3.** If $\int |H|^2 = 6\pi$ holds for $P^2$ immersed in $R^n$, then the surface is the image of a Veronese surface by a stereographic projection.

### 5. Minimal cones of minimal immersions of $S^2$ into $S^{2m}(1)$

Let $\chi$ be a full minimal immersion of $S^2$ into $S^{2m}(1)$. Then the cone $C_\chi$ is given by

$$\{ s\chi(x) \in R^{2m+1} : s \in [0, 1] \text{ and } x \in S^2 \}.$$

It is well known that $C_\chi$ is minimal in $R^{2m+1}$ and hence is called a minimal cone.

Using the fact [8] that the first eigenvalue of the Jacobi operator of minimal immersions of $S^2$ fully into $S^{2m}(1)$ is equal to $-2$, by the method of J. Simons [15], we see that $C_\chi$ is stable for variations which fix the boundary of $C_\chi$.

It is interesting to consider whether $C_\chi$ is homologically volume minimizing. With respect to this problem, an interesting result is known that the cones of the holomorphic curves in $S^6$ with the almost complex structure constructed by Cayley numbers are homologically volume minimizing. The proof is given as follows.

Let $(S^6(1), J, \langle \cdot, \cdot, \cdot \rangle)$ be the Tachibana space (nearly Kaehler manifold) constructed by using Cayley numbers and $\omega(X, Y, Z)$ the parallel 3-form defined by $\langle X, Y \cdot Z \rangle$ on $R^7$, where $\cdot$ is the product on $R^7$ defined by Cayley numbers. Then $\omega$ (any 3-plane) $\leq 1$

holds. For the cone $C_\chi$ of a holomorphic curve $S^2$ in $S^6(1)$, we get $\omega(T(C_\chi)) = 1$, where $T(C_\chi)$ is the tangent bundle (see, for example, [4, 13]). It follows from Stokes' formula that $C_\chi$ is homologically area minimizing. It is known that there exist many holomorphic curves of $S^2$ in $S^6(1)$ [4, 14].

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Therefore it is natural to pose a problem:

*Classify minimal immersions of $S^2$ into $S^{2m}(1)$ with the property such that there exist a parallel 3-form $W$ which satisfies*

\[(5.1) \quad W(T(C\chi)) = 1 \quad \text{and} \quad W(\text{any 3-plane}) \leq 1.\]

We give the answer to this problem.

**Theorem 5.1.** A full minimal immersion of $S^2$ into $S^{2m}(1)$ satisfies (5.1) if and only if $m = 3$ and $\kappa_2 = \frac{1}{2}$. If this is the case, there is an orthogonal transformation $T$ of $R^7$ such that $T \cdot \chi$ is a holomorphic curve and $W$ is $T^*\omega$.

**Proof.** We use the notations in §2. Let \( \{ x, e_1, e_2, \ldots, e_{2m-1}, e_{2m} \} \) be an orthogonal basis. Then \( \{ x, e_1, e_2 \} \) spans the tangent space of $C\chi$. Since $\omega$ attains its maximum at \( \{ x, e_1, e_2 \} \), that is, $W(x, e_1, e_2) = 1$ and $W(\text{any 3-plane}) \leq 1$, we obtain

\[W(x, e_1, e_2) = 0, \quad W(x, e_1, e_2) = 0 \quad \text{and} \quad W(x, e_1, e_2) = 0 \quad \text{for } \alpha \geq 3.\]

We rewrite these in terms of $x, E_j, E_k$, etc., as follows:

\[(5.2) \quad W(x, E_1, \overline{E_1}) = -2i,\]

\[(5.3) \quad W(E_\alpha, E_1, \overline{E_1}) = 0 \quad \text{for } \alpha \geq 2,\]

\[(5.4) \quad W(x, E_1, E_\alpha) = 0 \quad \text{for } \alpha \geq 2,\]

\[(5.5) \quad W(x, E_1, \overline{E_\alpha}) = 0 \quad \text{for } \alpha > 2.\]

Differentiating (5.3) by $E_1, E_1$ and using (2.1), we obtain

\[(5.6) \quad W(E_2, \overline{E_1}, E_\alpha) = 0 \quad \text{for } \alpha \geq 2,\]

\[(5.7) \quad W(E_1, \overline{E_2}, E_\alpha) = 0 \quad \text{for } \alpha \geq 2.\]

For (5.4), we have

\[(5.8) \quad W(x, E_2, E_\alpha) = 0 \quad \text{for } \alpha \geq 2.\]

Differentiating (5.5) by $\overline{E_1}$ and using (2.1), we have

\[(5.9) \quad W(x, E_2, \overline{E_2}) = -2i,\]

\[(5.10) \quad W(x, E_2, \overline{E_\alpha}) = 0 \quad \text{for } \alpha \geq 3.\]

For (5.6), we get

\[(5.11) \quad W(E_3, \overline{E_1}, E_\alpha) = 0 \quad \text{for } \alpha \geq 2,\]

\[(5.12) \quad W(E_2, \overline{E_2}, E_\alpha) = 0 \quad \text{for } \alpha \geq 2.\]

Differentiating (5.7) by $\overline{E_1}$, we obtain

\[(5.13) \quad W(E_1, E_3, E_2) = 2i/\kappa_2,\]

\[(5.14) \quad W(E_1, \overline{E_3}, E_\alpha) = 0 \quad \text{for } \alpha \geq 3.\]

If $m = 2$, (5.13) implies that there exists no $W$ which satisfies (5.1). Hence assume that $m \geq 3$. Differentiating (5.8) by $E_1$, we get

\[(5.15) \quad W(E_1, E_2, E_\alpha) + 2\kappa_2 W(x, E_3, E_\alpha) = 0 \quad \text{for } \alpha \geq 3.\]
For (5.10) differentiated by $E_1$, the case $\alpha = 3$ implies

\[ W(\chi, E_3, E_3) = i/\left(\kappa_2^2\right) - 2i. \]

Differentiating (5.11) by $E_1$, we obtain

\[ W(E_2, E_3, E_a) = 0 \quad \text{for} \quad \alpha \geq 4, \]

\[ - W(\chi, E_3, E_a) + \kappa_3 W(E_1, E_4, E_a) = 0 \quad \text{for} \quad \alpha \geq 3. \]

Differentiating (5.12) and (5.13) by $E_1$, we have

\[ W(E_2, E_3, E_a) = 0 \quad \text{for} \quad \alpha \geq 3, \]

\[ \frac{2i}{\left(\kappa_2^2\right)^2} \frac{E_1 \kappa_2}{\kappa_2} = \frac{2}{\kappa_2} \left( \omega_{5,6}(E_1) - \omega_{3,4}(E_1) - \omega_{1,2}(E_1) \right) + 2 \kappa_3 W(E_1, E_2, E_4). \]

Since, by (2.1), we have $\omega_{5,6} - \omega_{3,4} - \omega_{1,2} = d^c \log \kappa_2$,

\[ W(E_1, E_2, E_4) = 2i E_1 \kappa_2 / \left(\kappa_2^2\right) \kappa_3 \]

holds. Differentiating (5.4) by $E_1$, we get

\[ W(E_1, E_4, E_3) = \left(-i/\left(\kappa_2^2\right)^2 + 4i\right)/\kappa_3, \]

\[ - W(\chi, E_3, E_a) + \kappa_3 W(E_1, E_4, E_a) = 0 \quad \text{for} \quad \alpha \geq 4. \]

Differentiate (5.16) by $E_1$ and (5.17) by $E_1, E_1$, respectively. Then we get

\[ W(\chi, E_4, E_3) = -\frac{i}{\left(\kappa_2^2\right)^2} \left(\kappa_1 \kappa_2\right), \]

\[ W(E_3, E_3, E_a) = 0 \quad \text{for} \quad \alpha \geq 4, \]

\[ W(E_2, E_4, E_a) = 0 \quad \text{for} \quad \alpha \geq 4. \]

Differentiating (5.19) by $E_1$, we have

\[ W(E_2, E_4, E_a) = 0 \quad \text{for} \quad \alpha \geq 3. \]

When we differentiate (5.21) by $E_1$, using (5.26), we get

\[ E_1 \left( \frac{1}{\kappa_3} \left(-\frac{i}{\left(\kappa_2^2\right)^2} + 4i\right) \right) = i \left( \omega_{7,8}(E_1) - \omega_{5,6}(E_1) - \omega_{1,2}(E_1) \right) \]

\[ \times \left( \frac{1}{\kappa_3} \left(-\frac{i}{\left(\kappa_2^2\right)^2} + 4i\right) \right) + 2 \kappa_3 W(E_1, E_4, E_4), \]

which, together with (2.1), implies

\[ E_1 \left( \frac{1}{\kappa_3} \left(-\frac{i}{\left(\kappa_2^2\right)^2} + 4i\right) \right) = i \left( \omega_{1,2}(E_1) + i E_1 \log \kappa_3 \right) \]

\[ \times \left( \frac{1}{\kappa_3} \left(-\frac{i}{\left(\kappa_2^2\right)^2} + 4i\right) \right) + 2 \kappa_3 W(E_1, E_4, E_4). \]
If \( L = (-i/(\kappa_2)^2 + 4i)/\kappa_3 = 0 \), then
\[
\omega_{1,2}(E_1) = \frac{1}{iL} \left\{ E_1L - 2\kappa_3W(\bar{E}_4, E_4) \right\} + iLE_1 \log \kappa_3.
\]
The right-hand side is determined by the value of \( E_1, E_4 \) at each point. Let \( \tilde{e}_1, \tilde{e}_2 \) be other orthonormal vector fields tangent to \( S^2 \) such that \( e_j(x) = \tilde{e}_j(x) \) at a fixed point \( x \). Then we obtain
\[
\langle \nabla_x e_1, e_2 \rangle = \langle \nabla_x \tilde{e}_1, \tilde{e}_2 \rangle \quad \text{at } x
\]
and hence \( \omega_{1,2} = 0 \). This implies that \( S^2 \) is flat, which contradicts (2.2) or [7]. Thus we obtain \( L = 0 \). If \( m > 4 \), then \( \kappa_2 = \frac{1}{2} \). Differentiating (5.20) by \( E_1 \), we get \( \kappa_3 = 0 \), which contradicts the fact that the immersion is full. Therefore \( m = 3 \), and (5.21) implies \( \kappa_2 = \frac{1}{2} \). Furthermore, we know values of \( W \) for a basis \( \{ x, e_1, \ldots, e_6 \} \), i.e.,
\[
\]
and other values are zero. For \( x \in S^2 \), \( T_x(R^7) \) has a product defined by (5.27)

\[
\begin{array}{ccccccc}
  x & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  x & 0 & e_2 & -e_1 & e_4 & -e_3 & -e_6 & e_5 \\
  e_1 & -e_2 & 0 & x & e_6 & -e_5 & e_4 & -e_3 \\
  e_2 & e_1 & -x & 0 & -e_5 & -e_6 & e_3 & e_4 \\
  e_3 & -e_4 & -e_6 & e_5 & 0 & x & -e_2 & e_1 \\
  e_4 & e_3 & e_5 & e_6 & -x & 0 & -e_1 & -e_2 \\
  e_5 & e_6 & -e_4 & -e_3 & e_2 & e_1 & 0 & -x \\
  e_6 & -e_5 & e_3 & -e_4 & -e_1 & e_2 & x & 0
\end{array}
\]

This product is the same as the product “•.” Under an appropriate orthogonal transformation, the two products are equal. Consequently we obtain \( W = \langle \cdot, \cdot \rangle \) at \( x \). Since \( W \) is parallel, \( W = \langle \cdot, \cdot \rangle \) on \( S^2 \).

Conversely let \( \chi \) be a minimal immersion of \( S^2 \) into \( S^6(1) \) with \( \kappa_2 = \frac{1}{2} \). For \( x \in S^2 \), there is a 3-form \( W \) on \( T_x(R^7) \) which satisfies (5.27). (2.1) implies that \( W \) is a parallel form on \( S^2 \) and hence we may consider \( W = \langle \cdot, \cdot \rangle \) and that \( S^2 \) is a holomorphic curve in \( S^6(1) \). Q.E.D.

6. Equivariant minimal immersions of \( S^2 \) into \( S^6(1) \) with \( \kappa_2 = \frac{1}{2} \). Let \( \chi \) be an equivariant minimal immersion of \( S^2 \) into \( S^6(1) \) of type \( (m_1, m_2, m_3) \) and \( \xi = G_3/[G_3]^2 \) which gives the directrix curve of \( \chi \). Then by the definition of \( G_1, G_2, G_3, E_1, E_2, E_3 \), we have
\[
G_1 = \frac{\lambda}{2} E_1, \quad G_2 = \frac{\lambda^2}{2} \kappa_1 E_2, \quad G_3 = \frac{\lambda^3}{2} \kappa_1 \kappa_2 E_3
\]
and hence
\[ \frac{(G_3, G_3)}{(G_2, G_2)} = \lambda^2 \kappa_2^2. \]

Since \( \xi = G_3/|G_3|^2 \), we get
\[ |\xi|^2 |G_3|^2 = 1 \quad \text{and} \quad |\partial G_3|^2 = \frac{1}{|\xi|^4} \left( |\xi|^2 |\partial \xi|^2 - |(\partial \xi, \xi)|^2 \right). \]

It follows from Lemma 2.1 that \( \partial G_3 = -|G_3|^2 G_2/|G_2|^2 \) and hence \( |\partial G_3|^2 = |G_3|^4/|G_2|^2 \). Consequently we obtain
\[ \lambda^2 \kappa_2^2 = \frac{1}{|\xi|^4} \left( |\xi|^2 |\partial \xi|^2 - |(\partial \xi, \xi)|^2 \right) = \partial \bar{\partial} \log |\xi|^2. \]

On the other hand, Proposition 2.1 yields \( \lambda^2 = 2 \partial \bar{\partial} \log |\xi_2|^2 \). Thus
\[ (6.1) \quad \kappa_2 = \frac{1}{2} \quad \text{if and only if} \quad \partial \bar{\partial} \log |\xi|^4 = \partial \bar{\partial} \log |\xi_2|^2. \]

Note that \( |\xi|^4 = |\phi|^4 \) and \( |\xi_2|^2 = |\phi_2|^2 \) for \( \phi \) constructed in §3. By a simple calculation, we get
\[ (6.2) \quad |\phi|^2 = 2 |A_1|^2 |z|^{6-2m_1} + 2 |B_1|^2 |z|^{6+2m_1} \\
+ 2 |A_3|^2 |z|^{6-2m_2} + 2 |B_3|^2 |z|^{6+2m_2} \\
+ 2 |A_5|^2 |z|^{6-2m_3} + 2 |B_5|^2 |z|^{6+2m_3} + |\kappa|^2 |z|^6. \]

By using Lemma 3.6, the coefficients \( A_{jkl} \) of (3.6) are functions of \( |z|^2 \). Furthermore we have
\[ \min \left\{ \deg A_{jkl} \text{ with respect to } |z| \right\} = 6 - m_1 - m_2 - m_3, \]
\[ \max \left\{ \deg A_{jkl} \text{ with respect to } |z| \right\} = 6 + m_1 + m_2 + m_3. \]

Comparing \( |\phi|^4 \) with \( |\phi_2|^2 \) for degrees of \( |z|^2 \) and using (6.1) and Liouville’s theorem for harmonic functions on a complex plane, we get
\[ (6.3) \quad m_3 = m_1 + m_2 \]
and hence a positive real number \( \epsilon \) such that
\[ (6.4) \quad \epsilon |\phi|^4 = |\phi_2|^2. \]

By a simple but long calculation, we see that (6.4) is equivalent to
\[ \frac{|B_1|^2 |B_2|^2}{|B_3|^2} = \frac{|A_1|^2 |A_2|^2}{|A_3|^2}, \]
\[ \frac{1}{4} |\kappa|^2 m_3^2 = \frac{|B_1|^2 |B_3|^2}{|B_3|^2} (m_1 - m_2)^2, \]
\[ \frac{1}{4} |\kappa|^2 m_2^2 = \frac{|A_1|^2 |B_3|^2}{|B_2|^2} (m_1 + m_3)^2, \]
\[ \frac{1}{4} |\kappa|^2 m_1^2 = \frac{|A_2|^2 |B_3|^2}{|B_1|^2} (m_2 + m_3)^2, \]
which gives the following

**Theorem 6.1.** Let $\chi$ be an equivariant minimal immersion of $S^2$ fully into $S^6(1)$ of type $(m_1, m_2, m_3)$. Then $\kappa_2 = \frac{1}{3}$ is equivalent to the following:

1. $m_3 = m_1 + m_2$,
2. there exist real numbers $\alpha > 0$, $\beta < 0$, $\gamma > 0$ such that $\alpha \beta = -\gamma$ and

   $|A_1|^2 = \frac{\kappa^2 m_2 m_3}{4\alpha(m_2 - m_1)(m_1 + m_3)}$,

   $|A_2|^2 = -\frac{\kappa^2 m_1 m_3}{4\beta(m_2 - m_1)(m_2 + m_3)}$,

   $|A_3|^2 = \frac{\kappa^2 m_1 m_2}{4\gamma(m_1 + m_3)(m_2 + m_3)}$.

**Proof.** Setting $B_1 = \alpha A_1$, $B_2 = \beta A_2$, and $B_3 = \gamma A_3$ for complex numbers $\alpha$, $\beta$, and $\gamma$, we have Theorem 6.1. Q.E.D.

**Corollary 6.1.** For positive integers $m_1 < m_2$, there exists an equivariant holomorphic immersion of $S^2$ fully into $S^6(1)$ of type $(m_1, m_2, m_1 + m_2)$.

7. **Totally real submanifolds in $S^6(1)$**. Let $\chi$ be a full holomorphic immersion of $S^2$ into $S^6(1)$. Note that the first and normal bundles are well defined on $S^2$. Therefore we can construct the tubes of radius $\gamma$ ($0 < \gamma < \pi$) in the direction of the first and normal bundles. Except at isolated points of $S^2$ where $l_{s_0} = 0$, points of $S^2$ each have an open neighborhood $U$ where an orthonormal basis $e_1, \ldots, e_6$ can be constructed by the method described in §2. Using this basis, the tube of radius $\gamma$ ($0 < \gamma < \pi$) in the direction of the second normal bundle on $U$ is given by

$$F_\gamma: U \times S^1(1) \to S^6(1),$$

$$(x, \theta) \to (\cos \gamma)\chi(x) + (\sin \gamma)((\cos \theta)e_5 + (\sin \theta)e_6).$$

By (2.1), we obtain

$$F_{\gamma*}(e_1) = (\cos \gamma)e_1 - \kappa_2(\sin \gamma)(\cos \theta)e_3 - \kappa_2(\sin \gamma)(\sin \theta)e_4$$

$$- (\sin \gamma)(\sin \theta)\omega_{56}(e_1)e_5 + (\sin \gamma)(\cos \theta)\omega_{56}(e_1)e_6,$$

and $F_{\gamma*}(e_2) = \cdots$, $F_{\gamma*}(\partial/\partial \theta) = \cdots$. It follows from (5.27) that

$$JF_{\gamma*}(e_1) = F \cdot F_{\gamma*}(e_1)$$

$$= -(\sin \gamma)^2\omega_{56}(e_1)\chi + [(\cos \gamma)^2 - \kappa_2(\sin \gamma)^2]e_2$$

$$+ (\kappa_2 + 1)(\sin \gamma)(\cos \gamma)(\sin \theta)e_3$$

$$- (\kappa_2 + 1)(\sin \gamma)(\cos \gamma)(\cos \theta)e_4$$

$$+ (\sin \gamma)(\cos \gamma)(\cos \theta)\omega_{56}(e_1)e_5$$

$$+ (\sin \gamma)(\sin \gamma)(\cos \theta)\omega_{56}(e_1)e_6,$$

etc.
The condition that \( F \) gives a totally real submanifold is equivalent to \( (\tan \gamma)^2 = \frac{4}{3} \), because \( \kappa_2 = \frac{1}{3} \).

Next, let \( \chi \) be the holomorphic immersion of \( S^2(\frac{1}{6}) \) into \( S^6(1) \). Then \( \kappa_1 = \sqrt{5/12} \).

By the same calculation, we see that the tube of radius \( \gamma \) in the direction of the first normal space of \( \chi \) gives a totally real submanifold if and only if \( \gamma \) satisfies

\[
(7.1) \quad 27(\cos \gamma)^3 + 5(\cos \gamma)^2 - 15(\cos \gamma) - 5 = 0. 
\]

Consequently we obtain

**Theorem 7.1.** Let \( \chi \) be a full holomorphic immersion of \( S^2 \) into \( S^6(1) \). Then the tube of radius \( \gamma \) such that \( (\tan \gamma)^2 = \frac{4}{3} \) in the direction of the second normal space of \( \chi \) gives a totally real submanifold in \( S^6(1) \).

**Theorem 7.2.** Let \( \chi \) be the holomorphic immersion of \( S^2(\frac{1}{6}) \) into \( S^6(1) \). Then the tube of radius \( \gamma \) which satisfies (7.1) in the direction of the first normal space of \( \chi \) gives a totally real submanifold \( S^6(1) \).

We can calculate the Chern number \( c_1 \) of the second normal bundle of a full holomorphic immersion of \( S^2 \) into \( S^6(1) \). By (2.1),

\[
d\omega_{5,6} = 3d\omega_{1,2} + d\theta_2 \quad \text{and} \quad d\theta_2 = \Delta(\log \kappa_1) \omega_1 \wedge \omega_2. 
\]

Therefore the curvature of the second normal bundle of \( \chi \) is given by \( \frac{1}{2} \) which implies

\[
c_1 = \frac{1}{4\pi} \text{volume}(S^2). 
\]

Using Corollary 6.1 and Theorem 3.1, we obtain a full holomorphic immersion \( S^2 \) into \( S^6(1) \) with \( c_1 = 2k \) for a positive integer \( k \geq 3 \). Similarly, we see that the Chern number of the first normal bundle of \( S^2(\frac{1}{6}) \rightarrow S^6(1) \) is 4.

**Corollary 7.1.** There exists a minimal (totally real) immersion of the circle bundle of \( S^2 \) with positive even Chern number \( \geq 4 \) into \( S^6(1) \).

Bryant [4] gives a holomorphic map of any Riemann surface into \( S^6(1) \). Since they have the same properties as a full holomorphic map of \( S^2 \) into \( S^6(1) \), we obtain many 3-dimensional totally real submanifolds in \( S^6(1) \) with singularities.

In [8], we construct the totally real (minimal) immersion of \( S^6(\frac{1}{16}) \) into \( S^6(1) \). Calculating the curvature tensor of the tube in the direction of the second normal bundle of the holomorphic immersion of \( S^2(\frac{1}{6}) \) into \( S^6(1) \), we obtain the minimal immersion \( S^3(\frac{1}{16}) \) into \( S^6(1) \).

**Remark.** Let \( T_\gamma \) be the tube of radius \( \gamma \) \((0 < \gamma < \pi)\) in the direction of the second normal bundle of a full holomorphic immersion of \( S^2 \) into \( S^6(1) \). We denote by \( \mathcal{F}_\gamma \) the mean curvature vector of \( T_\gamma \). Then we easily see

\[
(1) \quad |\mathcal{F}_\gamma| = \frac{(\sin \gamma)(\cos \gamma)((\cotan \gamma)^2 - 5/4)}{(\cos \gamma)^2 + (\sin \gamma)^2 / 4}.
\]

(2) \( \mathcal{F}_\gamma \) is not parallel for the normal connection.
(3) $\mathcal{T}$ is the scalar multiple of the variation vector field in the direction of $\gamma$.

(4) $T_\gamma$ (not minimal) are Chen submanifolds [17] in $S^6(1)$.

(5) Let $V$ be the 4-dimensional submanifold defined by attaching the totally geodesic submanifold $S^2(1)$ for each point of the holomorphic immersion of $S^2$ into $S^6(1)$, where the tangent space of $S^2(1)$ is spanned by the second normal space of the holomorphic immersion. Then $V$ is minimal in $S^6(1)$ and contains $T_\gamma$.

(6) We obtain the analogous result for some holomorphic curve in the 3-dimensional complex projective space (in preparation).

REFERENCES

5. ____, Conformal and minimal immersions of compact surfaces into the 4-sphere, J. Differential Geom. 17 (1982), 455–473.

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