BOUNDS FOR PRIME SOLUTIONS
OF SOME DIAGONAL EQUATIONS. II

MING-CHIT LIU

ABSTRACT. Let $b_j$ and $m$ be certain integers. In this paper we obtain a bound for
prime solutions $p_j$ of the diagonal equations of order $k$, $b_1 p_1^k + \cdots + b_s p_s^k = m$.
The bound obtained is $C(k^\varepsilon + m^{1/k})$ where $B = \max_j \{ e, |b_j| \}$ and $C$ are
positive constants depending at most on $k$.

1. Introduction. Throughout $p$ denotes a prime number and $k \geq 2$ is an integer.
Let $\theta \geq 0$ be the largest integer such that $p^\theta$ divides $k$. We write $p^\theta \| k$. Let

\begin{equation}
(1.1) \quad s_0 = \begin{cases} 3k - 1 & \text{if there is a } p \text{ satisfying } p \| k \text{ and } k = (\frac{p - 1}{2}) p^\theta, \\ 2k & \text{otherwise}. \end{cases}
\end{equation}

\begin{equation}
(1.2) \quad s_1 = \begin{cases} 2k + 1 & \text{if } 2 \leq k \leq 11, \\ 2k^2(2 \log k + \log \log k + 2.5) - 1 & \text{if } k \geq 12. \end{cases}
\end{equation}

\begin{equation}
(1.3) \quad \nu = \begin{cases} \theta + 2 & \text{if } p = 2 \text{ and } 2 \| k, \\ \theta + 1 & \text{otherwise}. \end{cases}
\end{equation}

\begin{equation}
(1.4) \quad K = \prod_{(p-1) \| k} p^n.
\end{equation}

In this paper we shall prove

THEOREM 1. Let $b_1, \ldots, b_s$ be any nonzero integers which do not have the same sign.
Let $m$ be any integer satisfying

\begin{equation}
(1.5) \quad \sum_{j=1}^s b_j \equiv m \pmod{K}.
\end{equation}

If $s$ is the least integer with $s \geq s_1$ and if no prime can divide more than $s - s_0$ $b_j$
then there are constants $C_j(k)$ depending on $k$ only such that the equation

\begin{equation}
(1.6) \quad \sum_{j=1}^s b_j p_j^k = m
\end{equation}

Received by the editors November 19, 1984.
1980 Mathematics Subject Classification (1985 Revision). Primary 10J15; Secondary 10B15.
Key words and phrases. Bounds for prime solutions, diagonal equations, trigonometric sums, Dirichlet’s
characters, the Hardy-Littlewood method.

©1986 American Mathematical Society
0002-9947/86 $1.00 + .25 per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
always has a solution in odd primes \( p_j \) satisfying

\[
\max_{1 \leq j \leq s} p_j < C_1 |m|^{1/k} + C_2 (\log B)^2
\]

where \( B = \max\{|b_1|, \ldots, |b_s|, e\} \).

Investigations on bounds for integral solutions of diagonal equations similar to type (1.6) were made by Cassels [3], Birch and Davenport [2], Pitman and Ridout [11], Pitman [12]. On the other hand, results on bounds for prime solutions of (1.6) were obtained by Baker [1] and the author [9]. In all previous works on prime solutions, bounds obtained are of the form \( C(k, \delta)_{\max |b_j|}^\delta \) for any \( \delta > 0 \). So (1.7) in Theorem 1 gives an essentially better bound than the previous one [9, (1.6)] and our Theorem 1 improves Theorem 1 in [9]. The new bound, \( C(\log B)^2 \) is obtained by using [5, Theorem 6] a zero density estimate for \( L \)-functions which, as a consequence, replaces the Siegel-Walfisz theorem on prime distribution applied in both [1, Lemma 1] and [9, Lemma 6]. By this zero density estimate we can obtain a better error estimate as shown in our Lemma 2 which enables us to treat terms belonging to category (A) in §4 below. This change causes not only an improvement on the bound but also a greatly different emphasis in methods.

By (1.1) and (1.2) we see that the divisibility condition on \( b_j \) in Theorem 1 is better than (for \( k \geq 4 \)) the condition, \( (b_j, b_l) = 1 \) for \( j \neq l \), which is usually assumed in additive problems involving primes. By (1.4) and (1.5) our condition on \( m \) coincides with that in the Waring-Goldbach problem [7, p. 100 and p. 108] where the case \( b_j = 1 \) was considered.

2. Notation. Throughout we assume that \( N \) satisfies

\[
\log N \geq N_0 (\log B)^2
\]

where \( N_0 > 0 \) is a large constant depending on \( k \) only.

\( \chi \pmod{q} \) denotes a Dirichlet character and \( \chi_0 \pmod{q} \) denotes the principal character. \( \chi^* \pmod{r} \) is a primitive character, \( \tilde{\chi} \pmod{r} \) is the exceptional primitive character and \( \tilde{\beta} \) is the exceptional zero (see Lemma 1 below). Throughout the constants \( c_j \) and all implicit constants in the Vinogradov symbols \( \ll \), the O-symbols are positive and depend at most on \( k \). The constants \( A_j \) are positive absolute. \( \phi(q) \) is the Euler function and for real \( \alpha \) write \( e(\alpha) = \exp(i2\pi \alpha) \). Let

\[
P = P(N) = \exp(\sqrt{A_1 \log N} / 10), \quad Q = N^k P^{-1},
\]

where \( A_1 \) is given in Lemma 1. The constant \( \sqrt{A_1} / 10 \) in (2.2) will be needed in the proof of Lemma 2. Let

\[
W(a, \chi) = \sum_{n=1}^{q} \chi(n)e\left(\frac{an^k}{q}\right),
\]

\[
S(b\alpha) = \sum_{G < p \leq N} \log p \, e(bap^k), \quad S(b\alpha, \chi) = \sum_{G < p \leq N} \chi(p)\log p \, e(bap^k),
\]

where

\[
G = N(6^k |b|)^{-1/k}.
\]
For $1 \leq a \leq q \leq P$, $(a, q) = 1$ let $\mathcal{M}(q, a)$ be the major arc which is the set of real $\alpha$ satisfying $|\alpha - a/q| \leq \delta_q$ with

$$\delta_q = \left(qQ\right)^{-1}.$$  

These major arcs are disjoint. Let $\mathcal{M}$ be the union of all major arcs and $m$ denote minor arcs which is the complement of $\mathcal{M}$ with respect to the set of $\alpha$ satisfying $Q^{-1} \leq \alpha \leq 1 + Q^{-1}$.

For $\alpha \in \mathcal{M}(q, a)$ write $\alpha = a/q + \eta$. If $p > P$ then $(q, p) = 1$, since $q \leq P$. It follows from the orthogonal relation of characters that

$$S(b \alpha) = \phi(q)^{-1} \sum_{\chi} W(ab, \bar{\chi}) S(b \eta, \chi).$$

Note that if $p > P$ then

$$S(b \eta, \chi) = S(b \eta, \chi^*)$$

where $\chi^* (\mod r)$ induces $\chi (\mod q)$. Put

$$I(b \eta) = \sum_{|b|^G < n \leq |b|N^k} e(\pm \eta n)n^{-1+1/k}(k|b|^{1/k})^{-1},$$

$$\tilde{I}(b \eta) = -\sum_{|b|^\tilde{G} < n \leq |b|N^k} e(\pm \eta n)n^{-1+\tilde{\beta}/k}(k|b|^{\tilde{\beta}/k})^{-1},$$

where $\pm$ denotes the sign of $b$. $\tilde{I}(b \eta)$ is defined only if there is $\tilde{\beta}$. Let

$$\Delta(b \eta, \chi) = \begin{cases} S(b \eta, \chi_0) - I(b \eta) & \text{if } \chi = \chi_0, \\ S(b \eta, \tilde{\chi} \chi_0) - \tilde{I}(b \eta) & \text{if } \chi = \tilde{\chi} \chi_0, \\ S(b \eta, \chi) & \text{if } \chi \neq \chi_0 \text{ and } \chi \neq \tilde{\chi} \chi_0. \end{cases}$$

By (2.5) we have

$$\Delta(b \eta, \chi) = \Delta(b \eta, \chi^*).$$

3. Lemmas.

**Lemma 1.** Let $z = \sigma + it$. There is $A_1$ such that the Dirichlet $L$-function $L(z, \chi^*) \neq 0$ whenever $\sigma \geq 1 - A_1/\log(P(|t| + 2))$ for all primitive characters $\chi^* (\mod r)$ and $r \leq P$ with the possible exception of at most one primitive character, $\tilde{\chi}$ (mod $\tilde{r}$). If there is such an exceptional character then it is quadratic and the unique exceptional zero $\tilde{\beta}$ of $L(z, \tilde{\chi})$ is real and simple and satisfies

$$A_2/\tilde{r}^{1/2} (\log \tilde{r})^2 \leq 1 - \tilde{\beta} \leq A_1/\log P.$$

**Proof.** See [4, §14].

**Lemma 2.** For any real $\lambda \geq 1$ we have

$$\sum_{r \leq P} \sum_{\chi^*} \left( \int_{-\delta_0}^{\delta_0} \left| \Delta(b \eta, \chi^*) \right|^\lambda d\eta \right)^{1/\lambda} \ll |b|N^{1-k/\lambda} P^{-2},$$

where the summation $\sum_{\chi^*}$ is taken over all $\chi^* (\mod r)$.
Proof. The proof is essentially the same as Theorem 7 [5]. In the proof we apply Theorem 6 [5] and put the $T$ there to be $P^7$.

Lemma 3. Let $q = q_1 \cdots q_t$ with $(q_j, q_i) = 1$ for $j \neq i$. Let $\chi$ (mod $q$) be factorized into $\prod_{j=1}^{t} \chi_j$ (mod $q_j$). If $(a, q) = 1$ then there exist uniquely $a_j$ (mod $q_j$) with

\begin{equation}
(a_j, q_j) = 1 \quad (j = 1, \ldots, t), \quad a = \sum_{j=1}^{t} \frac{a_j q_j}{q_j}
\end{equation}

and

$$W(ab, \chi) = \prod_{j=1}^{t} W(a_j b, \chi_j).$$

Proof. This is essentially Theorem 4.1 in [8, p. 159].

Lemma 4. Let $h_1 = h/(h, q)$ and $q_1 = q/(h, q)$. Let $\chi^*$ (mod $r$) induce $\chi$ (mod $q$). Then

$$W(h, \chi) = \begin{cases} 0 & \text{if } r \nmid q_1, \\
\phi(q)\phi(q_1)^{-1}W(h_1, \chi_1) & \text{if } r \mid q_1 \text{ where } \chi_1 \text{ (mod } q_1) \\
\text{is induced by } \chi^* \text{ (mod } r) & \end{cases}$$

Remarks. Lemma 4 is parallel to the known result on the Ramanujan sum and its generalization [6, p. 450]. In fact, we can also prove that $W(h, \chi) = 0$ if $r \mid q_1$ and $(r, q_1/r) \nmid k$.

Proof. Write $q_2 = q/q_1$ and $n = uq_1 + v$ with $u = 0, 1, \ldots, q_2 - 1; v = 1, 2, \ldots, q_1$. Then

\begin{equation}
\sum_{n=1}^{q_1} \chi(n)e\left(\frac{hn^k}{q_1}\right) = \sum_{v=1}^{q_1} e\left(\frac{h_1 v^k}{q_1}\right) T(v)
\end{equation}

where $T(v) = \sum_{u=1}^{q_2-1} \chi(uq_1 + v)$.

Let $r \nmid q_1$. By the same argument as in showing $S(v) = 0$ in [4, p. 66] we can prove that $T(v) = 0$ and hence $W(h, \chi) = 0$.

Next consider $r \mid q_1$. Let $d = \prod_{p \mid q_1} p + q_1 p$ and $\mathcal{J} = \{uq_1 + v: 1 \leq u \leq q_2\}$. If $(v, q_1) = 1$ then

\begin{equation}
\sum_{j \in \mathcal{J}} 1 = \sum_{j \in \mathcal{J}} \sum_{n \mid (j, d)} \mu(n) = \sum_{n \mid d} \frac{\mu(n) q_2}{n} = q_2 \prod_{p \mid d} (1 - p^{-1}).
\end{equation}

It follows from $\chi^*(uq_1 + v) = \chi^*(v)$ and (3.4) that if $(v, q_1) = 1$ then

$$T(v) = \chi^*(v) \sum_{u=1}^{q_2} 1 = \chi^*(v) \phi(q)\phi(q_1)^{-1}.$$
Lemma 5. (a) If \((a, p) = 1\) and \(p'\) is the modulus of \(\chi\) then \(|W(a, \chi)| \leq 2kp^{1/2}\).
(b) If \((a, q) = 1\) and \(q\) is the modulus of \(\chi\) then for any \(\varepsilon > 0\) there is a positive constant \(C(\varepsilon, k)\) depending at most on \(\varepsilon, k\) such that
\[|W(ab, \chi)| \leq C(k, \varepsilon)(q, b)^{1/2}q^{1/2 + \varepsilon}.\]

Proof. Part (a) follows from a similar argument as part 2 of the proof of Lemma 8.5 [7].

(b) Let \(\chi^* \pmod r\) induce \(\chi \pmod q\), \(q' = q/(b, q)\), \(b' = b/(b, q)\). Suppose that \(r \mid q'\). Put \(q'' = \prod_j p_j^{h_j}\) and factorize \(\chi' \pmod {q''}\) into \(\prod_j \chi_j \pmod {p_j^{h_j}}\), where \(\chi' \pmod {q''}\) is induced by \(\chi^* \pmod r\). Then by Lemmas 4, 3, and Lemma 5(a)
\[
|W(ab, \chi)| \leq \left|\phi(q)\phi(q')^{-1}|W(ab', \chi')| \leq (b, q) \prod_{j=1}^t |W(a_j b', \chi_j)|
\leq (b, q)^{1/2}((b, q)q')^{1/2}2k.
\]
This proves Lemma 5(b).

4. Major arcs. I. Write
\[
\left\{
\begin{align*}
\mathcal{W}_j &= \phi(q)^{-1} \sum_{\chi} W(ab_j, \bar{\chi}) \Delta(b_j \eta, \chi), \\
\mathcal{F}_j &= \phi(q)^{-1} I(b_j \eta) W(ab_j, \chi_0), \\
\mathcal{F}_j' &= \phi(q)^{-1} \bar{I}(b_j \eta) W(ab_j, \bar{\chi} \chi_0),
\end{align*}
\right.
\]
where \(\mathcal{F}_j\) is defined only when the exceptional character exists. By (2.4), (2.7) we have
\[
R_1(m) = \sum_{q \leq p} \sum_a \int_{-\delta_q}^{\delta_q} e\left(-m\left(\frac{a}{q} + \eta\right)\right) \prod_{j=1}^s S(b_j a) \, d\eta
\]
\[
= \sum_{q \leq p} \sum_a \left(\frac{-ma}{q}\right) \int_{-\delta_q}^{\delta_q} e\left(-m\eta\right) \prod_{j=1}^s \left(\mathcal{W}_j + \mathcal{F}_j + \mathcal{F}_j'\right) \, d\eta
\]
where the sum \(\sum_a\) is taken over all \(a\) with \(1 \leq a \leq q\) and \((a, q) = 1\).

There are two categories of terms in the last product of (4.2), namely, (A) terms having at least a factor \(\mathcal{W}_j\); (B) terms having no factor \(\mathcal{W}_j\). We shall treat category (A) in this section and category (B) in \(\S\)6.

Let \(\mathcal{F}_j'\) denote either \(\mathcal{F}_j\) or \(\mathcal{F}_j'\). In category (A) for each fixed \(h = 1, 2, \ldots, s\) we choose \(\prod_{j=1}^h \mathcal{W}_j \prod_{j=1}^{s-h+1} \mathcal{F}_j\) as the representative of those terms having exactly \(h\) factors \(\mathcal{W}_j\). Put
\[
T_h(m) = \sum_{q \leq p} \sum_a \left(\frac{-ma}{q}\right) \prod_{j=1}^h \mathcal{W}_j \prod_{j=1}^{s-h+1} \mathcal{F}_j e(-m\eta) \, d\eta
\]
\[
(h = 1, \ldots, s).
\]

Let
\[
\chi' \pmod q = \chi_0 \pmod q \quad \text{or} \quad \bar{\chi} \chi_0 \pmod q.
\]
\[
I'(b_j \eta) = I(b_j \eta) \quad \text{or} \quad \bar{I}(b_j \eta).
\]
Then by Schwarz’s inequality and (4.1), (4.3) we have

\begin{equation}
|T_h(m)| \leq \sum_{p \leq P} \phi(q)^{-s} \sum_{x_0 \leftrightarrow x} \left| \sum_{a} \phi(q) \frac{-ma}{q} \prod_{j=1}^{h} W(ab_j, x_j) \prod_{j=h+1}^{s} W(ab_j, \chi_j^*) \right|
\end{equation}

\times \prod_{j=1}^{h} \left( \int_{\delta_q}^{\infty} |1 - \Delta(b_j \eta, \chi_j^*)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^{s} \left( \int_{\delta_q}^{\infty} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j},

where \( \sum_{x_j \leftrightarrow x_0} \) denotes \( h \) summations each of which is taken over all \( \chi \) (mod \( q \)) and \( n_j > 1 \) are integers satisfying \( \sum_{j=1}^{s} 1/n_j = 1 \). Note that each \( \chi_j \) (mod \( q \)) is induced by a unique \( \chi_j^* \) (mod \( r_j \)) with \( r_j \mid q \) and that each \( \chi_j^* \) (mod \( r_j \)) and each \( q \) with \( r_j \mid q \) induce a unique \( \chi \) (mod \( q \)). Then by (2.8), (4.6) we have

\begin{equation}
|T_h(m)| \leq \sum_{r_j \leq P} \sum_{\chi_j^* \leftrightarrow \chi} \left( \sum_{q=1}^{\infty} \phi(q)^{-s} \sum_{a} \phi(q) \frac{-ma}{q} \prod_{j=1}^{h} W(ab_j, \chi_0 \chi_j^*) \right)
\end{equation}

\times \prod_{j=h+1}^{s} W(ab_j, \chi_j^*) \left( \prod_{j=1}^{h} \left( \int_{\delta_q}^{\infty} |1 - \Delta(b_j \eta, \chi_j^*)|^{n_j} d\eta \right)^{1/n_j} \prod_{j=h+1}^{s} \left( \int_{\delta_q}^{\infty} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j} \right).

By Lemma 5 with \( \epsilon = (10s)^{-1} \), the infinite sum inside the curly brackets of (4.7) is

\begin{equation}
\ll \sum_{q=1}^{\infty} \phi(q)^{-s+1} \prod_{j=1}^{s} |b_j|^{1/2} q^{1/2} + 1/10s \ll B^{s/2}
\end{equation}

since by (1.2) we have \( s \geq 5 \) for any \( k \geq 2 \). Also by (2.6) we have \( I'(b_j \eta) \ll N \) and then by (2.3), (2.2)

\begin{equation}
\left( \int_{\delta_q}^{\infty} |I'(b_j \eta)|^{n_j} d\eta \right)^{1/n_j} \ll N^{1-k/n_j, P^{1/n_j}}.
\end{equation}

It follows from (4.7), (4.8), (4.9) and Lemma 2 that

\begin{equation}
T_h(m) \ll B^{s/2} \left( \prod_{j=1}^{h} |b_j|^N 1-k/n_j, P^{1/k} \right) \left( \prod_{j=h+1}^{s} N^{1-k/n_j, P^{1/n_j}} \right)
\end{equation}

\ll B^{3s/2} N^{s-k} P^{-1} = E_1, \quad \text{say},

since \( \sum_{j=1}^{s} 1/n_j = 1 \).

5. Singular series.

**Lemma 6.** For a given \( p \) let \( p^k \mid k \) and \( p^b \mid b \). Suppose that \( p^i \) and \( p^j \) are the moduli of \( \chi_0 \) and \( \chi_1 \) respectively and

\begin{equation}
u = 2\phi + \theta + \begin{cases} 3 & \text{if } p = 2, \\ 1 & \text{if } p > 3. \end{cases}
\end{equation}
If \( 1 \leq j \leq u - 2\phi - \theta, \ t \geq u + 1 \) and \((a, p) = 1\) then
\[
W(ab, \chi_0) = W(ab, \chi_1X_0) = 0.
\]

**Proof.** The proof is essentially the same as Lemma 1 [9].

**Lemma 7.** Let \( q = q_1q_2, \ (q_1, q_2) = 1 \) and factorize \( \chi_j \ (mod \ q) \) into \( \prod_{i=1}^{2} \chi_j (mod \ q_i) \) \((j = 1, 2, \ldots, s)\). If
\[
B(m, q) = \phi(q)^{-s} \sum_{a} \left( \frac{-ma}{q} \right) \prod_{j=1}^{s} W(ab_j, \chi_j),
\]
then
\[
B(m, q) = B(m, q_1)B(m, q_2).
\]

**Proof.** Apply Lemma 3.

By Lemma 1 the exceptional character \( \tilde{\chi} \ (mod \ r) \) is real and primitive. Then it is known [8, p. 159] that
\[
\tilde{r} = 2^l p_2 \cdots p_t
\]
where \( p_j \) are distinct odd primes and \( l = 0 \) or 2 or 3. If \( \tilde{r} \mid q \) write
\[
q = q_1q_2, \ (q_1, q_2) = 1 \quad \text{and} \quad q_1 = 2^{l_1} p_2^{l_2} \cdots p_t^{l_t}
\]
where \( l_j \geq 1 \ (j = 2, \ldots, t) \); \( l_1 \geq l \) if \( l \neq 0 \) and \( l_1 = 0 \) if \( l = 0 \). Put
\[
B_h(m, q) = \phi(q)^{-s} \sum_{a} \left( \frac{-ma}{q} \right) \prod_{j=1}^{s} W(ab_j, \chi_j') \quad (h = 0, 1, \ldots, s),
\]
where \( \chi_j' \) is defined in (4.4) and there are exactly \( h \chi_j' = \tilde{\chi}X_0 \ (mod \ q) \) in the last product of (5.3). Define singular series \((h = 0)\) and pseudosingular series \((h = 1, 2, \ldots, s)\) by
\[
\mathcal{S}_0(m) = \sum_{q=1}^{\infty} B_0(m, q) \quad \text{and} \quad \mathcal{S}_h(m) = \sum_{q=1}^{\infty} B_h(m, q).
\]

By Lemma 5(b) all series in (5.4) are absolutely convergent.

**Lemma 8.** Let \( \tilde{r} \) and \( q_1 \) be defined as in (5.1), (5.2). If \( B_h(m, q_1) \neq 0 \) then \( q_1 = d_k \tilde{r} \) or \( 2d_k \tilde{r} \) where \( d_k \) is a divisor of \( k \).

**Proof.** For each \( p_j \ (j = 1, \ldots, t) \) in (5.1) with \( p_1 = 2 \) let \( p_j^{\theta_j} \mid k \). Suppose that 
\[
l_1 \geq 4 + \theta_1 \text{ or } \ l_j \geq 2 + \theta_j \text{ for some } j \geq 2.
\]
For simplicity we only give the details for the case \( j = 2 \). Let
\[
l_2 \geq \theta_2 + 2.
\]
Since no prime can divide all \( b_j \), we may assume that \( p_2 \mid b_1 \). Factorizing the exceptional character \( \tilde{\chi} \) and the character \( \chi_1' \) in (5.3) we have
\[
\tilde{\chi} \ (mod \ \tilde{r}) = \tilde{\chi}_1 \ (mod \ 2^{l_2}) \prod_{j=2}^{t} \tilde{\chi}_j \ (mod \ p_j),
\]
\[
\chi_1' \ (mod \ q_1) = \prod_{j=1}^{t} \chi_1' \ (mod \ p_j^{l_j}).
\]
where each $\chi'_{ij}$ is either $\chi_0 \mod p_j$ or $\tilde{\chi}_0 \mod p_j$. By (3.2) for each $a$ with $(a, q_1) = 1$ there are $a_j$ ($j = 1, \ldots, t$) with $(a_j, p_j) = 1$ such that $W(ab, \chi_0) = \prod_{j=1}^t W(a_jb, \chi'_j)$. Then by (5.5) and Lemma 6 with $\phi = 0$, for each $a$ in $\Sigma_\alpha$ of (5.3) we have $W(a_2b_1, \chi_{12}) = 0$. So by (5.3) if $B_h(m, q_1) \neq 0$ then $l \leq l_j \leq l + 1 + \theta$ and $1 \leq l_j \leq 1 + \theta_j$ ($j = 2, 3, \ldots, s$). This proves Lemma 8.

LEMMA 9. (a) $\mathcal{S}_0(m) \gg B^{s(1-s)}$ and (b) $\mathcal{S}_h(m) \ll \mathcal{S}_0(m)(\log N)^{-1/2}$ ($h = 1, 2, \ldots, s$).

PROOF. Part (a) is Lemma 5 in [9].

We come now to prove part (b). For each $q$ with $\tilde{r} | q$ define $q_1$ and $q_2$ as in (5.2). Since, by the hypothesis on $b_j$, no prime can divide more than $s - s_0$ $b_j$, we have

$$\prod_{j=1}^s (q_1, b_j) \leq q_1^{-s_0}.$$ 

Then by (5.3) and Lemma 5 with $\epsilon = (10s)^{-1}$ we have

$$B_h(m, q_1) \ll \phi(q_1)^{-s+1} q_1^{s/2 + 1/10} q_1^{(s-s_0)/2} \ll q_1^{6/5 - s_0/2}.$$ 

Then by Lemma 8 and $s_0 \geq 2k \geq 4$ (see (1.1)) we have

$$B_h(m, q_1) \ll \tilde{r}^{-4/5}.$$ 

So by Lemma 8 again we have

$$\sum_{q_1=1}^\infty B_h(m, q_1) \ll \tilde{r}^{-4/5}. \tag{5.6}$$ 

On the other hand, by Lemma 5(a), the divisibility hypothesis on $b_j$ and $\lvert W(ab, \chi'_j)\rvert \leq \phi(p')$, we see that the product in $B_0(m, p')$ in (5.3) satisfies

$$\left| \prod_{j=1}^s W(ab_j, \chi'_j) \right| \leq (2k)^{s_0} p^{ts_0/2} \phi(p')^{s-s_0}.$$ 

So by (5.3) and $s_0 \geq 4$ we have

$$\left| B_0(m, p') \right| \leq \phi(p')^{-s_0+1}(2k)^{s_0} p^{ts_0/2} \leq (4k)^s p^{(1-s_0/2)} \ll c_1 p^{-t}. \tag{5.7}$$ 

For each $p$ there exists some $b_j = b_1$, say, which is not divisible by $p$. By Lemma 6 for each $a$ with $(a, p) = 1$ we have $W(ab, \chi_0) = 0$ if $t \geq \nu + 2$ where $\nu$ is defined in (1.3) and $p'$ is the modulus of $\chi_0$. So by (5.3) we have $B_0(m, p') = 0$ if $t \geq \nu + 2$. Then by Lemma 7 and ($\tilde{r}, q_2$) = 1

$$\sum_{q_2=1}^\infty B_0(m, q_2) = \prod_{p|\tilde{r}} \left( 1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right)$$

$$= \sum_{q=1}^\infty B_0(m, q) / \prod_{p|\tilde{r}} \left( 1 + \sum_{t=1}^{\nu_1} B_0(m, p^t) \right), \tag{5.8}$$

where $\nu_1 = \nu + 1$. Separate the last product $\prod_{p|\tilde{r}}$ into $\prod_{p|\tilde{r}, p \leq c_2}$ and $\prod_{p|\tilde{r}, p > c_2}$ where $c_2 = 4c_1$. Same as that in the proof of Lemma 5 in [9, see (4.16) and the product $\prod_{l}$ on p. 197] which depends essentially on (1.1)–(1.5) and the divisibility
condition on \( b_j \) in Theorem 1, we have that the first product \( \prod_{p \mid r, p \leq c_2} \) satisfies
\[
\prod_{p \mid r, p \leq c_2} \left( 1 + \sum_{i=1}^{r_i} B_0(m, p^i) \right) \geq \prod_{p \mid r, p \leq c_2} \frac{\phi(p^{r_i})^{-s} p^{r_i}}{p^{c_2}} \\
\geq \prod_{p \leq c_2} p^{r_i(1-s)} = c_3 > 0.
\]
For the second product \( \prod_{p > c_2} \) by (5.7) we have
\[
(5.9) \quad \prod_{p > c_2} \left( 1 + \sum_{i=1}^{r_i} B_0(m, p^i) \right) > \prod_{c_2 < p \leq r} \left( 1 - c_1 \sum_{i=1}^{\infty} p^{-i} \right) \\
> \prod_{c_2 < p < r} \left( 1 - c_2/2p \right) \gg (\log r)^{-c_2}.
\]
The last inequality is a simple modification of Theorem 9.3 in [8, p. 92]. Now by (5.8), (5.9) we have
\[
(5.10) \quad \sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathcal{S}_0(m)(\log r)^{-c_2}.
\]
Finally, by (5.2) we see that \( \tilde{x} \chi_0 \pmod q \) can be factorized as the product of \( \tilde{x} \chi_0 \pmod q_1 \) and \( \chi_0 \pmod q_2 \). Then by (5.4), Lemma 7, (5.6), (5.10) we have
\[
\mathcal{S}_k(m) = \sum_{q_1=1}^{\infty} B_k(m, q_1) \sum_{q_2=1}^{\infty} B_0(m, q_2) \ll \mathcal{S}_0(m)(\log N)^{-1/2}
\]
since by (3.1) we have
\[
\tilde{r}^{4/5}(\log r)^{-c_2} \gg (\log N)^{1/2}.
\]
This proves Lemma 9.

6. Major arcs. II.

**Lemma 10.** We have
\[
\int_{(qQ)^{-1}}^{1/2} \left| \prod_{j=1}^{s} I'(b_j \eta) \right| \, d\eta \ll (qQ)^{s-1} N^{s(1-k)}
\]
where \( I'(b_j \eta) \) is defined in (4.5).

**Proof.** If \( 0 < \eta \leq 1/2 \) then for any \( n \geq 1 \) we have \( \Sigma_n^* e(ln) \ll |n|^{-1} \). Let \( \phi = 1/k \) or \( \tilde{B}/k \). Then by Abel’s partial summation formula and (2.6)
\[
b^{\phi} I'(b \eta) \ll |\eta|^{-1} \left( |bN^k|^{\phi-1} + \int_{|b|G^k}^{1} \frac{dy}{|b|G^k} y^{\phi-1} \right)
\ll |\eta|^{-1} (|b|G^k)^{\phi-1} \ll |\eta|^{-1} N^{1-k}.
\]
So the lemma follows.

Let
\[
(6.1) \quad J_h(m) = \int_{-1/2}^{1/2} \prod_{j=1}^{h} I'(b_j \eta) \prod_{j=h+1}^{s} I(b_j \eta) e(-m\eta) \, d\eta \quad (h = 0, 1, \ldots, s).
\]
Lemma 11. (a) $|J_h(m)| \leq J_0(m)$ ($h = 1, 2, \ldots, s$).

(b) If

$$|m| \leq (N/4)^k s^{-1}$$

then

$$J_0(m) \gg B^{-s/k} N^{s-k}.$$

Proof. Part (a) follows from (6.1) and part (b) is essentially Lemma 8 [9].

We come now to treat those terms in category (B) defined in §4. In category (B) we choose $\prod_{j=1}^h \mathcal{S}_j \prod_{j=h+1}^s \mathcal{S}_j$ ($h = 0, 1, \ldots, s$) to represent those terms $\prod_{j=1}^s \mathcal{S}_j$ having exactly $h$ factors $\mathcal{S}_j$. Put

$$T_0(m) = \sum_{q \leq \rho} \sum_{a} e\left(\frac{-ma}{q}\right) \int_{\delta_a}^{1/2} \left( \prod_{j=1}^h \mathcal{S}_j \prod_{j=h+1}^s \mathcal{S}_j \right) e(-m\eta) d\eta,$$

$$\hat{T}_h(m) = \sum_{q \leq \rho} \sum_{a} e\left(\frac{-ma}{q}\right) \int_{\delta_a}^{1/2} \left( \prod_{j=1}^h \mathcal{S}_j \prod_{j=h+1}^s \mathcal{S}_j \right) e(-m\eta) d\eta$$

(6.3)

By (4.1), $s \geq 5$, Lemmas 10 and 5 with $\epsilon = (10s)^{-1}$ we have

$$\sum_{q \leq \rho} \sum_{a} e\left(\frac{-ma}{q}\right) \int_{\delta_a}^{1/2} \left( \prod_{j=1}^h \mathcal{S}_j \prod_{j=h+1}^s \mathcal{S}_j \right) e(-m\eta) d\eta$$

(6.4)

$$\ll N^{s(1-k)} Q^{s-1} \sum_{q \leq \rho} \phi(q)^{-1} \left( \prod_{j=1}^s (q, b_j)^{1/2} q^{1/2+\epsilon} \right) q^{s-1}$$

$$\ll N^{s-k} B_{s/2} P^{-3/10} = E_2,$$

say.

So, if we replace the integral $\int_{\delta_a}^{1/2}$ in (6.3) by $\int_{\delta_a}^{1/2}$ we have the error $E_2$ given in (6.4). Then by (6.1), (6.3), (4.1) we have

(6.5)

$$\hat{T}_h(m) = J_h(m)$$

$$+ E_2.$$

Similarly, by Lemma 5, if we replace the sum $\sum_{q \leq \rho, \rho | q}$ in (6.5) by $\sum_{q=1, \rho | q}$ we have an error $\ll B^{s/2} P^{-3/10}$. So by (5.4), (6.5) we have

(6.6)

$$\hat{T}_h(m) = J_h(m)(\mathcal{S}_h(m) + E_3) + E_2 \quad (h = 1, 2, \ldots, s),$$

where $E_3 = O(E_2 N^{k-s})$. By the same argument we have

(6.7)

$$T_0(m) = J_0(m)(\mathcal{S}_0(m) + E_3) + E_2.$$

Note that each representative in either category (A) or (B) defined in §4 represents at most $O(1)$ terms. It follows from (4.2), (4.10), (6.7), (6.6) that

(6.8)

$$R_1(m) = J_0(m)(\mathcal{S}_0(m) + E_3) + O\left( \sum_{h=1}^s J_h(m) \{ \mathcal{S}_h(m) + E_3 \} \right)$$

$$+ O(E_2 + E_3),$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
By (4.10), (6.4), Lemmas 9(a) and 11(b) we see that for \( j = 1, 2 \)

\[
(6.9) \quad E_j/\mathcal{L}_0(m) \text{ and } E_j/J_0(m) \mathcal{L}_0(m) \ll B^{2s^2 + 3/10}.
\]

It follows from (6.8), (6.9), (2.1), Lemmas 11(a) and 9(b) that

\[
(6.10) \quad R_1(m) > \frac{1}{2} J_0(m) \mathcal{L}_0(m).
\]

7. Minor arcs.

**Lemma 12.** If \( \alpha \in m \) then

\[
\sum_{p \leq N} e(abp^k) \ll N|b|P^{-\omega(k)}
\]

where \( \omega(k)^{-1} = 4(k+2)(k+1) \).

**Proof.** This is essentially Lemma 11 [9] (see also Lemma 5 [1]).

Let

\[
R_2(m) = \int_{m}^{N} \prod_{j=1}^{s} S(b_j \alpha) e(-m \alpha) d \alpha.
\]

Then by Lemma 12 and the same argument as Lemma 12 in [9] we have

\[
R_2(m) \ll N^{s-k} B^{-s} P^{-\omega(k)} (\log N)^{c_4}.
\]

By (4.2), (6.10), Lemmas 9(a), 11(b) and (2.1)

\[
(7.1) \quad \int_{Q^{-1}}^{1} \prod_{j=1}^{s} S(b_j \alpha) e(-m \alpha) d \alpha = R_1(m) + R_2(m)
\]

\[
\gg N^{s-k} B^{-s^2} \left[ 1 - c_2 B^{s(s+1)} P^{-\omega(k)} (\log N)^{c_4} \right] > 0.
\]

Choose the least \( N \) satisfying (2.1) and (6.2). So (7.1) implies the existence of a solution of \( \Sigma_{j=1}^{s} b_j p_j^k = m \) in primes \( p_j \) and

\[
\max_{1 \leq j \leq s} p_j \leq N \leq C_1(k) m^{1/k} + C_2(k) (\log B)^2.
\]

This completes the proof of Theorem 1.

**Remark.** Combining the Circle Method with the Sieve Method, when \( k = 1 \) and \( s = 3 \), the author [10] is able to obtain a bound for solutions of (1.6) to be \( B^A \) where \( A > 0 \) is an absolute constant. However, for \( k \geq 2 \) it seems that these two methods do not combine well to replace the \( (\log B)^2 \) in (1.7) by \( \log B \).

**References**


License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG, HONG KONG