

SHAPE PROPERTIES OF WHITNEY MAPS FOR HYPERSPACES

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ABSTRACT. In this paper, some shape properties of Whitney maps for hyperspaces are investigated. In particular, the following are proved:

(1) Let X be a continuum and let \mathfrak{S} be the hyperspace 2^X or $C(X)$ of X with the Hausdorff metric. Then if ω is any Whitney map for \mathfrak{S} , for any $0 \leq s \leq t \leq \omega(X)$ $\omega^{-1}(t)$ is an approximate strong deformation retract of $\omega^{-1}([s, t])$. In particular, $\text{Sh}(\omega^{-1}(t)) = \text{Sh}(\omega^{-1}([s, t]))$.

(2) Pointed 1-movability is a Whitney property.

(3) For any given $n < \infty$, the property of (cohomological) dimension $\leq n$ is a sequential strong Whitney-reversible property.

(4) The property of being chainable or circle-like is a sequential strong Whitney-reversible property.

(5) The property of being an FAR is a Whitney property for 1-dimensional continua.

Property (2) is an affirmative answer to a problem of J. T. Rogers [16, 112]. Properties (3) and (4) are affirmative answers to problems of S. B. Nadler [20, (14.57) and 21].

0. Introduction. By a *continuum* we mean a compact connected metric space. For a given continuum X , by the *hyperspaces* of X we mean

$$2^X = \{A \subset X \mid A \text{ is a nonempty closed subset of } X\},$$

$$C(X) = \{A \in 2^X \mid A \text{ is connected}\},$$

which have the Hausdorff metric $\rho_{\mathfrak{S}}$ (see [20]). The term *mapping* is a continuous function. Let $\mathfrak{S} = 2^X$ or $C(X)$. A mapping $\omega: \mathfrak{S} \rightarrow [0, \omega(X)]$ is called a *Whitney map* for \mathfrak{S} [20 or 31] provided that the following conditions are satisfied;

(a) if $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$,

(b) $\omega(\{x\}) = 0$ for each $x \in X$.

In 1932, H. Whitney [31] showed that Whitney maps always exist. The notion of Whitney map is a convenient and important tool for hyperspace theory. A topological property P is said to be a *Whitney property* [20 or 21] provided that whenever a continuum X has property P , so does $\omega^{-1}(t)$ for each Whitney map ω for $C(X)$ and $0 \leq t < \omega(X)$. It is well known that connectivity is a Whitney property. If ω is a

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Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$, then Whitney levels $\omega^{-1}(t)$ are coverings of X which, as t gets close to zero, converge to $\omega^{-1}(0) \cong X$. Hence, it is of interest to obtain information about the structure of Whitney levels and determine those properties which are preserved by the convergence of positive Whitney levels to the zero level. A topological property P is said to be a *strong Whitney-reversible property* [20 or 21] provided that whenever X is a continuum such that $\omega^{-1}(t)$ has property P for some Whitney map ω for $C(X)$ and for $0 < t < \omega(X)$, then X has property P . The notions of Whitney property and strong Whitney-reversible property are “converses” to each other. A topological property P is said to be a *sequential strong Whitney-reversible property* [20 or 21] provided that if X is a continuum such that there is a Whitney map ω for $C(X)$ and a decreasing sequence $\{t_n\}_{n=1,2,\dots} \rightarrow 0$ as $n \rightarrow \infty$ such that $\omega^{-1}(t_n)$ has property P for each n , then X has property P . Clearly, sequential strong Whitney-reversible property implies strong Whitney-reversible property.

The purpose of this paper is to investigate Whitney properties, strong Whitney-reversible properties and sequential strong Whitney-reversible properties.

In [20], S. B. Nadler proved that the property of being arcwise connected is a Whitney property. In relation to this result, J. T. Rogers had the following problem [16, 112]: Is the property of being pointed 1-movable a Whitney property? In the first section, we show that for any continuum X and any Whitney map ω for 2^X or $C(X)$, $\omega^{-1}(t)$ is an approximate strong deformation retract of $\omega^{-1}([s, t])$ for $0 \leq s \leq t \leq \omega(X)$. As a corollary, it is shown that Rogers’ problem has an affirmative answer. Also, it is shown that some shape properties are sequential strong Whitney-reversible properties, e.g., the property of being approximately n -connected (= AC^n) or fundamental dimension $\leq n$, etc. In [21], S. B. Nadler proved that for any given $n < \infty$, the property of dimension $\leq n$ is a strong Whitney-reversible property. He had the following question [21, (2.10)]: Is the property of dimension $\leq n$ a sequential strong Whitney-reversible property? In the second section, we show that for a given ANR M , the property of being extendable with respect to M is a sequential strong Whitney-reversible property. As a corollary, we obtain that the property of (cohomological) dimension $\leq n$ is a sequential strong Whitney-reversible property. In [10], J. Krasinkiewicz showed that the property of being chainable or proper circle-like is a Whitney property. In [20, (14.57)], S. B. Nadler asked if the converses to Krasinkiewicz’ results are true, i.e., is the property of being chainable or circle-like a (sequential) strong Whitney-reversible property? In [21, (5.5), (5.7) and 33], it was shown that the answer to this question is “yes” in the hereditarily indecomposable case and in the hereditarily decomposable case. In §3, we show that the property of being chainable or circle-like is a sequential strong Whitney-reversible property. In [22], A. Petrus showed that the property of being an FAR is not a Whitney property. More precisely, she showed that there exist a 2-dimensional FAR X (in fact, $X =$ a disk) and a Whitney map ω for $C(X)$ such that $\omega^{-1}(t)$ is not an FAR for some $t \in (0, \omega(X))$. Also, she proved that if X is a dendrite (= 1-dimensional AR), then for any Whitney map ω for $C(X)$, $\omega^{-1}(t)$ is contractible for each $t \in [0, \omega(X)]$. She had the following question: Is the property of being contractible a Whitney property for 1-dimensional continua? In §4, we prove an “approximation theorem for

Whitney maps." By using this theorem, we show that the property of being an FAR is a Whitney property for 1-dimensional continua. In the last section, we give some problems that are related to this paper.

Almost all of the results in this paper are proved by the ideas and techniques of ANR and shape theories. The author believes that those theories are also very useful when studying the structures of Whitney maps.

We refer readers to [1, 4, 8 and 18] for shape theory and strong shape theory, and to [20] for hyperspaces theory.

1. Shapes of $\omega^{-1}(t)$ and $\omega^{-1}([s, t])$. In this section, we show that for any Whitney map $\omega: \mathfrak{S} \rightarrow [0, \omega(X)]$ for $\mathfrak{S} = 2^X$ or $C(X)$, $\omega^{-1}(t)$ is an approximate strong deformation retract of $\omega^{-1}([s, t])$ for $0 \leq s \leq t \leq \omega(X)$. In particular, $\text{Sh}(\omega^{-1}(t)) = \text{Sh}(\omega^{-1}([s, t]))$. As a corollary, it is shown that pointed 1-movability is a Whitney property. Also, it is shown that some shape properties are sequential strong Whitney-reversible properties.

A subcompactum X of a compactum Y is said to be an *approximate strong deformation retract* of Y provided that for any ANR M containing Y , there exist mappings $r: Y \times [0, \infty) \rightarrow M$ and $D: Y \times [0, \infty) \times I \rightarrow M$ satisfying the conditions

(*) $r(x, t) = x$ for $x \in X$ and $t \in [0, \infty)$, and for any neighborhood U of X in M there is a real $t_0 \geq 0$ such that $r(Y \times [t_0, \infty)) \subset U$,

(**) $D(x, t, s) = x$ for $x \in X$, $t \in [0, \infty)$, and $s \in I$, $D(y, t, 0) = y$, $D(y, t, 1) = r(y, t)$ for $y \in Y$ and $t \in [0, \infty)$, and for any neighborhood V of Y in M there is a real $t_1 \geq 0$ such that $D(Y \times [t_1, \infty) \times I) \subset V$.

Clearly, the inclusion $i: X \hookrightarrow Y$ is a (strong) shape equivalence (e.g., see [1, 4, 8 or 18]). Hence $\text{Sh}(X) = \text{Sh}(Y)$. A continuum X is said to be *pointed 1-movable* [13 or 18] provided that for any ANR M containing X and each neighborhood U of X in M , there is a neighborhood $V \subset U$ of X in M such that each loop in (V, x_0) can be deformed within (U, x_0) into any neighborhood of X in M . It is well known that planar continua are pointed 1-movable but nonplanar circle-like continua are not pointed 1-movable. Let X be a continuum and $x, y \in X$. Then we say that x and y are *joinable* in X [13 or 18] provided that for any ANR M containing X , there is a mapping $\varphi: I \times [0, \infty) \rightarrow M$ satisfying the condition

$\varphi(0, t) = x$, $\varphi(1, t) = y$ for each $t \in [0, \infty)$, and for each neighborhood U of X in M there is a real $t_0 \geq 0$ such that $\varphi(I \times [t_0, \infty)) \subset U$.

Such a map φ is called an *approximate path* from x to y in X . If each pair of points of X is joinable in X , then X is said to be *joinable*.

Note that the properties of being "approximate strong deformation retract" and being "joinable" are not dependent on the choice of ANR M and embeddings of X and Y into M (cf. [1 and 13]).

In relation to pointed 1-movability and joinability, J. Krasinkiewicz and P. Minc proved

(1.1) (J. Krasinkiewicz and P. Minc [13]). *A continuum X is pointed 1-movable if and only if X is joinable. In particular, arcwise connected continua (more generally, λ -connected continua) are pointed 1-movable.*

The following result will be used many times in this paper.

(1.2) (L. E. Ward, Jr. [30]). *If Y is a continuum and X is a subcontinuum of Y , then any Whitney map for $C(X)$ (resp. 2^X) can be extended to a Whitney map for $C(Y)$ (resp. 2^Y).*

The main result in this section is the following

(1.3) THEOREM. *Let X be a continuum and let $\omega: \mathfrak{S} \rightarrow [0, \omega(X)]$ be any Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Then $\omega^{-1}(t)$ is an approximate strong deformation retract of $\omega^{-1}([s, t])$ for $0 \leq s \leq t \leq \omega(X)$. In particular, $\text{Sh}(\omega^{-1}(t)) = \text{Sh}(\omega^{-1}([s, t]))$.*

(1.4) THEOREM. *Let X be a continuum and let $\omega: C(X) \rightarrow [0, \omega(X)]$ be any Whitney map for $C(X)$. Suppose that $A_1, A_2 \in \omega^{-1}([0, t_0])$ for some $t_0 \in [0, \omega(X)]$ and A_1 and A_2 are joinable in $\omega^{-1}([0, t_0])$. If $B_1, B_2 \in \omega^{-1}(t_1)$ for $t_1 \geq t_0$ and $A_i \cap B_i \neq \emptyset$ ($i = 1, 2$), then B_1 and B_2 are joinable in $\omega^{-1}(t_1)$.*

As a corollary of (1.1) and (1.4), we have

(1.5) COROLLARY. *For any Whitney map ω for $C(X)$, if $\omega^{-1}(t)$ is pointed 1-movable for some $t \in [0, \omega(X)]$, then $\omega^{-1}(t')$ is also pointed 1-movable for $t' \geq t$. In particular, the property of being pointed 1-movable is a Whitney property.*

PROOF OF (1.3). For simplicity, we assume that $s = 0$. The case $s > 0$ will be proved by the same method. Suppose that X is lying in the Hilbert cube $Q = \prod_{i=0}^\infty [-1, 1]$. Choose a decreasing sequence $X_1 \supset X_2 \supset X_3 \supset \dots$ of Peano continua such that for each n , X_n is a closed neighborhood of X in Q and $X = \bigcap_{n=1}^\infty X_n$. By (1.2), there is a Whitney map $\mu: \mathfrak{S}' \rightarrow [0, \mu(Q)]$ which is an extension of ω , where $\mathfrak{S}' = 2^Q$ if $\mathfrak{S} = 2^X$, and $\mathfrak{S}' = C(Q)$ if $\mathfrak{S} = C(X)$. By [32], \mathfrak{S}' is an ANR. Set $\mu_{X_n} = \mu|_{\mathfrak{S}_n}$, where $\mathfrak{S}_n = 2^{X_n}$ if $\mathfrak{S} = 2^X$, and $\mathfrak{S}_n = C(X_n)$ if $\mathfrak{S} = C(X)$. Clearly, μ_{X_n} is a Whitney map for \mathfrak{S}_n . Note that $\mu^{-1}(t) \supset \mu_{X_1}^{-1}(t) \supset \mu_{X_2}^{-1}(t) \supset \dots$ and $\omega^{-1}(t) = \bigcap_{n=1}^\infty \mu_{X_n}^{-1}(t)$. Now, for each $n = 1, 2, \dots$, we shall construct mappings $r_n: \omega^{-1}([0, t]) \rightarrow \mu_{X_n}^{-1}(t)$ and homotopies $F_n: \omega^{-1}([0, t]) \times I \rightarrow \mu_{X_n}^{-1}(t)$ such that $r_n|_{\omega^{-1}(t)} =$ the identity mapping on $\omega^{-1}(t)$ and $F_n: r_n = r_{n+1}$ rel. $\omega^{-1}(t)$ in $\mu_{X_n}^{-1}(t)$. Since X_n is a Peano continuum, X_n has a convex metric d_n . As in [5, (1.2) Theorem], define a homotopy $K_{d_n}: \mathfrak{S}_n \times [0, \infty) \rightarrow \mathfrak{S}_n$ by letting

$$K_{d_n}(A, s) = \{x \in X_n \mid d_n(x, a) \leq s \text{ for some } a \in A\}.$$

For each $n = 1, 2, \dots$, define a function $r_n: \omega^{-1}([0, t]) \rightarrow \mu_{X_n}^{-1}(t)$ by

$$r_n(A) = K_{d_n}(A, \theta^n(A)), \quad \text{where } \mu_{X_n}(K_{d_n}(A, \theta^n(A))) = t.$$

Then we can easily see that r_n is continuous (cf. [5, (3.1) Proposition]).

Next, we shall construct homotopies F_n ($n = 1, 2, \dots$). For this purpose, we need to define homotopies $R_n: \omega^{-1}([0, t]) \times I \rightarrow \mu_{X_n}^{-1}([0, t])$ ($n = 1, 2, \dots$) by

$$R_n(A, s) = K_{d_n}(A, \theta_s^n(A)), \quad \text{where } \mu_{X_n}(K_{d_n}(A, \theta_s^n(A))) = s \cdot t + (1 - s)\mu_{X_n}(A).$$

By a way similar to [5, (3.1) Proposition], R_n is continuous. Note that $R_n(A, 1) = r_n(A)$ for each $A \in \omega^{-1}([0, t])$. Also, define homotopies $S_n: \omega^{-1}([0, t]) \times I \rightarrow \mu_{X_n}^{-1}([0, t])$

($n = 1, 2, \dots$) by

$$S_n(A, s) = \begin{cases} R_n(A, 1 - 2s) & \text{if } 0 \leq s \leq 1/2, \\ R_{n+1}(A, 2s - 1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Set $F_n = r_n \circ S_n: \omega^{-1}([0, t]) \times I \rightarrow \mu_{X_n}^{-1}(t)$. Then we have

$$F_n(A, 0) = r_n R_n(A, 1) = r_n r_n(A) = r_n(A),$$

$$F_n(A, 1) = r_n R_{n+1}(A, 1) = r_n r_{n+1}(A) = r_{n+1}(A) \quad \text{for } A \in \omega^{-1}([0, t]),$$

and

$$F_n(A, s) = A \quad \text{for } A \in \omega^{-1}(t) \text{ and } 0 \leq s \leq 1.$$

Hence F_n ($n = 1, 2, \dots$) are desired homotopies. By using F_n , we can easily define a mapping $r: \omega^{-1}([0, t]) \times [0, \infty) \rightarrow \mathfrak{S}'$ satisfying condition (*) (see Figure 1).

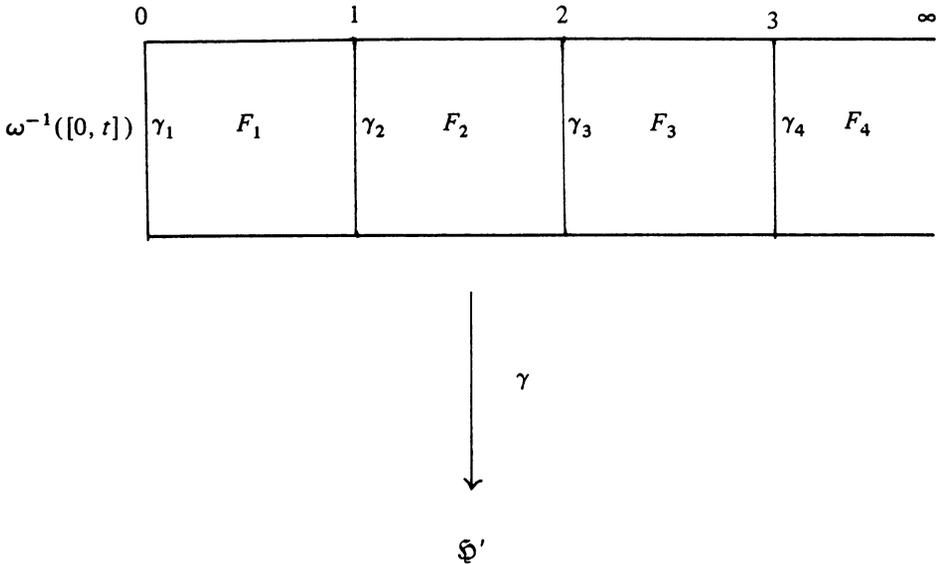


FIGURE 1

Next, we shall construct a mapping $D: \omega^{-1}([0, t]) \times [0, \infty) \times I \rightarrow \mathfrak{S}'$ satisfying condition (**). For $n = 1, 2, \dots$, define mappings $T_n: \omega^{-1}([0, t]) \times I \times I \rightarrow \mu_{X_n}^{-1}([0, t])$ by

$$T_n(A, s, s') = K_{d_n}(S_n(A, s), \theta_s^n(S_n(A, s))),$$

where

$$\mu_{X_n}(K_{d_n}(S_n(A, s), \theta_s^n(S_n(A, s)))) = s' \cdot t + (1 - s')\mu_{X_n}(S_n(A, s)).$$

By using the homotopies T_n , we can easily obtain homotopies $D_n: \omega^{-1}([0, t]) \times I \times I \rightarrow \mu_{X_n}^{-1}([0, t])$ such that

$$D_n(A, s, 0) = A, \quad D_n(A, s, 1) = F_n(A, s),$$

$$D_n(A, 0, s') = R_n(A, s'), \quad D_n(A, 1, s') = R_{n+1}(A, s')$$

for $A \in \omega^{-1}([0, t])$, and

$$D_n(A, s, s') = A \quad \text{for } A \in \omega^{-1}(t) \text{ and } 0 \leq s, s' \leq 1.$$

By using D_n , we can define a mapping $D: \omega^{-1}([0, t]) \times [0, \infty) \times I \rightarrow \mathfrak{S}'$ satisfying condition (**) (see Figure 2).

To prove (1.4), we need the following lemma.

(1.6) LEMMA. *Let X be a subcontinuum of a continuum Y and let $x, y \in X$. If X is an approximate strong deformation retract of Y , and x and y are joinable in Y , then x and y are also joinable in X .*

PROOF. Assume that M is an ANR which contains Y . Let $\gamma: Y \times [0, \infty) \rightarrow M$ be a mapping satisfying condition (*). Since x and y are joinable in Y , there is an approximate path $\varphi: I \times [0, \infty) \rightarrow M$ from x to y in Y . Since M is an ANR, there is a neighborhood W of $Y \times [0, \infty)$ in $M \times [0, \infty)$ and an extension $\tilde{\gamma}: W \rightarrow M$ of γ . By changing the parameter $t \in [0, \infty)$, we may assume that $(\varphi(I \times \{t\}), t) \subset W$ for each $t \in [0, \infty)$. Define a mapping $\varphi': I \times [0, \infty) \rightarrow M$ by $\varphi'(s, t) = \tilde{\gamma}(\varphi(s, t), t)$ for $(s, t) \in I \times [0, \infty)$. Clearly, φ' is an approximate path from x to y in X .

PROOF OF (1.4). By [7], there are segments $\alpha_i: I \rightarrow \omega^{-1}([0, t_1])$ such that $\alpha_i(0) = A_i$ and $\alpha_i(1) \in \omega^{-1}(t_1)$ for $i = 1, 2$. Then $\alpha_i(1) \cap B_i \neq \emptyset$ ($i = 1, 2$), hence by [20, (14.8.1)] there exist arcs β_i in $\omega^{-1}(t_1)$ from $\alpha_i(1)$ to B_i ($i = 1, 2$). Since A_1 and A_2 are joinable in $\omega^{-1}([0, t_0])$, B_1 and B_2 are joinable in $\omega^{-1}([0, t_1])$. By (1.3) and (1.6), B_1 and B_2 are joinable in $\omega^{-1}(t_1)$. This completes the proof.

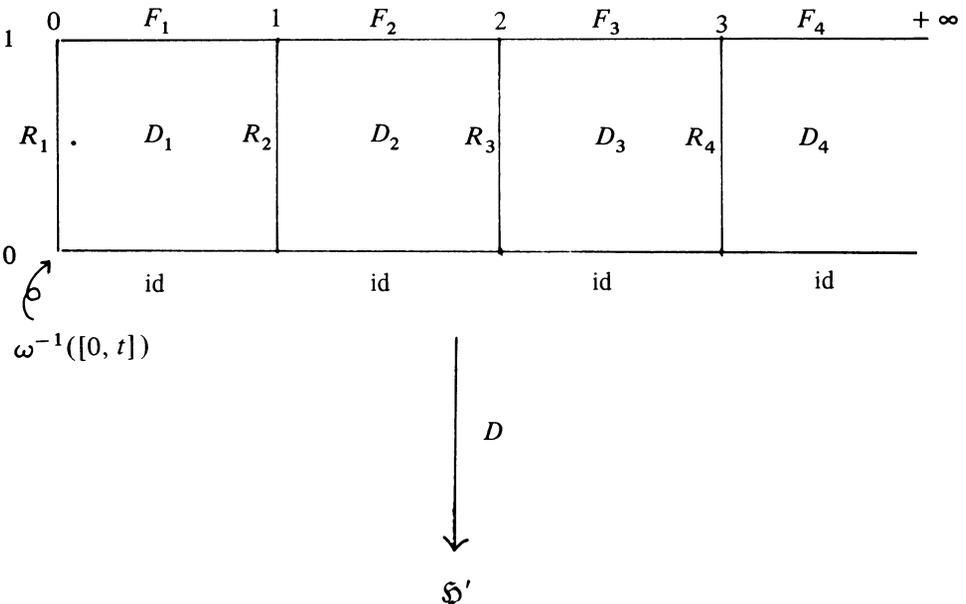


FIGURE 2

(1.7) REMARK. In (1.4) and (1.5), we cannot replace “Whitney map for $C(X)$ ” by “Whitney map for 2^X .” In fact, it is well known that there is a Whitney map $\omega: 2^I \rightarrow [0, \omega(I)]$ such that for some $t \in (0, \omega(I))$, $\omega^{-1}(t)$ is not connected (see [20, (14.61)]).

(1.8) EXAMPLE. It is easily seen that if X is a Peano continuum, then for any Whitney map ω for $\mathfrak{S} = 2^X$ or $C(X)$, $\omega^{-1}(t)$ is a strong deformation retract of $\omega^{-1}([s, t])$ for $0 \leq s \leq t \leq \omega(X)$. For such a case, this is a stronger result than (1.3). But, if X is not a Peano continuum, $\omega^{-1}(t)$ is not always a strong deformation retract of $\omega^{-1}([s, t])$. Moreover, $\omega^{-1}(t)$ is not always homotopy equivalent to $\omega^{-1}([s, t])$. Consider the following subsets X , Y in the plane E^2 and Z in the Euclidean 3-dimensional space E^3 (see Figure 3).

$$X = \text{Cl}\{(x, \sin[1/x]) \in E^2 \mid 0 < x \leq 1\}$$

$$\cup \text{Cl}\{(x, 2 + \sin[1/x]) \in E^2 \mid -1 \leq x < 0\},$$

$$Y = \text{Cl}\{(X, \sin[1/x]) \in E^2 \mid 0 < x \leq 1\} \cup \{(0, y) \in E^2 \mid 1 \leq y \leq 2\}$$

$$\cup \text{Cl}\{(x, 3 + \sin[1/x]) \in E^2 \mid -1 \leq x < 0\},$$

$$Z = \text{the mapping cylinder of the mapping } f: Y \rightarrow X \text{ defined by } f|_A$$

$$= \{(0, y) \in E^2 \mid 1 \leq y \leq 2\}$$

$$= \text{the constant map to the point } (0, 1) \in E^2 \text{ and } f|_{Y-A} \text{ is one-to-one.}$$

Let ω be any Whitney map for $C(X)$. Then there is $t_0 \in (0, \omega(X))$ such that if $0 < t \leq t_0$, then $\omega^{-1}(t)$ is homeomorphic to Y (see [20, (14.43.2)]). Also, $\omega^{-1}([0, t])$ is homeomorphic to Z . Since $\omega^{-1}([0, t])$ is homotopy equivalent to $\omega^{-1}(0) \cong X$ and X and Y are not homotopy equivalent, $\omega^{-1}(t) \cong Y$ is not homotopy equivalent to $\omega^{-1}([0, t])$. Note that for some $\varepsilon > 0$ and any $t \in (0, \omega(X))$ there is no mapping $g: \omega^{-1}(0) \rightarrow \omega^{-1}(t)$ such that $\rho_{\mathfrak{S}}(\{x\}, g(\{x\})) < \varepsilon$ for $x \in X$ (cf. (3.1)).

In (1.3), if $s = 0$ and $t = \omega(X)$, we have the following well-known result.

(1.9) (J. KRASINKIEWICZ [12]). *If X is a continuum, then $\text{Sh}(C(X)) = \text{Sh}(2^X) = *$.*

(1.10) COROLLARY. *Let X be a continuum and let ω be a Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Suppose that $t_1 > t_2 > t_3 > \dots$ is a decreasing sequence of positive numbers in $[0, \omega(X)]$ and $t = \lim_{n \rightarrow \infty} t_n$. Then*

(1) (S. B. NADLER [21] OR A. KOYAMA [9]). *If $\omega^{-1}(t_n)$ is contractible with respect to an ANR M , then ω^{-1} is contractible with respect to M . In particular, the property of being an FAR is a sequential strong Whitney-reversible property.*

(2) (S. B. NADLER [21]). *If $\check{H}^i(\omega^{-1}(t_n)) = 0$, then $\check{H}^i(\omega^{-1}(t)) = 0$.*

(3) *If $\omega^{-1}(t_n)$ is approximately m -connected ($= \text{AC}^m$), then $\omega^{-1}(t)$ is AC^m .*

(4) *If $\text{pro-}H_i(\omega^{-1}(t_n)) = 0$, then $\text{pro-}H_i(\omega^{-1}(t)) = 0$. Also, if $\check{H}_i(\omega^{-1}(t_n)) = 0$, then $\check{H}_i(\omega^{-1}(t)) = 0$.*

(5) *If $\text{Fd}(\omega^{-1}(t_n)) \leq m$, then $\text{Fd}(\omega^{-1}(t)) \leq m$.*

(See [1 and 18] for the definitions in statements (3), (4), and (5).)

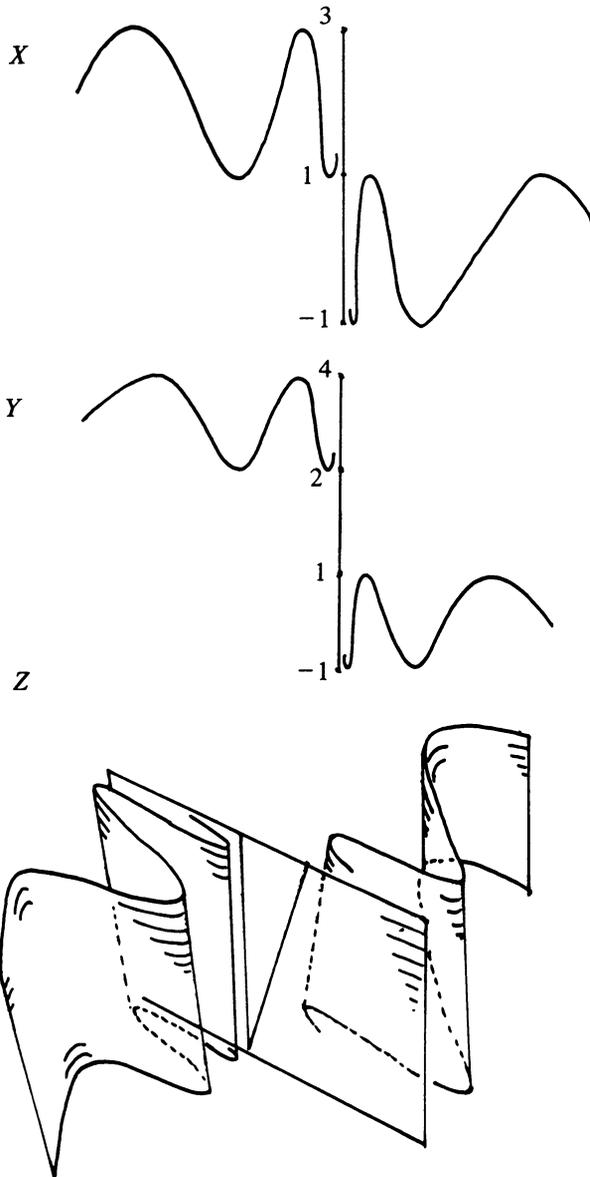


FIGURE 3

PROOF. Note that $\omega^{-1}(t) = \bigcap_{n=1}^{\infty} \omega^{-1}([t, t_n])$. The proof follows from the following commutative diagram in shape category (see [1, 18] for the definition of shape category):

$$\begin{array}{ccccccc}
 \omega^{-1}([t, t_1]) & \leftarrow & \omega^{-1}([t, t_2]) & \leftarrow & \omega^{-1}([t, t_3]) & \leftarrow & \dots \\
 \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \\
 \omega^{-1}(t_1) & \leftarrow & \omega^{-1}(t_2) & \leftarrow & \omega^{-1}(t_3) & \leftarrow & \dots \\
 & & \downarrow \uparrow & & & & \\
 & & \omega^{-1}(t_2) & & \omega^{-1}(t_3) & &
 \end{array}$$

$\downarrow \uparrow$ $\omega^{-1}(t_2)$ $\downarrow \uparrow$ $\omega^{-1}(t_3)$

where $r_n: \omega^{-1}([t, t_n]) \rightarrow \omega^{-1}(t_n)$ is a (strong) shape equivalence that is induced by the mapping $r_n: \omega^{-1}([t, t_n]) \times [0, \infty) \rightarrow \mathfrak{S}'$ as in the proof of (1.3). By continuity of shape, (1)–(5) are easily proved.

(1.11) COROLLARY. *Let X be a continuum and let ω be any Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. If X is an FANR, there is $t_0 \in (0, \omega(X))$ such that $\text{Sh}(X) \leq \text{Sh}(\omega^{-1}(t))$ for each $t \in [0, t_0]$.*

PROOF. Since X is an FANR, by [3], there is $t_0 \in (0, \omega(X))$ such that $\omega(0)$ is an approximate retract of $\omega^{-1}([0, t_0])$, i.e., there is a mapping $r: \omega^{-1}([0, t_0]) \times [0, \infty) \rightarrow M$ satisfying condition (*), where M is an ANR that contains $\omega^{-1}(0)$. Hence $\omega^{-1}(0)$ is (strong) shape dominated by $\omega^{-1}([0, t_0])$. Since $\omega^{-1}([0, t])$ is (strong) shape equivalent to $\omega^{-1}(t)$, X is (strong) shape dominated by $\omega^{-1}(t)$ for $0 \leq t \leq t_0$.

(1.12) EXAMPLE. In the statement of (1.11), we cannot conclude that $\text{Sh}(X) = \text{Sh}(\omega^{-1}(t))$ for $0 \leq t \leq t_0$. Consider the following sets (see [22, Example 5]). In the Euclidean 3-dimensional space E^3 , let $A_0 = \{(x, y, z) \in E^3 \mid x^2 + y^2 = 1, z = 0\}$ and $A_n = \{(x, y, z) \in E^3 \mid x^2 + y^2 = 1, z = 1/2^{n-1}\}$ ($n = 1, 2, \dots$). Let X_n be the annulus that is contained in the sphere with center $(0, 0, 3/2^{n+1})$ and radius $\sqrt{1 + (1/2^{n+1})^2}$ and is bounded by circles A_n and A_{n+1} . Let $Y = \bigcup_{n=1}^{\infty} X_n$, $Z = \{(x, y, z) \in E^3 \mid x^2 + y^2 \leq 1 \text{ and } z = 1\}$, and $T = Y \cup Z$. Let P_n be paths in E^3 and let T_n be a reduced copy of T in E^3 such that $\lim \text{diam } T_n = 0$, $\lim \text{diam } P_n = 0$, $P_i \cap P_j = \emptyset$, $T_i \cap T_j = \emptyset$ ($i \neq j$), and $P_i \cap T_i = \{*\}$, $P_i \cap T_{i+1} = \{*\}$ as below (see Figure 4). Set $X = \bigcup_{n=1}^{\infty} P_n \cup \bigcup_{n=1}^{\infty} T_n$. Note that X is an AR, hence an FAR. Let $S_n = \{(x, y, z) \in E^3 \mid (x - (3/2^{n+1}))^2 + y^2 = (1/2^{n+1})^2\}$ ($n = 1, 2, \dots$) and let $S = \bigcup_{n=1}^{\infty} S_n$ (see Figure 4). Let $\omega: C(X) \rightarrow [0, \omega(X)]$ be the Whitney map for $C(X)$ as defined by [31] and the usual Euclidean metric. Then, by similar argument as in [22, Example 5], there is a decreasing sequence $t_1 > t_2 > \dots$ of positive numbers in $[0, \omega(X)]$ such that $\lim t_n = 0$ and for each n there exist mappings $f_n: \omega^{-1}(t_n) \rightarrow S$ and $g_n: S \rightarrow \omega^{-1}(t_n)$ such that $f_n g_n = 1_S$. Since S is not an FANR, $\omega^{-1}(t_n)$ is not an FANR, i.e., the property of being FANR is not a Whitney property. Also, $\text{Sh}(\omega^{-1}(t_n)) \neq \text{Sh}(X) = *$.

In [29], J. T. Rogers showed that there is always a monomorphism $\check{H}^1(\omega^{-1}(t)) \rightarrow \check{H}^1(X)$ for any continuum X , any Whitney map ω for $C(X)$, and $0 \leq t \leq \omega(X)$. In [15], A. Y. Lau showed that if X is a 1-dimensional continuum and $\check{H}^1(X)$ is finitely generated, then there is $0 < t \leq \omega(X)$ such that $\check{H}^1(X) \cong H^1(\omega^{-1}(s))$ for each $s \leq t$. Now, we shall give more precise information about the morphism $\check{H}^1(\omega^{-1}(t)) \rightarrow \check{H}^1(X)$ and show that in the result of Lau the condition that X is 1-dimensional is not essential.

Let $r: \omega^{-1}([s, t]) \times [0, \infty) \rightarrow \mathfrak{S}'$ be the approximate retraction from $\omega^{-1}([s, t])$ to $\omega^{-1}(t)$ as in the proof of (1.3). Let $f_{st}: \omega^{-1}(s) \rightarrow \omega^{-1}(t)$ be the shape morphism induced by the restriction $r|_{\omega^{-1}(s) \times [0, \infty)}$. Note that $f_{tu} f_{st} = f_{su}$ for $s \leq t \leq u$. Then we have the following

(1.13) COROLLARY (CF. J. T. ROGERS [29] AND A. Y. LAU [15]). *Let X be a continuum and let ω be any Whitney map for $C(X)$. Then $\check{H}(f_{st}): \check{H}^1(\omega^{-1}(t)) \rightarrow \check{H}^1(\omega^{-1}(s))$ is a monomorphism for $0 \leq s \leq t \leq \omega(X)$.*

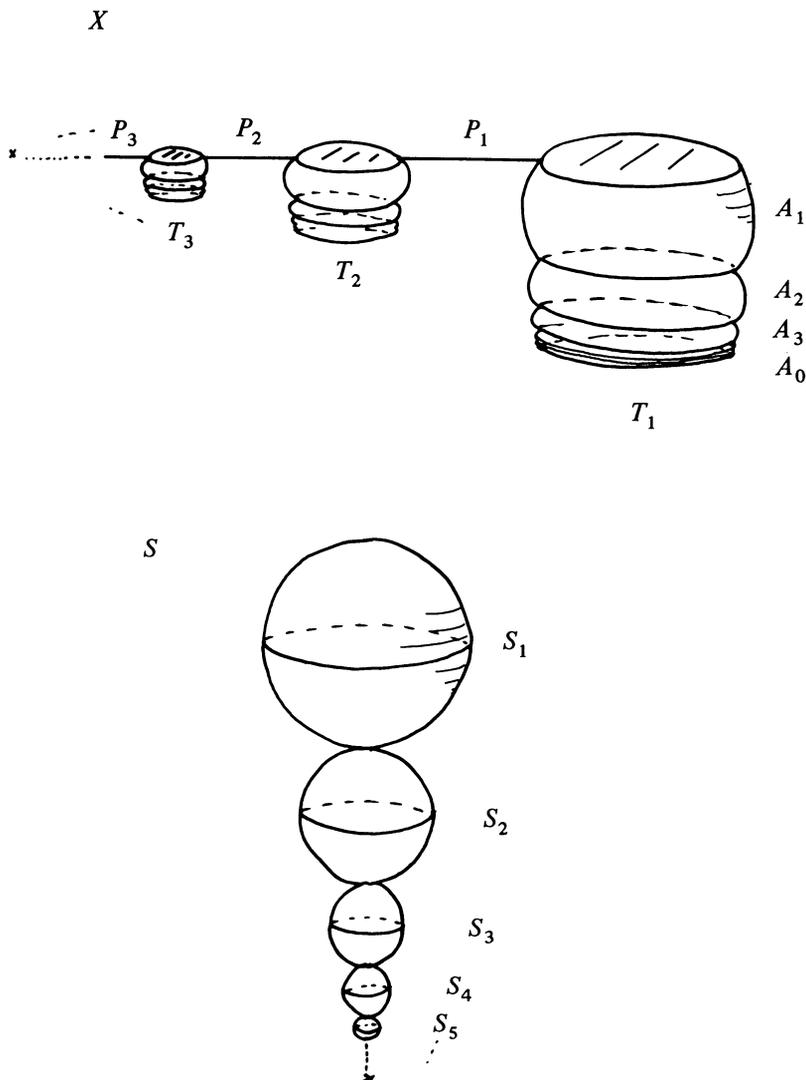


FIGURE 4

PROOF. Consider the following exact sequence of Čech cohomology:

$$\check{H}^1(\omega^{-1}([0, t]), \omega^{-1}(0)) \rightarrow \check{H}^1(\omega^{-1}([0, t])) \xrightarrow{\check{H}^1(i)} \check{H}^1(\omega^{-1}(0)),$$

where $i: \omega^{-1}(0) \hookrightarrow \omega^{-1}([0, t])$ is the inclusion mapping. By [15, Theorem 2], $\check{H}^1(\omega^{-1}([0, t]), \omega^{-1}(0)) = 0$. Hence $\check{H}^1(i)$ is a monomorphism. By using the homotopy D (see the proof of (1.3)), we can easily see that $jf_{0t} = i$, where $j: \omega^{-1}(t) \hookrightarrow \omega^{-1}([0, t])$ is the inclusion mapping. Since j is a shape equivalence, $\check{H}^1(f_{0t})$ is a monomorphism. Let $0 \leq s \leq t$. Since $f_{0t} = f_{st}f_{0s}$, $\check{H}^1(f_{st})$ is a monomorphism.

(1.14) COROLLARY. *Let X be a continuum and let ω be any Whitney map for $C(X)$. Suppose that $\check{H}^1(X)$ is finitely generated. Then for any $s \in [0, \omega(X))$, there is $s' > s$ such that $\check{H}^1(f_{s'}) : \check{H}^1(\omega^{-1}(t)) \rightarrow \check{H}^1(\omega^{-1}(s))$ is an isomorphism for each $s \leq t \leq s'$.*

PROOF. Since $\check{H}^1(f_{0s}) : \check{H}^1(\omega^{-1}(s)) \rightarrow \check{H}^1(X)$ is a monomorphism, $\check{H}^1(\omega^{-1}(s))$ is also finitely generated. We may assume that $\check{H}^1(\omega^{-1}(s)) = [\omega^{-1}(s), S^1]$, where S^1 is a circle and $[Y, Z]$ denotes the homotopy classes of mappings from Y to Z . Let $[g_1], [g_2], \dots, [g_n]$ be generators of $\check{H}^1(\omega^{-1}(s))$, where $g_i : \omega^{-1}(s) \rightarrow S^1$ is a mapping ($i = 1, 2, \dots, n$). Since S^1 is an ANR, there is $s' > s$ and extensions $\tilde{g}_i : \omega^{-1}([s, s']) \rightarrow S^1$ ($i = 1, 2, \dots, n$) of g_i . Let $s \leq t \leq s'$. Then $\check{H}^1(i)([\tilde{g}_i]) = [g_i]$ ($i = 1, 2, \dots, n$), where $i : \omega^{-1}(s) \hookrightarrow \omega^{-1}([s, t])$ is the inclusion mapping. Hence $\check{H}^1(i)$ is an epimorphism. Note that $i = jf_{st}$, where $j : \omega^{-1}(t) \hookrightarrow \omega^{-1}([s, t])$ is the inclusion mapping. Hence $\check{H}^1(i) = \check{H}^1(f_{st})\check{H}^1(j)$, which implies that $\check{H}^1(f_{st})$ is an epimorphism. By (1.13), $\check{H}^1(f_{st})$ is an isomorphism. This completes the proof.

(1.15) EXAMPLE. In the statement of (1.14), we cannot omit the condition that $\check{H}^1(X)$ is finitely generated. Consider the set $X = \bigcup_{n=1}^{\infty} S_n$ in the plane E^2 , where S_n is the circle in E^2 with center $((n-1)/n, 0)$ and radius $1/n$. Note that $\check{H}^1(X)$ is not finitely generated. Now, we shall show that for each $t \in (0, \omega(X)]$, $\check{H}^1(f_{0t}) : \check{H}^1(\omega^{-1}(t)) \rightarrow \check{H}^1(X)$ is not an epimorphism. Consider the morphism $\check{H}^1(i) : [\omega^{-1}([0, t]), S^1] \rightarrow [X, S^1]$, where $i : X \hookrightarrow \omega^{-1}([0, t])$ is the natural inclusion mapping. Since $\lim_{n \rightarrow \infty} \text{diam } S_n = 0$, there is n_0 such that $C(S_n) \subset \omega^{-1}([0, t])$ for $n \geq n_0$. Let $g_n : X \rightarrow S^1$ be a mapping such that $g_n|_{S_n}$ is a homeomorphism. Since $C(S_n)$ is a disk, $\text{Image } \check{H}^1(i) \not\ni [g_n]$ ($n \geq n_0$). Since $i = j\check{f}_{0t}$ and j is a shape equivalence, $\text{Image } \check{H}^1(f_{0t}) \not\ni [g_n]$ ($n \geq n_0$). Hence $\check{H}^1(f_{0t})$ is not an epimorphism.

2. Extendability with respect to an ANR. In [21], it is shown that for any given $n < \infty$, the property of dimension $\leq n$ is a strong Whitney-reversible property. Also, S. B. Nadler had the following question [21, (2.10)]: Is the property of dimension $\leq n$ is a sequential strong Whitney-reversible property? In this section, we prove that the property of being extendable with respect to an ANR M is a sequential strong Whitney-reversible property. As a corollary, it is shown that Nadler's question has an affirmative answer.

Let M be an ANR. A continuum X is said to be *extendable with respect to M* provided that for any closed subset A of X and any mapping $f : A \rightarrow M$, there exists an extension $\tilde{f} : X \rightarrow M$ of f .

The following is the main theorem in this section.

(2.1) THEOREM. *Let M be an ANR and let X be a continuum. Suppose that $\omega : \mathfrak{S} \rightarrow [0, \omega(X)]$ is a Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$ and $t_1 > t_2 > \dots$ is a decreasing sequence of positive numbers in $[0, \omega(X)]$ such that $\lim t_n = t$. If $\omega^{-1}(t_n)$ is extendable with respect to M , then $\omega^{-1}(t)$ is extendable with respect to M .*

If we consider the case $\mathfrak{S} = C(X)$ and $M = n$ -sphere S^n or $M =$ the Eilenberg-Mac Lane complex $K(Z, n)$, then we have the following

(2.2) COROLLARY. *For any given $n < \infty$, the property of (cohomological) dimension $\leq n$ is a sequential strong Whitney-reversible property.*

To prove (2.1), we need the following

(2.3) (J. L. KELLEY [7, (1.5)]). Let X be a continuum and let ω be any Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Then for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that if $A, B \in \mathfrak{S}$, $A \subset B$, and $\omega(B) - \omega(A) < \eta(\varepsilon)$, then $\rho_{\mathfrak{S}}(A, B) < \varepsilon$, where $\rho_{\mathfrak{S}}$ denotes the Hausdorff metric on \mathfrak{S} .

(2.4) THEOREM. Let X be a continuum and let ω be any Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Suppose that M is an ANR that contains \mathfrak{S} . Then, for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that if $0 \leq t \leq t' \leq \omega(X)$ and $t' - t < \eta(\varepsilon)$, then $\omega^{-1}(t')$ is an ε -approximate strong deformation retract of $\omega^{-1}([t, t'])$ in M .

Let Y be a compactum in an ANR M . A subcompactum X of Y is said to be an ε -approximate strong deformation retract of Y in M provided that there exist mappings $r: Y \times [0, \infty) \rightarrow M$ and $D: Y \times [0, \infty) \times I \rightarrow M$ satisfying conditions (*), (**), and

$$(***) \quad d(y, D(y, s, t)) < \varepsilon \quad \text{for any } y \in Y, s \in [0, \infty) \text{ and } t \in I.$$

Note that Y is contained in the ε -neighborhood of X in M .

SKETCH OF THE PROOF OF (2.4). Assume that X is lying in the Hilbert cube Q and $\mu: \mathfrak{S}' \rightarrow [0, \mu(Q)]$ is an extension of ω , where $\mathfrak{S}' = 2^Q$ or $C(Q)$. Let $i: U \rightarrow M$ be an extension of $i_{\mathfrak{S}}: \mathfrak{S} \rightarrow \mathfrak{S}$, where U is a neighborhood of \mathfrak{S} in \mathfrak{S}' . Choose $\varepsilon' > 0$ such that if $\rho_{\mathfrak{S}'}(A, B) < \varepsilon'$, then $d(i(A), i(B)) < \varepsilon$. By (2.3), there is $\eta(\varepsilon') > 0$ as in (2.3) for $\mu: \mathfrak{S}' \rightarrow [0, \mu(Q)]$. We shall show that $\eta(\varepsilon')$ is a desired positive number. Let $t, t' \in [0, \omega(X)]$, $t \leq t'$, and $t' - t < \eta(\varepsilon')$. In the proof of (1.3), we can see that $r_n(A) \supset A$, $R_n(A, s) \supset A$, $S_n(A, s) \supset A$, and $F_n(A, s) \supset A$ for each $n = 1, 2, \dots$, $A \in \omega^{-1}([t, t'])$, and $0 \leq s \leq 1$. Hence we conclude that $D(A, s, s') \supset A$ for $A \in \omega^{-1}([t, t'])$, $s \in [0, \infty)$, and $s' \in I$. Note that $\mu(D(A, s, s')) - \mu(A) < \eta(\varepsilon')$. By (2.3), $\rho_{\mathfrak{S}'}(D(A, s, s'), A) < \varepsilon'$ for $A \in \omega^{-1}([t, t'])$. Without loss of generality, we may assume that $D(\omega^{-1}([t, t']) \times [0, \infty) \times I) \subset U$. Then $\tilde{D} = i \circ D: \omega^{-1}([t, t']) \times [0, \infty) \times I \rightarrow M$ is a desired mapping.

PROOF OF (2.1). Let N be an ANR that contains H . Let Λ be any closed subset of $\omega^{-1}(t)$ and $f: \Lambda \rightarrow M$ be any mapping. Since M is an ANR, there is a closed neighborhood U of Λ in N and there is an extension $f_1: U \rightarrow M$ of f . Take a positive number ε such that $\{A \in N \mid d(A, B) \leq \varepsilon \text{ for some } B \in \Lambda\} \subset \text{Int } U$. Let $\eta(\varepsilon)$ be the positive number as in (2.4). Choose t_n such that $t_n - t < \eta(\varepsilon)$. Since $\omega^{-1}(t_n)$ is extendable with respect to M , there is an extension $g: \omega^{-1}(t_n) \rightarrow M$ of $f_1|_{U \cap \omega^{-1}(t_n)}$. Since M is an ANR, there is a neighborhood W of $\omega^{-1}(t_n)$ in N and an extension $\tilde{g}: U \cup W \rightarrow M$ of g and f_1 . By (2.4), there is a mapping $D: \omega^{-1}([t, t_n]) \times [0, \infty) \times I \rightarrow N$ satisfying conditions (*), (**), and (***)). Choose $s_0 \in [0, \infty)$ such that $D(\omega^{-1}([t, t_n]) \times [s_0, \infty) \times \{1\}) \subset W$. Consider the homotopy $H: \omega^{-1}(t) \times I \rightarrow N$ defined by $H = D|_{\omega^{-1}(t) \times \{s_0\} \times I}$. Then $H(A, 0) = A$, $H(A, 1) \in W$, and $d(H(A, s), A) < \varepsilon$ for $A \in \omega^{-1}(t)$ and $0 \leq s \leq 1$. Note that $H(\Lambda \times I) \subset \text{Int } U$. Let V be a neighborhood of Λ in $\omega^{-1}(t)$ such that $H(V \times I) \subset U$. Choose

a mapping $\alpha: \omega^{-1}(t) \rightarrow I$ such that $\alpha(A) = 0$ if $A \in \Lambda$ and $\alpha(A) = 1$ if $A \in \omega^{-1}(t) - V$. Then $H(A, \alpha(A)) \in U \cup W$ for $A \in \omega^{-1}(t)$. Define a mapping $\tilde{f}: \omega^{-1}(t) \rightarrow M$ by

$$\tilde{f}(A) = \tilde{g}H(A, \alpha(A)) \quad \text{for } A \in \omega^{-1}(t).$$

Clearly, \tilde{f} is an extension of f . Hence $\omega^{-1}(t)$ is extendable with respect to M . This completes the proof.

3. The property of being chainable or circle-like is a sequential strong Whitney-reversible property. In [10], J. Krasinkiewicz proved that the property of being chainable or proper circle-like is a Whitney property. In [20, (14.57)], S. B. Nadler asked if the converses are true, i.e., is the property of being (a) chainable or (b) circle-like a (sequential) strong Whitney-reversible property? In [21, (5.5), (5.7) and 33], it was shown that the answer is “yes” in the hereditarily indecomposable case and hereditarily decomposable case. In this section, we show that the property of being chainable or circle-like is a sequential strong Whitney-reversible property.

In this section, we need the following definitions. A mapping $f: X \rightarrow Y$ between compacta is said to be an ε -mapping provided that for each $y \in Y$, $\text{diam } f^{-1}(y) < \varepsilon$ ($f^{-1}(y)$ may be empty). In this paper, “ ε -mapping” does not imply “surjective.” Let \mathfrak{B} be a given ANR-family. A compactum X is said to be *weak \mathfrak{B} -like* provided that for each $\varepsilon > 0$ there is an ε -mapping from X to some $P \in \mathfrak{B}$. A compactum is said to be *\mathfrak{B} -like* provided that for any $\varepsilon > 0$ there is an onto ε -mapping from X onto some $P \in \mathfrak{B}$.

First, we show the following

(3.1) THEOREM. *Let X be a continuum and let ω be any Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Suppose that $t_1 > t_2 > \dots$ is a decreasing sequence of positive numbers in $[0, \omega(X)]$ such that $t = \lim t_n$. If $\omega^{-1}(t_n)$ is weak \mathfrak{B} -like, then $\omega^{-1}(t)$ is also weak \mathfrak{B} -like.*

For the case $\mathfrak{S} = C(X)$, we have the following

(3.2) COROLLARY. *The property of being weak \mathfrak{B} -like is a sequential strong Whitney-reversible property.*

Note that X is weak arc-like if and only if X is arc-like (chainable), and X is weak tree-like if and only if X is tree-like. Hence by (3.2), we have the following which is an affirmative answer to [20, (14.57) and 21, (3.6)].

(3.3) COROLLARY. *The property of being chainable or tree-like is a sequential strong Whitney-reversible property.*

To prove (3.1), we need the following simple lemma. For completeness, we give the proof.

(3.4) LEMMA. *Let X be a subcompactum of a compactum Y . Let $f: X \rightarrow Z$ be an ε -mapping and let $\tilde{f}: Y \rightarrow Z$ be an extension of f . Then there is a closed neighborhood U of X in Y such that $\tilde{f}|U: U \rightarrow Z$ is an ε -mapping.*

PROOF. By the compactness of X , there is a positive number ε_1 such that $\varepsilon_1 < \varepsilon$ and $f: X \rightarrow Z$ is an ε_1 -mapping. Choose $\delta > 0$ such that if $A \subset Z$ and $\text{diam } A < \delta$, then $\text{diam } f^{-1}(A) < \varepsilon_1$. Take $\delta' > 0$ such that if $x, y \in Y$ and $d(x, y) < \delta'$, then $d(\tilde{f}(x), \tilde{f}(y)) < \delta/2$. Define a neighborhood U of X in Y by letting

$$U = \{y \in Y \mid d(y, x) \leq \min\{(\varepsilon - \varepsilon_1)/2, \delta'\} \text{ for some } x \in X\}.$$

We shall show that $\tilde{f}|U: U \rightarrow Z$ is an ε -mapping. Let $x, y \in U$ and $\tilde{f}(x) = \tilde{f}(y)$. Take points x' and y' of X such that $d(x, x') \leq \delta'$ and $d(y, y') \leq \delta'$. Then we have

$$d(f(x'), f(y')) \leq d(\tilde{f}(x'), \tilde{f}(x)) + d(\tilde{f}(y), \tilde{f}(y')) < \delta,$$

which implies that $d(x', y') < \varepsilon_1$. Also, we have

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y') + d(y', y) \\ &< (\varepsilon - \varepsilon_1)/2 + \varepsilon_1 + (\varepsilon - \varepsilon_1)/2 = \varepsilon. \end{aligned}$$

Hence $\tilde{f}|U: U \rightarrow Z$ is an ε -mapping.

PROOF OF (3.1). Let $\varepsilon > 0$. Assume that M is an ANR that contains \mathfrak{S} . By (2.4), there is a positive number $\eta(\varepsilon/3)$ such that if $t, t' \in [0, \omega(X)]$ and $0 \leq t' - t < \eta(\varepsilon/3)$, then $\omega^{-1}(t')$ is an $(\varepsilon/3)$ -approximate strong deformation retract of $\omega^{-1}([t, t'])$ in M . Choose t_n such that $t_n - t < \eta(\varepsilon/3)$. Since $\omega^{-1}(t_n)$ is weak \mathfrak{B} -like, there is an $(\varepsilon/3)$ -mapping $f: \omega^{-1}(t_n) \rightarrow P$ for some $P \in \mathfrak{B}$. Since P is an ANR, by (3.4) there is a neighborhood U of $\omega^{-1}(t_n)$ in M and there is an extension $\tilde{f}: U \rightarrow P$ of f which is an $(\varepsilon/3)$ -mapping. Let $r: \omega^{-1}([t, t_n]) \times [0, \infty) \rightarrow M$ be a mapping satisfying conditions (*), (**), and (***) . Choose a real s in $[0, \infty)$ such that

$$r(\omega^{-1}([t, t_n]) \times [s, \infty)) \subset U.$$

Set $g = r|_{\omega^{-1}(t) \times \{s\}}: \omega^{-1}(t) \rightarrow U$. Now, we shall show that $\tilde{f} \circ g: \omega^{-1}(t) \rightarrow P$ is an ε -mapping. Let $A, B \in \omega^{-1}(t)$ and $\tilde{f}g(A) = \tilde{f}g(B)$. Since \tilde{f} is an $(\varepsilon/3)$ -mapping, $d(g(A), g(B)) < \varepsilon/3$. Then we have

$$\begin{aligned} \rho_{\mathfrak{S}}(A, B) &\leq d(A, g(A)) + d(g(A), g(B)) + d(g(B), B) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Hence $\omega^{-1}(t)$ is weak \mathfrak{B} -like. This completes the proof.

Next, we shall prove the following

(3.5) COROLLARY. *The property of being circle-like is a sequential strong Whitney-reversible property. More precisely, the property of being proper circle-like or chainable and circle-like is a sequential strong Whitney-reversible property.*

To prove (3.5), we need the following definitions. A continuum X is said to be n -decomposable provided that X is the essential sum of n continua, i.e., $X = X_1 \cup X_2 \cup \cdots \cup X_n$, where X_i is a subcontinuum of X such that $X_i - \bigcup_{j \neq i} X_j \neq \emptyset$. A continuum X is said to be n -indecomposable [2] provided that X is n -decomposable

and not $(n + 1)$ -decomposable. Clearly, X is indecomposable if and only if X is 1-indecomposable.

(3.6) (C. E. BURGESS [2, THEOREM 7]). *If X is chainable, then in order that X should be circle-like, it is necessary and sufficient that X is either indecomposable or 2-indecomposable.*

(3.7) PROPOSITION. *Let X be n -decomposable and let w be any Whitney map for $C(X)$. Then there is $t_0 \in (0, \omega(X))$ such that if $t \leq t_0$, then $\omega^{-1}(t)$ is n -decomposable.*

PROOF. Let $X = A_1 \cup A_2 \cup \dots \cup A_n$ be the essential sum of n continua. Choose points a_i ($i = 1, 2, \dots, n$) of X such that $a_i \in A_i - \bigcup_{j \neq i} A_j$. Set $\varepsilon = \min\{d(a_i, \bigcup_{j \neq i} A_j) \mid i = 1, 2, \dots, n\} > 0$. Choose $t_0 \in (0, \omega(X))$ such that $A \in \omega^{-1}([0, t_0])$, then $\text{diam } A < \varepsilon$. Consider the following sets in $\omega^{-1}(t)$ for $t \leq t_0$.

$$X(A_i, \omega, t) = \left\{ K \in \omega^{-1}(t) \mid K \cap A_i \neq \emptyset \right\} \quad (i = 1, 2, \dots, n).$$

By [14, 3.2 Corollary], $X(A_i, \omega, t)$ is a subcontinuum of $\omega^{-1}(t)$. Then $\omega^{-1}(t) = \bigcup_{i=1}^n X(A_i, \omega, t)$ is the essential sum of n continua, which implies that $\omega^{-1}(t)$ is n -decomposable.

PROOF OF (3.5). By (3.2), we see that X is chainable or circle-like. Since the property of being chainable is a Whitney property [10], the property of being proper circle-like is a sequential strong Whitney-reversible property. Now, we must show that the property of being chainable and circle-like is a sequential strong Whitney-reversible property. By (3.6), it is sufficient to prove that (a) the property of being indecomposable and chainable or (b) the property of being 2-indecomposable and chainable is a sequential strong Whitney-reversible property. By (3.3) and (3.7), the properties (a) and (b) are sequential strong Whitney-reversible properties. This completes the proof.

J. Krasinkiewicz [12, 3.3 Corollary], proved that if X is circle-like, then for any Whitney map ω for $C(X)$, $\text{Sh}(X) = \text{Sh}(\omega^{-1}(t))$ for each $t \in [0, \omega(X))$. Hence, we have

(3.8) COROLLARY. *Let X be a continuum and let ω be any Whitney map for $C(X)$. Suppose that $t_1 > t_2 > \dots$ is a decreasing sequence of positive numbers in $[0, \omega(X))$ such that $\lim t_n = 0$. If $\omega^{-1}(t_n)$ is circle-like, then X is circle-like and $\text{Sh}(X) = \text{Sh}(\omega^{-1}(t))$ for each $0 \leq t < \omega(X)$.*

(3.9) EXAMPLE. In the statement of (3.1), we cannot conclude that if $\omega^{-1}(t_n)$ is \mathfrak{B} -like, then $\omega^{-1}(t)$ is \mathfrak{B} -like in the case $\mathfrak{S} = 2^X$. Consider the arc X in the plane as below (see Figure 5). Let $\omega: 2^X \rightarrow [0, \omega(X)]$ be the Whitney map for 2^X as defined by [31] and the usual Euclidean metric. Then there is a decreasing sequence $t_1 > t_2 > \dots$ of positive numbers in $[0, \omega(X))$ such that $\lim t_n = 0$ and $\omega^{-1}(t_n)$ is not connected. Hence there is a family \mathfrak{B} of compact polyhedra such that each member of \mathfrak{B} is not connected and $\omega^{-1}(t_n)$ is \mathfrak{B} -like ($n = 1, 2, \dots$) (see [17]). Since X is connected, X is not \mathfrak{B} -like.

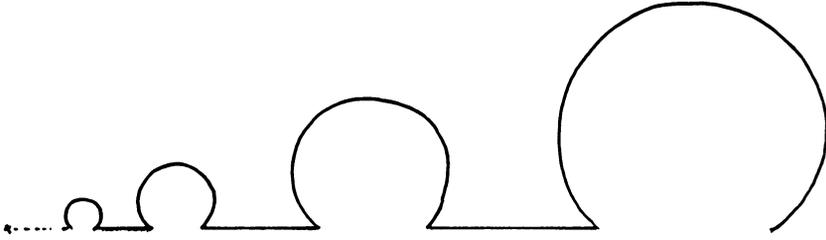


FIGURE 5

4. Inverse limits and approximations for Whitney maps. In this section, we consider the relation between inverse limits and Whitney maps.

In [22], A. Petrus showed that the property of being an FAR is not a Whitney property, more precisely, there exist a 2-dimensional FAR X (in fact, $X =$ a disk) and a Whitney map ω for $C(X)$ such that $\omega^{-1}(t)$ is not an FAR for some $t \in (0, \omega(X))$ (cf. (1.12)). Also, she proved that if X is a dendrite (= 1-dimensional AR), then for any Whitney map ω for $C(X)$, $\omega^{-1}(t)$ is contractible for each $t \in [0, \omega(X)]$. She had the following question [22]: Is the property of being contractible a Whitney property for 1-dimensional continua? In relation to this question, we prove the following

(4.1) THEOREM. *The property of being an FAR is a Whitney property for 1-dimensional continua.*

To prove (4.1), we need the following approximation theorem for Whitney map.

(4.2) THEOREM (APPROXIMATION FOR WHITNEY MAP). *Let $\underline{X} = \{X_n, p_{n,n+1}\}$ be an inverse sequence of continua and let $X = \varprojlim \underline{X}$. Suppose that ω is a Whitney map for $\mathfrak{S} = 2^X$ or $C(X)$. Then there exist Whitney maps ω_n ($n \geq 1$) for $\mathfrak{S}_n = 2^{X_n}$ or $C(X_n)$ satisfying the conditions: for any $\epsilon > 0$, there is n_0 such that*

$$d(\omega, \omega_n p_n^*) = \sup\{|\omega(A) - \omega_n p_n^*(A)| \mid A \in \mathfrak{S}\} < \epsilon \quad \text{and}$$

$$d(\omega_m p_{m,n}^*, \omega_n) < \epsilon \quad \text{for each } n \geq m \geq n_0,$$

where $p_n: X \rightarrow X_n$ denotes the projection and $p_n^*: \mathfrak{S} \rightarrow \mathfrak{S}_n$ denotes the mapping defined by $p_n^*(A) = p_n(A)$ for $A \in \mathfrak{S}$.

PROOF. Consider the Freudenthal space σX of X , i.e., σX is the set $X \cup \bigcup_{n=1}^\infty X_n$ with the topology defined by assuming that the totality of open subsets of the spaces X_n and sets of the form $p_n^{-1}(U) \cup \bigcup_{m \geq n} p_{n,m}^{-1}(U)$, where U is an open subset of X_n ($n \geq 1$), is an open base of σX . Then σX is a compact metric space (e.g., see J. Krasinkiewicz, *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. **92** (1976), p. 98). Assume that σX is lying in the Hilbert cube Q . By (1.2), there is a Whitney map $\mu: \mathfrak{S}' \rightarrow [0, \mu(Q)] \hookrightarrow [0, \infty)$ for $\mathfrak{S}' = 2^Q$ or $C(Q)$, which is an extension of ω . Set $\omega_n = \mu|_{\mathfrak{S}_n}$, where $\mathfrak{S}_n = 2^{X_n}$ or $C(X_n)$. Note that for

any $\varepsilon' > 0$, there is n_1 such that if $n \geq m \geq n_1$, then $d(i_X, p_n) < \varepsilon'$ and $d(i_{X_n}, p_{m,n}) < \varepsilon'$, where $i_X: X \rightarrow X \hookrightarrow \sigma X$ is the natural inclusion and $p_n: X \rightarrow X_n \hookrightarrow \sigma X$, etc. By using this fact and the continuity of μ , we can easily see that there exist n_0 satisfying the desired conditions.

PROOF OF (4.1). Since X is tree-like, there is an inverse sequence $\underline{X} = \{X_n, P_{n,n+1}\}$ such that each X_n is a tree and $X = \varprojlim \underline{X}$. Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ be a decreasing sequence of positive numbers such that $\sum \varepsilon_i < \infty$. By (4.2), without loss of generality, we may assume that there are Whitney maps ω_n for $C(X_n)$ such that

$$d(\omega, \omega_n p_n^*) < \varepsilon_n \quad \text{and} \quad d(\omega_n p_{n,m}^*, \omega_m) < \varepsilon_n \quad \text{for } m \geq n.$$

Let $t \in (0, \omega(X))$. Let $A \in \omega_{n+1}^{-1}([t - \sum_{i=n+1}^{\infty} \varepsilon_i, t + \sum_{i=n+1}^{\infty} \varepsilon_i])$. Then we have

$$\begin{aligned} |\omega_n p_{n,n+1}^*(A) - t| &\leq |\omega_n p_{n,n+1}^*(A) - \omega_{n+1}(A)| + |\omega_{n+1}(A) - t| \\ &\leq \varepsilon_n + \sum_{i=n+1}^{\infty} \varepsilon_i = \sum_{i=n}^{\infty} \varepsilon_i. \end{aligned}$$

Hence

$$p_{n,n+1}^*(A) \in \omega^{-1}\left(\left[t - \sum_{i=n}^{\infty} \varepsilon_i, t + \sum_{i=n}^{\infty} \varepsilon_i\right]\right).$$

This implies that

$$p_{n,n+1}^* \omega_{n+1}^{-1}\left(\left[t - \sum_{i=n+1}^{\infty} \varepsilon_i, t + \sum_{i=n+1}^{\infty} \varepsilon_i\right]\right) \subset \omega_n^{-1}\left(\left[t - \sum_{i=n}^{\infty} \varepsilon_i, t + \sum_{i=n}^{\infty} \varepsilon_i\right]\right).$$

Consider the following inverse sequence $\omega^{-1}(t)$;

$$\omega_1^{-1}([t - \alpha_1, t + \alpha_1]) \xleftarrow{p_{12}^*} \omega_2^{-1}([t - \alpha_2, t + \alpha_2]) \xleftarrow{p_{23}^*} \omega_3^{-1}([t - \alpha_3, t + \alpha_3]) \leftarrow \dots,$$

where $\alpha_n = \sum_{i=n}^{\infty} \varepsilon_i$.

Let $A \in \omega^{-1}(t)$. Then we have

$$|\omega_n p_n^*(A) - t| = |\omega_n p_n^*(A) - \omega(A)| < \varepsilon_n,$$

which implies that

$$p_n^*(\omega^{-1}(t)) \subset \omega_n^{-1}([t - \varepsilon_n, t + \varepsilon_n]) \subset \omega^{-1}([t - \alpha_n, t + \alpha_n]).$$

Let $(A_n)_n \in \varprojlim \omega^{-1}(t)$. Set $A = \varprojlim \{A_n, p_{n,n+1} | A_{n+1}\}$. Then $A \in C(X)$. Also,

$$\begin{aligned} |\omega(A) - t| &\leq |\omega(A) - \omega_n p_n^*(A)| + |\omega_n p_n^*(A) - t| \\ &< \varepsilon_n + \alpha_n \quad \text{for each } n. \end{aligned}$$

If $n \rightarrow \infty$, then $\varepsilon_n + \alpha_n \rightarrow 0$. Hence $\omega(A) = t$. Therefore, we conclude that $\omega^{-1}(t) = \varprojlim \omega^{-1}(t)$. Since each X_n is a tree, by the proof of [22, Proposition 12 or 5, (2.17)], we can see that $\omega_n^{-1}([t - \alpha_n, t + \alpha_n])$ is an AR for each $n = 1, 2, \dots$. Hence $\omega^{-1}(t)$ is an FAR. This completes the proof.

5. Problems. In this section, we give the following problems.

(5.1) Is it true that the movability is a Whitney property?

(5.2) Is it true that the property of being pointed 1-movable, or movable is a (sequential) strong Whitney-reversible property?

(5.3) Is it true that the property of being \mathfrak{B} -like is a sequential strong Whitney-reversible property?

(5.4) Is the property of being an FANR a Whitney property for 1-dimensional continua?

ADDED IN PROOF. Recently, the author proved that (5.1) has a negative answer.

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