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ON $R^\infty(Q^\infty)$-MANIFOLD BUNDLES OVER CW COMPLEXES

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ABSTRACT. Let $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ be a pseudo CW complex generated either by Hausdorff compact spaces or by metric spaces. In the theory of manifolds modeled on R^∞ or Q^∞, we will prove the Λ-fiber-preserving versions of the following: Equivalences among the notions of D-sets, D^*-sets and infinite deficient sets; relative stability theorems; relative deformation of homotopy equivalences to homeomorphisms; strong unknotting theorem for D-embeddings; and α-approximation theorems.

0. Introduction. Let $E^\infty = \operatorname{dir\,lim}\{E^n\}$ where E denotes either the real line R or the Hilbert cube Q. Separable paracompact manifolds modeled on R^∞ and Q^∞ have been studied by R. E. Heisey, K. Sakai and the author. Many results which are similar to those in the l_2-manifold theory have been established such as: Open Embedding Theorem, Stability Theorem, Collaring Theorem, Unknotting Theorem, Classification Theorem, α-Approximation Theorem, etc.

In this note, first we will prove in §3 a canonicality of the deformation of homotopy equivalences to homeomorphisms (Theorem 3.4) and a fibered stability theorem (Corollary 3.7) as a special case of Theorem 3.6; then, following K. Sakai [**Sa**$_2$], we will use the notions of f.p. D-sets and f.p. D^*-sets introduced in §2 to prove a fibered collaring theorem (Theorem 4.3); and a strong fibered unknotting theorem (Theorem 5.3); finally, we will obtain a fibered α-approximation theorem (Thoerem 6.1).

Given a CW complex X, let $\operatorname{Homeo}(X)$ and $\operatorname{Hom}(X)$ denote the group of homeomorphisms and the semigroup of homotopy equivalences of X, respectively, endowed with the compact-open topology. Some time ago, P. J. Kahn asked whether $\operatorname{Hom}(X \times R^\infty)$ has $\operatorname{Homeo}(X \times R^\infty)$ as a deformation retract or whether the inclusion is a weak homotopy equivalence. Lemma 3.3 (or Theorem 3.4) provides an affirmative answer to the second question for countable CW complexes. Also, by use of Lemma 3.3, the author can prove in [**L**$_5$] a local strong connectedness of $\operatorname{Homeo}(X \times E^\infty)$ if X is a finite complex.

All spaces are Hausdorff. For a subset A of X, let i_A denote the inclusion map $A \hookrightarrow X$ and id_X the identity of X. For undefined notions and notations, refer to [**D**].

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By a *G_δ-expanding system* $\Lambda_0 \subset \Lambda_1 \subset \cdots$, we mean that each Λ_n is a closed G_δ-set in Λ_{n+1}. We will write

$\mathscr{N}$ = the class of all normal spaces,

$\mathscr{M}$ = the class of all metric spaces,

$\mathscr{C}$ = the class of all compact spaces,

$\mathscr{PC}$ = the class of all perfectly normal compact spaces,

$\Sigma(\mathscr{X})$ = the class of all topological sums of spaces in the class $\mathscr{X}$,

$\mathscr{DL}(\mathscr{X})$ = the class of all direct limit spaces of G_δ-expanding systems of spaces in $\mathscr{X}$,

$\mathscr{CW}(\mathscr{X})$ = the class of all pseudo CW complexes generated by the class $\mathscr{X}$.

By a *pseudo* CW *complex generated by the class* $\mathscr{X}$, we mean the direct-limit space of a G_δ-expanding system $\Lambda_0 \subset \Lambda_1 \subset \cdots$ where $\Lambda_0 \in \Sigma(\mathscr{X})$ and Λ_n is obtained inductively by attaching a member Y of $\Sigma(\mathscr{X})$ to Λ_{n-1} with a characteristic map η: $(Y, Z) \to (\Lambda_n, \Lambda_{n-1})$ where Z is a closed subset of Y.

Observe that every ordinary CW complex endowed with the Whitehead topology belongs to $\mathscr{CW}(\mathscr{PC}) \cap \mathscr{CW}(\mathscr{M})$, and $\mathscr{DL}(\mathscr{X}) \subset \mathscr{CW}(\mathscr{X})$ for $\mathscr{X} = \mathscr{N}$, $\mathscr{M}$, $\mathscr{C}$ or $\mathscr{PC}$. Moreover, that $\mathscr{CW}(\mathscr{M})$ is the class of M-spaces in [**Hy**].

Throughout this note, $I = [0,1]$ and $\mathring{I} = (0,1)$. By an *E^∞-manifold bundle* (X, π, Λ), we mean a locally trivial bundle map π: $X \to \Lambda$ whose fiber is an E^∞-manifold. If it is not specified otherwise, the base space Λ will belong to $\mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$. If $\Lambda \in \mathscr{CW}(\mathscr{PC}) \cup \mathscr{CW}(\mathscr{M})$, all involved spaces in our proofs will be perfectly normal by Corollary 0.4; therefore, the G_δ-properties of closed subsets are surplus in the definitions of D-sets and D^*-sets in §2, and in our main results. However, we will state our results with G_δ-properties to cover the case $\Lambda \in \mathscr{CW}(\mathscr{C}) \setminus \mathscr{CW}(\mathscr{PC})$. By $X \cong Y$, we mean that X and Y are homeomorphic. If we do not specify otherwise, we will work with the category $\mathscr{CG}$ whose objects are compactly generated spaces and whose morphisms are continuous maps [**Gr**, 8.1]. The category $\mathscr{CG}$ contains locally compact spaces and metrizable spaces [**Gr**, Proposition 8.3]. Given two spaces X and Y in $\mathscr{CG}$, following [**Gr**, 8.9] we let $X \times Y$ and $X \times_c Y$ denote their $\mathscr{CG}$-product space and their usual Cartesian product space, respectively. Note that if either X or Y is locally compact, then $X \times Y \cong X \times_c Y$ [**Gr**, Theorem 8.11]. Moreover, it follows from Theorem 8.8(i) [**Gr**] that each open cover of $X \times_c Y$ is also an open cover of $X \times Y$; so, we usually refer to open covers of $X \times Y$ if we do not indicate otherwise. Recall that each member of $\mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ is a $\mathscr{CG}$ space (by use of [**Ha**, Lemma 2.4]), e.g., E^∞-manifolds are $\mathscr{CG}$ spaces. We will refer to [**Gr**] for standard results in the category $\mathscr{CG}$.

If there is no danger of confusion, we will let $(E, 0)$ denote either $(R, 0)$ or $(I, 0)$ or $(Q, 0)$; and for simplicity, identify E^n with $E^n \times 0$ in $E^\infty = \operatorname{dir\,lim} E^n$. Recall that $(R^\infty, 0) \cong (I^\infty, 0)$ [**L**$_3$]. Given an R^∞ (or Q^∞)-manifold M, following [**L**$_2$ and **L**$_4$], we will write $M = \operatorname{dir\,lim} M_n$, where M_n is a flat compact submanifold of $\mathring{M}_{n+1}$ (or a compact-Q-manifold Z-set in M_{n+1}) if we do not indicate otherwise.

Given two spaces S and T, let π_S and π_T denote the projections from $S \times T$ to S and T, respectively. If (X, π, Λ) is a bundle, A a subset of X and P a subset of Λ, we write $A_P = A \cap \pi^{-1}(P)$; specially, $X_P = \pi^{-1}(P)$. For simplicity we will write A_λ instead of $A_{\{\lambda\}}$. Let (X, π, Λ) and (Y, π', Λ) be bundles and A a subset of X. A map $f\colon A \to Y$ is said to be Λ-*fiber preserving* (f.p.) if $\pi' f = \pi|A$. If $f\colon A \to \Lambda \times Z$ is an f.p. map, for each $\lambda \in \Lambda$, let $f_\lambda\colon A_\lambda \to Z$ denote the map $\pi_Z f|A_\lambda$. By an f.p. *ambient invertible isotopy* of X, we mean an $(I \times \Lambda)$-f.p. homeomorphism G of $I \times X$ with $G_0 = \mathrm{id}_X$. A subset A of X is said to be *compactly proper* if A_P is compact for each compact subset P of Λ. For example, if K is a compact subset of an E^∞-manifold M, then $\Lambda \times K$ is a compactly proper G_δ-set in $\Lambda \times M$.

LEMMA 0.1. *Let X and Y be total spaces of bundles in $\mathscr{CG}$.*

(1) *A compactly proper set in X is a closed subset of X.*

(2) *The finite union of compactly proper sets is a compactly proper set.*

(3) *A closed subset of a compactly proper set is a compactly proper set.*

(4) *If $f\colon A \to Y$ is an f.p. map where A is a compactly proper subset of X, then f is a closed map and $f(A)$ is compactly proper.*

(5) *Let $\eta\colon \Gamma \to \Lambda$ be a continuous map and A a compactly proper subset of X, the total space of (X, π, Λ). Then the pull-back $\eta^*(A) = \{(\gamma, x) \in \Gamma \times A | \eta(\gamma) = \pi(x)\}$ is compactly proper over Γ.*

PROOF. The proof of (1), (2), (3) and (5) are straightforward. To prove (4), given a closed subset C of A and a compact subset P of Λ, since f is Λ-fiber preserving, we have $[f(C)]_P = f(C_P)$. By (3), it is compact. Hence, $f(C)$ is a compactly proper set; so, it is closed in Y by (1). □

Let $f\colon C \to D$ be a map such that $f|A$ is an embedding where A is a subset of C, and let B be a subset of $f(A)$. We will let $f^{-1}|B$ denote the inverse of the homeomorphism $f|\colon f^{-1}(B) \cap A \to B$ if no ambiguity occurs.

Given an open cover $\mathscr{U}$ of a space Z, a homotopy $H\colon W \times I \to Z$ is a $\mathscr{U}$-*homotopy* if for each $w \in W$ the set $H(w \times I)$ is contained in some $U \in \mathscr{U}$. A homotopy H is *stationary on* A or rel A, where $A \subset W$, if $H(w \times I) = \{H(w, 0)\}$ for each $w \in A$. Two maps $f, g\colon X \to Z$ are $\mathscr{U}$-*close* if for each $x \in X$ there is a $U \in \mathscr{U}$ such that $\{f(x), g(x)\} \subset U$. Let (X, π, Λ) and (Y, π', Λ) be fiber bundles. Let $\mathscr{U}$ be an open cover of Y and $A \subset X$. Two f.p. maps $f, g\colon A \to Y$ are f.p. $\mathscr{U}$-*homotopic* ($f \overset{\mathscr{U}}{\simeq} g$ (f.p.)) if there is an f.p. $\mathscr{U}$-homotopy from f to g. An f.p. map $f\colon X \to Y$ is an f.p. rel A $\mathscr{U}$-*homotopy equivalence* provided that $f|A$ is an f.p. embedding and that there is an f.p. map $g\colon Y \to X$ such that (a) $g|f(A) = f^{-1}|f(A)$, (b) $fg \overset{\mathscr{U}}{\simeq} \mathrm{id}$ (f.p.) rel $f(A)$ and (c) $gf \overset{f^{-1}(\mathscr{U})}{\simeq} \mathrm{id}$ (f.p.) rel A where $f^{-1}(\mathscr{U}) = \{f^{-1}(U) | U \in \mathscr{U}\}$. If f is an f.p. rel A $\mathscr{U}$-homotopy equivalence for every open cover $\mathscr{U}$ of Y, it is called an f.p. rel A *fine homotopy equivalence*. For each open cover $\mathscr{U}$ of Y, if there is an f.p. homeomorphism $h\colon X \to Y$ such that h is $\mathscr{U}$-close to f and $h|A = f|A$, the map f is called an f.p. rel A *near homeomorphism*. For more details, refer to [F].

Let $\mathscr{U}, \mathscr{V}$ be families of subsets of X and $U \in \mathscr{U}$; we write $\mathrm{st}(U, \mathscr{V}) = \bigcup\{V | V \in \mathscr{V},\ U \cap V \neq \varnothing\}$, $\mathrm{st}(\mathscr{U}, \mathscr{V}) = \{\mathrm{st}(U, \mathscr{V}) | U \in \mathscr{U}\}$, and inductively $\mathrm{st}^{n+1}(\mathscr{U}, \mathscr{V}) = \mathrm{st}(\mathrm{st}^n(\mathscr{U}, \mathscr{V}), \mathscr{V})$ for $n \geqslant 1$. By $\mathscr{U} \overset{*}{<} \mathscr{V}$, we mean that $\mathrm{st}(\mathscr{U}) \equiv \mathrm{st}(\mathscr{U}, \mathscr{U})$ is a

refinement of $\mathscr{V}$. If A, B are subsets of X, $A \setminus B$ denotes the complement of $A \cap B$ in A. For the definition of nbd-*finite families*, refer to [**D**, p. 81].

LEMMA 0.2. *Let* $X = \operatorname{dir}\lim\{X_0 \subset X_1 \subset \cdots\} \in \mathscr{CW}(\mathscr{N})$. *If A is a closed subset of X such that each $A_n = A \cap X_n$ is a G_δ-set in X_n, then A is a G_δ-set in X.*

PROOF. Write $A_n^* = A_n \cup X_{n-1}$ $(n \geqslant 1)$. Let h_0: $X_0 \to I$ be a map such that $h_0^{-1}(0) = A_0$ [**D**, Corollary VII.4.2]. Since A_1^* is a G_δ-set in X_1 [**E**, p. 45], there is a map ϕ: $X_1 \to I$ with $\phi^{-1}(0) = A_1^*$.

Moreover, by use of Tietze's extension theorem, we can obtain a map f: $X_1 \to I$, an extension of h_0, such that $f(x) = 0$ if $x \in A_1$. Define h_1: $X_1 \to I$ by $h_1(x) = f(x) + \phi(x)$. Then, for $x \in X_0$, $h_1(x) = f(x) + \phi(x) = h_0(x)$; and $h_1^{-1}(0) = f^{-1}(0) \cap \phi^{-1}(0) = A_1$.

Similarly, we can define by induction a sequence of maps h_n: $X_n \to I$ such that h_n is an extension of h_{n-1} with $h_n^{-1}(0) = A_n$. Define h: $X \to I$ by $h(x) = h_n(x)$ if $x \in X_n$. Then, h is continuous and $h^{-1}(0) = A$. □

LEMMA 0.3. *If $M = \operatorname{dir}\lim\{M_n\}$ is an E^∞-manifold, then* $\operatorname{dir}\lim\{\Lambda \times M_n\} \cong \Lambda \times M \cong \operatorname{dir}\lim\{\Lambda_n \times M_n\}$. *Consequently, if $\Lambda \in \mathscr{CW}(\mathscr{C})$ ($\mathscr{CW}(\mathscr{PC})$, $\mathscr{CW}(\mathscr{M})$, resp.), then so does $\Lambda \times M$, respectively.*

PROOF. The first homeomorphism $\operatorname{dir}\lim\{\Lambda \times M_n\} \cong \Lambda \times M$ follows from [**Gr**, Ex. 11, p. 129]. For the second, observe that id: $\operatorname{dir}\lim\{\Lambda_n \times M_n\} \to \Lambda \times M$ is continuous; moreover, its inverse is also continuous since $\Lambda \times M \in \mathscr{CG}$ and each of its compact subsets is contained in some $\Lambda_n \times M_n$ [**Ha**, Lemma 2.4]. □

COROLLARY 0.4. *Let (X, π, Λ) be an E^∞-manifold bundle.*

(1) *If $\Lambda \in \mathscr{CW}(\mathscr{PC}) \cup \mathscr{CW}(\mathscr{M})$, then X is perfectly normal and hereditarily paracompact.*

(2) *If $\Lambda \in \mathscr{CW}(\mathscr{C})$, then X is paracompact.*

PROOF. It follows from [**Hu**, Proposition I.2.1 and 2.3(iv), (v)] and [$\mathbf{M_2}$, Theorem 8.2 and its remark] that Λ and $\Lambda \times M$ (see Lemma 0.3) are perfectly normal (paracompact, resp.) in (1) (in (2), resp.), where M is an E^∞-manifold. Let β denote the property that the given total space X over Λ is perfectly normal (paracompact, resp.). Then, from above, β holds true locally over Λ by the local triviality of (X, π, Λ). Applying [$\mathbf{M_1}$, Theorem 5.5] to Λ and using [**Hu**, Proposition I.2.3(iv), (v)], we can infer that β holds true globally over Λ; therefore, the corollary is proved. □

The following is an f.p. version of [**K1**, Theorem 3.3] and its Hilbert-cube version is [**TW**, Theorem A.2.9]. So, we will omit its proof.

LEMMA 0.5. *Let Λ be a paracompact space, K a compact subset of E^k and f: $\Lambda \times_c K \to \Lambda \times_c E^n$ an f.p. embedding. Then, there is an f.p. ambient isotopy G_t $(t \in I)$ of $\Lambda \times_c E^n \times_c E^k$ such that $G_1(f(\lambda, x), 0) = (\lambda, 0, x)$ for all $x \in K$.* □

1. Some technical lemmas. In this section we will prove some technical lemmas needed in the sequel. Recall that $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$.

LEMMA 1.1. *Let M and N be E^∞-manifolds and $\mathscr{U}$ an open cover of $\Lambda \times M$. Let A be a closed subset of $\Lambda \times N$ and Γ a closed subset of Λ. Let h: $A \times I \to \Lambda \times M$ be an f.p. $\mathscr{U}$-homotopy such that h_0: $A \to \Lambda \times M$ has an f.p. extension f: $\Lambda \times N \to \Lambda \times M$. Then, h has an f.p. $\mathscr{U}$-homotopy extension $\tilde{h}$: $\Lambda \times N \times I \to \Lambda \times M$ with $f = \tilde{h}_0$. Moreover, if h is stationary on $A \cap (\Gamma \times N)$, then $\tilde{h}$ can be chosen to be stationary on $\Gamma \times N$. If $M = N$, h is stationary on $A \cap (\Gamma \times N)$ and $f|\Gamma \times N = i_{\Gamma \times N}$, then $\tilde{h}_t$ can be chosen to be identity on $\Gamma \times N$.*

PROOF. Observe that E^∞ is an AE (absolute extensor) for $\mathscr{C} \cup \mathscr{M}$. It follows from the proof of [**Hy**, Theorem 10.1] that E^∞ is an AE($\mathscr{C}\mathscr{W}(\mathscr{C})$) and an AE($\mathscr{C}\mathscr{W}(\mathscr{M})$). Recall that $\Lambda \times N \in \mathscr{C}\mathscr{W}(\mathscr{C}) \cup \mathscr{C}\mathscr{W}(\mathscr{M})$ by Lemma 0.3. Now, the rest of the proof is an easy f.p. version of the proof of [**Hu**, Theorem IV.2.2]. □

In the following, we write $cQ = (Q \times [0,2])/(Q \times 0)$ (the cone over Q), and $\dot{c}Q = cQ \setminus (Q \times \{2\})$. As justified by [**Ch**, Theorem 12.2], $(cQ, c) \cong (Q, 0)$.

LEMMA 1.2. *Let M and N be E^∞-manifolds and $A \subset B$ compactly proper subsets of $\Lambda \times N$ such that A is a G_δ-set in B. If $\Lambda \in \mathscr{C}\mathscr{W}(\mathscr{M})$, assume in addition that $B \subset \Lambda \times K$ for some compact subset K of N. Let $\mathscr{U}$ be an open cover of $\Lambda \times M$ and g: $B \to \Lambda \times M$ an f.p. map. Then, there is an f.p. map f: $B \to \Lambda \times M$ such that*

(a) *$f|(B \setminus A)$ is an embedding,*

(b) *$f|A = g|A$, and,*

(c) *$f \underset{\mathscr{U}}{\simeq} g$ (f.p.) rel A.*

Moreover, if $g|A$ is an embedding, then so is f.

PROOF. *Case* 1. B is compact. (In this case, we only need $\Lambda \in \mathscr{C}\mathscr{G}$ and paracompact.) Let D^k denote $[-k,k]^k$ (or Q^k). Write $M = \operatorname{dir\,lim}\{M_k\}$, $N = \operatorname{dir\,lim}\{N_k\}$ and $\tilde{M}_k = M_k \times E^k$. By stability theorems [**He$_1$, He$_2$**], we have $M \cong M \times E^\infty \cong \operatorname{dir\,lim}\{M'_k\}$ where $M'_k = M_k \times D^k$.

Let q be an integer such that $B \subset \Lambda \times N_q$ and $g(B) \subset \Lambda \times M'_q \subset \Lambda \times \tilde{M}_q$ [**Ha**, Lemma 2.4]. Let θ: $N_q \to S$ be an embedding where $S = S^{n-1}$ for some large n (or $S = Q \times \{1\} \subset cQ$). Since A is a G_δ-set in B, there is a map ϕ: $B \to I$ such that $\phi^{-1}(0) = A$ [**D**, Corollary VII.4.2].

Define f': $B \to \Lambda \times \tilde{M}_q \times E^n$ (recall that $(Q^n, 0) \cong (cQ, c)$) by $f'(\lambda, x) = (\pi_{\Lambda \times \tilde{M}_q} g(\lambda, x),\ \phi(\lambda, x)\theta(x))$. Then, f' satisfies (a) and (b). Moreover, from the paracompactness of $\Lambda \times \tilde{M}_q$, there is a map α: $\Lambda \times \tilde{M}_q \to (0,1]$ such that $\{\{z\} \times B(0;\ \alpha(z)) | z \in \Lambda \times \tilde{M}_q\} < \mathscr{U}$, where $B(0;\ \varepsilon)$ denotes the closed ε-neighborhood of 0 in R^n (or in $Q^n \equiv cQ$). Let h be a $(\Lambda \times \tilde{M}_q)$-f.p. homeomorphism of $\Lambda \times \tilde{M}_q \times E^n$ such that $h(\{z\} \times B(0;\ 1)) = \{z\} \times B(0;\ \alpha(z))$ for each $z \in \Lambda \times \tilde{M}_q$. Then, $f = jhf'$, where j: $\Lambda \times \tilde{M}_q \times E^n \to \Lambda \times M \times E^\infty$ is the inclusion, is a desired f.p. map.

Case 2. $\Lambda \in \mathscr{M}$ and $B \subset \Lambda \times K$ for some compact set K of N. Since M is homeomorphic to an open subset of E^∞ [**He$_1$, He$_2$**] and $g(B)$ is closed in $\Lambda \times M$ by Lemma 0.1, without loss of generality, we can assume that $M = E^\infty$ and $B = \Lambda \times K$. Let $\mathscr{U} \overset{*}{>} \mathscr{U}_1 \overset{*}{>} \mathscr{U}_2 \overset{*}{>} \cdots$ and $\mathscr{V}_n = \operatorname{st}(\mathscr{V}_{n-1}, \mathscr{U}_n)$ with $\mathscr{V}_1 = \mathscr{U}_1$ sequences of open covers of $\Lambda \times M$. Observe that $\mathscr{V}_n < \mathscr{U}$ for each n. By [**D**, Theorem 4.4, p. 417], for each $(\lambda, x) \in \Lambda \times K$, there are an open neighborhood $\omega \times V$ of

(λ, x) in $\Lambda \times K$ and an integer n such that $g(\omega \times V) \subset \Lambda \times E^n$. Then, by use of the compactness of K, for each $\lambda \in \Lambda$, we can obtain an open neighborhood ω_λ of λ in Λ and an integer n_λ such that $g(\omega_\lambda \times K) \subset \Lambda \times E^{n_\lambda}$. Then, let $\Omega = \{U_\alpha | \alpha \in \mathscr{A}\}$ be a σ-discrete, nbd-finite open refinement of $\{\omega_\lambda | \lambda \in \Lambda\}$ [**K**, Theorem 5.28], and let $\mathscr{F} = \{W_\alpha | \alpha \in \mathscr{A}\}$ be a precise open refinement of Ω [**D**, Theorem VIII.2.3] such that $\overline{W}_\alpha \equiv \mathrm{cl}(W_\alpha) \subset U_\alpha$ for each $\alpha \in \mathscr{A}$. Let $\{\mathscr{A}_1, \mathscr{A}_2, \dots\}$ be a countable partition of $\mathscr{A}$ such that each $\Omega_n = \{U_\alpha | \alpha \in \mathscr{A}_n\}$ is a discrete open family. For each $\alpha \in \mathscr{A}_n$, define $\tilde{U}_\alpha = U_\alpha \setminus \bigcup_{j=1}^{n-1}(\bigcup_{\beta \in \mathscr{A}_j} \overline{W}_\beta)$; this is an open set by [**D**, Theorem III.9.2]. So, $\tilde{\Omega}_n = \{\tilde{U}_\alpha | \alpha \in \mathscr{A}_n\}$ is also a discrete open family. For each n, write $\mathcal{O}_n = \bigcup_{\alpha \in \mathscr{A}_n} U_\alpha$ and $\tilde{\mathcal{O}}_n = \bigcup_{\alpha \in \mathscr{A}_n} \tilde{U}_\alpha$ $(n \geqslant 2)$. Observe that $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n = (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{n-1}) \cup \tilde{\mathcal{O}}_n$ for $n \geqslant 2$.

SUBLEMMA. *For each open subset U of an ω_λ above and each $\mathscr{U}_j$, there is an f.p. map f^U: $\Lambda \times K \to \Lambda \times E^\infty$ such that $f^U|(U \times K) \setminus A$ is an embedding and $f^U \overset{\mathscr{U}_j}{\simeq} g$ (f.p.) rel $A \cup [(\Lambda \setminus U) \times K]$.*

PROOF. Assume that $K \subset R^{\tilde{p}}$ if $E = R$. Define $p = \tilde{p} + 1$ if $E = R$, and $p = 1$ if $E = Q$. As in the proof of Case 1, let θ: $K \to S \subset E^p$ be an embedding, and ϕ: $\Lambda \times K \to I$ a map with $\phi^{-1}(0) = A \cup [(\Lambda \setminus U) \times K]$. Define f^U: $\Lambda \times K \to \Lambda \times E^{n_\lambda + p}$ by

$$f_\lambda^U(x) = (\pi_{E^{n_\lambda}} g(\lambda, x), \phi(\lambda, x)\theta(x)).$$

If it is necessary, we can adjust ϕ to have $f^U \overset{\mathscr{U}_j}{\simeq} g$ (f.p.) rel $A \cup [(\Lambda \setminus U) \times K]$. Moreover, since F^U is a closed map by Lemma 0.1(4), $f^U|(U \times K) \setminus A$ is an embedding. □

Now, for each $\alpha \in \mathscr{A}_1$, let f^{U_α}: $\Lambda \times K \to \Lambda \times E^\infty$ be an f.p. map given by Sublemma for g and $\mathscr{V}_1 = \mathscr{U}_1$. Define f^1: $\Lambda \times K \to \Lambda \times E^\infty$ by

$$f^1(\lambda, x) = \begin{cases} f^{U_\alpha}(\lambda, x) & \text{if } \lambda \in U_\alpha \text{ for some } \alpha \in \mathscr{A}_1, \\ g(\lambda, x) & \text{otherwise.} \end{cases}$$

Then, since Ω_1 is discrete, f^1 is a well-defined f.p. map such that

$(1)_1$ $f^1|(\mathcal{O}_1 \times K) \setminus A$ is an embedding, and

$(2)_1$ $f^1 \overset{\mathscr{V}_1}{\simeq} g$ (f.p.) rel $A \cup [(\Lambda \setminus \mathcal{O}_1) \times K]$.

Next, assume by induction that we have already defined an f.p. map f^{n-1}: $\Lambda \times K \to \Lambda \times E^\infty$ such that

$(1)_{n-1}$ $f^{n-1}|[\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{n-1}) \times K] \setminus A$ is an embedding, and

$(2)_{n-1}$ $f^{n-1} \overset{\mathscr{V}_{n-1}}{\simeq} g$ (f.p.) rel $A \cup [(\Lambda \setminus (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{n-1})) \times K]$.

We now define f^n having the corresponding properties $(1)_n$ and $(2)_n$. For each $\alpha \in \mathscr{A}_n$, let $f^{\tilde{U}_\alpha}$: $\Lambda \times K \to \Lambda \times E^\infty$ be an f.p. map given by Sublemma for f^{n-1} and $\mathscr{U}_n$, and define f^n: $\Lambda \times K \to \Lambda \times E^\infty$ by

$$f^n(\lambda, x) = \begin{cases} f^{\tilde{U}_\alpha}(\lambda, x) & \text{if } \lambda \in \tilde{U}_\alpha \text{ for some } \alpha \in \mathscr{A}_n, \\ f^{n-1}(\lambda, x) & \text{otherwise.} \end{cases}$$

Then, since $\tilde{\Omega}_n$ is discrete, f^n is a well-defined f.p. map such that

$(1)_n$ $f^n|[(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n) \times K] \setminus A$ is an embedding (by use of Lemma 0.1(4)) and

$(2)'_n$ $f^n \overset{\mathcal{U}_n}{\simeq} f^{n-1}$ (f.p.) rel $A \cup [(\Lambda \setminus \tilde{\mathcal{O}}_n) \times K]$.

Consequently,

$(2)_n$ $f^n \overset{\mathcal{V}_n}{\simeq} g$ (f.p.) rel $A \cup [(\Lambda \setminus (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n)) \times K]$ since $\mathcal{V}_n = \mathrm{st}(\mathcal{V}_{n-1}, \mathcal{U}_n)$.

Finally, define f: $\Lambda \times K \to \Lambda \times E^\infty$ by $f(\lambda, x) = f^n(\lambda, x)$ if n is the least integer such that $\lambda \in \overline{W}_\alpha$ for some $\alpha \in \mathcal{A}_n$. Then, f is a well-defined (by $(2)'_n$) and continuous (by the nbd-finiteness of Ω) f.p. map, and $f \overset{\mathcal{U}}{\simeq} g$ rel A since $\mathcal{V}_n < \mathcal{U}$ for each n. Moreover, $f|(\Lambda \times K) \setminus A$ is an embedding since f is a closed map by Lemma 0.1(4).

Case 3. General case. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be a sequence of open covers of $\Lambda \times M$ such that $\mathcal{U} \overset{*}{>} \mathcal{U}_0 \overset{*}{>} \mathcal{U}_1 \overset{*}{>} \mathcal{U}_2 \overset{*}{>} \cdots$. Define inductively $\mathcal{V}_0 = \mathcal{U}_0$ and $\mathcal{V}_n = \mathrm{st}(\mathcal{V}_{n-1}, \mathcal{U}_n)$ for $n \geqslant 1$ as in Case 2. Let $\Lambda = \mathrm{dir\,lim}\{\Lambda_0 \subset \Lambda_1 \subset \cdots\}$ as in the Introduction, and write $B_n = B_{\Lambda_n} \equiv B \cap (\Lambda_n \times N)$ and $A_n = A_{\Lambda_n}$ (a G_δ-set in B_n). First, we will define, by induction on n, two sequences $\{\tilde{f}_n\}_{n=0}^\infty$, $\{f_n\}_{n=0}^\infty$ of f.p. maps where $\tilde{f}_n, f_n$: $B_n \to \Lambda_n \times M$, and three sequences $\{F^n\}_{n=0}^\infty$, $\{H^n\}_{n=1}^\infty$, $\{G^n\}_{n=0}^\infty$ of f.p. homotopies such that

$(1)_n$ $f_n|(B_n \setminus A_n)$ is an embedding,

$(2)_n$ $\tilde{f}_n$ is an extension of f_{n-1} for $n \geqslant 1$,

$(3)_n$ F^n: $B_n \times [1 - 1/2^n, 1 - 1/2^{n+1}] \to \Lambda_n \times M$ is an f.p. $\mathcal{U}_n$-homotopy from $\tilde{f}_n$ to f_n (rel $B_{n-1} \cup A_n$),

$(4)_n$ H^n: $B_n \times [0, 1 - 1/2^n] \to \Lambda_n \times M$ is an f.p. $\mathcal{V}_{n-1}$-homotopy from $g|B_n$ to $\tilde{f}_n$ (rel A_n), that is an extension of G^{n-1}, and

$(5)_n$ G^n: $B_n \times [0, 1 - 1/2^{n+1}] \to \Lambda_n \times M$ is the f.p. $\mathcal{V}_n$-homotopy defined by: $G_t^n = H_t^n$ for $0 \leqslant t \leqslant 1 - 1/2^n$ $(n \geqslant 1)$ and $G_t^n = F_t^n$ for $1 - 1/2^n \leqslant t \leqslant 1 - 1/2^{n+1}$ $(n \geqslant 0)$.

Next, define f: $B \to \Lambda \times M$ by $f(x) = f_n(x)$ if $x \in B_n$. Then, f satisfies (a) and (b) by use of $(1)_n$, $(2)_n$ and $(3)_n$. Moreover, the homotopy H: $B \times I \to \Lambda \times M$, defined by $H(x, 1) = f(x)$ and $H(x, t) = H^n(x, t)$ if $(x, t) \in B_n \times [0, 1 - 1/2^n]$, is a well-defined f.p. $\mathcal{U}$-homotopy by use of $(3)_n$, $(4)_n$ and $(5)_n$. Therefore, (c) is verified and the lemma will be proved.

For $k = 0$, define $\tilde{f}_0 = g|B_0$. Write $\Lambda_0 = \Sigma X_\alpha$ in either $\Sigma(\mathscr{C})$ or $\Sigma(\mathscr{M})$. Applying Case 1 or Case 2 to each $\tilde{f}_0|(B_0)_{X_\alpha}$ separately, we can define an f.p. map f_0: $B_0 \to \Lambda_0 \times M$ such that: $f_0|(B_0 \setminus A_0)$ is an embedding and $\tilde{f}_0 \overset{\mathcal{U}_0}{\simeq} f_0$ (f.p.) rel A_0 say through F^0. Define $G^0 = F^0$. Therefore, $(1)_0$, $(3)_0$ and $(5)_0$ are satisfied.

Now, for $n \geqslant 1$ assume that $\tilde{f}_{n-1}$, f_{n-1}, F^{n-1}, H^{n-1} (if $n > 1$) and G^{n-1} have been defined such that the properties $(1)_{n-1}$–$(5)_{n-1}$ are satisfied. We now define $\tilde{f}_n$, f_n, F^n, H^n and G^n. Since Λ is endowed with the weak topology, without loss of generality, we can assume that $\Lambda_n = \Lambda_{n-1} \cup_\eta Y$ where $Y \in \mathscr{C}$ if $\Lambda \in \mathscr{CW}(\mathscr{C})$ or $Y \in \mathscr{M}$ if $\Lambda \in \mathscr{CW}(\mathscr{M})$, with a characteristic map η: $(Y, Z) \to (\Lambda_n, \Lambda_{n-1})$. Write $\tilde{Y} = \eta(Y)$ and $\tilde{Z} = \eta(Z)$. Recall that $B_{\tilde{Y}}$ and $A_{\tilde{Y}} \cup B_{\tilde{Z}}$ are compact if $\Lambda \in \mathscr{CW}(\mathscr{C})$. From the $\mathcal{V}_{n-1}$-homotopy G^{n-1} of $(5)_{n-1}$, by use of Lemma 1.1, we can obtain an f.p. $\mathcal{V}_{n-1}$-homotopy H^n: $B_n \times [0, 1 - 1/2^n] \to \Lambda_n \times M$ that extends

G^{n-1} and deforms (f.p. rel A_n) $g|B_n$ to an f.p. map $\tilde{f}_n$: $B_n \to \Lambda_n \times M$, i.e. $(4)_n$ is satisfied. Observe that $B_{n-1} = B_n \cap (\Lambda_{n-1} \times N)$ is a G_δ-set in B_n since $\Lambda_{n-1} \times N$ is a G_δ-set in $\Lambda_n \times N$. So, $B_{n-1} \cup A_n$ is a G_δ-set in B_n [**E**, p. 45]. Then, we can define an f.p. $\mathscr{U}_n$-homotopy F^n: $B_n \times [1 - 1/2^n, 1 - 1/2^{n+1}] \to \Lambda_n \times M$ (rel $B_{n-1} \cup A_n$) from $\tilde{f}_n$ to an f.p. map f_n: $B_n \to \Lambda_n \times M$ such that $f_n|(B_n \setminus A_n)$ is an embedding (i.e. $(1)_n$, $(2)_n$ and $(3)_n$ are satisfied) as follows. If $\Lambda \in \mathscr{C}\mathscr{W}(\mathscr{C})$, use Case 1 directly; if $\Lambda \in \mathscr{C}\mathscr{W}(\mathscr{M})$, apply Case 2 to the induced map $\eta^*(\tilde{f}_n)|$: $\eta^*(B_{\tilde{Y}}) \to Y \times M$. (Recall that $(\eta^*(B_{\tilde{Y}}), \eta^*(A_{\tilde{Y}} \cup B_{\tilde{Z}}))$ is a pair of compactly proper sets by Lemmas 0.1(5) and (2).) Finally, an f.p. $\mathscr{V}_n$-homotopy G^n can be defined as indicated in $(5)_n$; so, $(5)_n$ is satisfied since $\mathscr{V}_n = \mathrm{st}(\mathscr{V}_{n-1}, \mathscr{U}_n)$. The proof is now complete. □

LEMMA 1.3. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles where $\Lambda \in \mathscr{C}\mathscr{W}(\mathscr{C})$, and let A be a compactly proper G_δ-set in X. If f: $A \to Y$ is an f.p. embedding, then $f(A)$ is a G_δ-set in Y.*

PROOF. Since Λ is paracompact by Corollary 0.4(2), from [**K**, Theorem 5.28], we can obtain an nbd-finite, σ-discrete closed cover $\{F_\alpha | \alpha \in \mathscr{A} = \bigcup_1^\infty \mathscr{A}_n\}$ of Λ such that each F_α is a closed G_δ-set in Λ, each $\{F_\alpha | \alpha \in \mathscr{A}_n\}$ is a discrete family in Λ, and for each $\alpha \in \mathscr{A}$, $(X_{F_\alpha}, \pi|, F_\alpha)$ and $(Y_{F_\alpha}, \pi'|, F_\alpha)$ are trivial bundles. Write $F(n) = \bigcup\{F_\alpha | \alpha \in \mathscr{A}_n\}$ and $F^*(n) = F(1) \cup \cdots \cup F(n)$. Observe that for each n, $F(n)$ and $F^*(n)$ are closed G_δ-sets in Λ [**E**, p. 45]. It follows from [**D**, p. 131, Example 2] that $\Lambda = \dim\lim\{F^*(1) \subset F^*(2) \subset \cdots\}$. Now, by Lemma 0.2 and the discreteness of each family $\{F_\alpha | \alpha \in \mathscr{A}_n\}$, we only need to prove the lemma for the case $X = \Lambda \times M$ and $Y = \Lambda \times N$ where M and N are E^∞-manifolds.

Case 1. $\Lambda \in \mathscr{C}$. Then, A is a compact G_δ-set in $\Lambda \times M$. By [$\mathbf{L}_2$, Theorem 4.2 and $\mathbf{L}_1$, Lemma 2.2], M and N can be embedded into E^∞ as closed G_δ-sets; hence, $\Lambda \times M$ and $\Lambda \times N$ can be identified with closed G_δ-sets in $\Lambda \times E^\infty$. By use of [**E**, Exercise 2.1.B(a)], without loss of generality, we can assume that $M = N = E^\infty$. Let K be a compact subset of E^∞ such that $A \subset \Lambda \times K$ [**D**, p. 417]. By Lemma 1.1, f has an f.p. extension over $\Lambda \times K$, and by Lemma 1.2, an f.p. embedding extension over $\Lambda \times K$. Therefore, by [**E**, Exercise 2.1.B(a)], we need only to prove for the case $A = \Lambda \times K$.

Assume that $K \subset E^k \subset E^\infty$ and $f(\Lambda \times K) \subset \Lambda \times E^m \subset \Lambda \times E^\infty$. Identify E^∞ with $E^m \times E^\infty_{m+1}$ and $E^k \times E^\infty_{k+1}$. By use of an f.p. homeomorphism of $\Lambda \times E^m \times 0 \times E^k \times 0$ given by Lemma 0.5, we can define an f.p. homeomorphism h of $\Lambda \times E^\infty \times E^\infty$ such that $h(\lambda, f_\lambda(x), 0) = (\lambda, 0, x)$ for all $\lambda \in \Lambda$, $x \in K$. Therefore, $f(\Lambda \times K) \times 0 = h^{-1}(\Lambda \times 0 \times K)$ is a G_δ-set in $\Lambda \times E^\infty \times E^\infty$; then, $f(\Lambda \times K)$ is a G_δ-set in $\Lambda \times E^\infty$ by [**E**, Exercise 2.1.B(a)].

Case 2. $\Lambda = \dim\lim \Lambda_n$. From Case 1 and the construction of Λ_n from Λ_{n-1}, by making use of Tietze's extension theorem, we can define inductively a sequence of maps θ_n: $\Lambda_n \times N \to I$ so that $\theta_n^{-1}(0) = [f(A)]_{\Lambda_n}$. Hence, $[f(A)]_{\Lambda_n}$ is a G_δ-set in $\Lambda_n \times N$ for each n. Therefore, $f(A)$ is a G_δ-set in $\Lambda \times N$ by Lemma 0.2 since $\Lambda \times N = \dim\lim(\Lambda_n \times N)$ by Lemma 0.3. □

2. Some properties of f.p.d.-sets. Throughout this section, if it is not specified otherwise, X and Y denote the total space of fiber bundles over a space Γ. Following [**Sa**$_2$] we define the following notions. A closed subset A of an open subset V of X is a Γ-*fiber preserving* D-set in V if the following condition holds: ($\mathscr{D}$) Given an open cover $\mathscr{U}$ of V and a pair (C, D) of compactly proper G_δ-sets in V, there is an f.p. embedding h: $C \to V$ such that h is $\mathscr{U}$-close to i_C, $h|D = i_D$ and $h(C \setminus D) \cap A = \varnothing$.

A closed subset A of an open subset V of X is a Γ-*fiber preserving* D^*-*set* in V if the following condition holds: ($\mathscr{D}^*$) Given a closed G_δ-set D in V and an open cover $\mathscr{U}$ of V, there is an f.p. embedding f: $V \to V$ such that $f|D = i_D$, f is $\mathscr{U}$-close to id_V and $f(V \setminus D) \cap A = \varnothing$.

A closed subset A of X is a *local f.p. D-set* (resp., a *local f.p. D*-set*) if each $x \in A$ has an open neighborhood U in X such that $A \cap U$ is an f.p. D-set (resp., f.p. D^*-set) in U. Let B be a closed subset of Y. An f.p. embedding f: $B \to X$ is an f.p. *D-embedding* if $f(B)$ is an f.p. D-set in X. By an f.p. (G_δ, D)-*set* in X, we mean an f.p. D-set that is G_δ in X. Similarly, we can define f.p. (G_δ, D)-*embeddings*. A subset A of X is f.p. *collared* in X if there is an f.p. open embedding f: $A \times [0,1) \to X$ such that $f(z,0) = z$ for each $z \in A$. Observe that each f.p. D^*-set is an f.p. D-set. Moreover, we have the following proposition.

PROPOSITION 2.1. *Let (X, π, Γ) be a fiber bundle and A a closed subset of X. If X is paracompact and A is contained in an f.p. collared set in X, then A is an f.p. D^*-set in X.*

PROOF. The proof is parallel to that of [**Sa**$_2$, Proposition 1.2]: The assumption of perfect normality is unnecessary because the closed G_δ-sets to be used are from the definition of f.p. D^*-sets and the open set W in the proof of [**Sa**$_2$, Proposition 1.2] can be taken as an open F_σ-set by use of the normality of X; and the map k in the proof of [**Sa**$_2$, Proposition 1.2] can be chosen to be f.p., then so is f. □

PROPOSITION 2.2. (1) *Any closed subset of an f.p. D-set (resp. f.p. D*-set) in an arbitrary total space X is also an f.p. D-set (resp. f.p. D*-set).*

(2) *A finite union of f.p. D*-sets in an arbitrary total space X is also an f.p. D*-set in X.*

(3) *If $\{A_\alpha | \alpha \in \mathscr{A}\}$ is a discrete family of f.p. D-sets (resp. f.p. D*-sets) in a paracompact total space X, then so is the union $A = \bigcup\{A_\alpha | \alpha \in \mathscr{A}\}$.*

(4) *If A is a closed subset of a normal total space X which is an f.p. D-set (f.p. D*-set) in an open set W of X, then A is an f.p. D-set (f.p. D*-set) in X.*

PROOF. The proofs of (1) and (2) follow directly from the conditions ($\mathscr{D}$) and ($\mathscr{D}^*$).

We now prove (3). Since X is paracompact and $\{A_\alpha\}$ is discrete, there is from Lemma 31 [**K**, p. 158] a family $\{V_\alpha\}$ of mutually disjoint open sets of X such that $V_\alpha \supset A_\alpha$ for each α. Furthermore, we can choose each V_α to be an F_σ-set. Let $\mathscr{U}$ be an open cover of X and (C, D) a pair of compactly proper G_δ-sets in X. Then for

each $\alpha \in \mathscr{A}$, since $(C \setminus V_\alpha) \cup D$ is also a G_δ-set, there is an f.p. embedding f_α: $C \to X$ such that $f_\alpha(x) = x$ for $x \in (C \setminus V_\alpha) \cup D$, f_α is $\mathscr{U}$-close to i_C, and $f_\alpha(C \setminus D) \cap A_\alpha = \varnothing$. Define f: $C \to X$ by $f(x) = f_\alpha(x)$ if $x \in V_\alpha$ for some α, and $f(x) = x$ otherwise. Then, f is an embedding showing that $(\mathscr{D})$ holds true for A. Similarly, we can prove that $(\mathscr{D}^*)$ holds true for A if $(\mathscr{D}^*)$ holds true for each A_α.

Finally, to prove (4), let $\mathscr{U}$ be an open cover of X and (C, D) a pair of compactly proper G_δ-sets in X. Let ϕ: $X \to I$ be a map such that $\phi^{-1}(0) \supset A$ and $W \supset \phi^{-1}([0, 1))$. Define $\tilde{C} = C \cap \phi^{-1}([0, \frac{3}{4}])$ and $\tilde{D} = [C \cap \phi^{-1}([\frac{1}{2}, \frac{3}{4}])] \cup (D \cap \tilde{C})$. Then, $(\tilde{C}, \tilde{D})$ is a pair of compactly proper G_δ-sets in W by Lemma 0.1. Let $\tilde{\mathscr{U}}$ be an open cover of W that is a refinement of both $\mathscr{U}$ and $\mathscr{V}$, where $\mathscr{V} = \{\phi^{-1}([0, \frac{3}{4})), \phi^{-1}((\frac{1}{2}, 1])\}$. From the hypothesis, there is an f.p. embedding $\tilde{f}$: $\tilde{C} \to W$ such that $\tilde{f}|\tilde{D} = i_{\tilde{D}}$, $\tilde{f}$ is $\tilde{\mathscr{U}}$-close to $i_{\tilde{C}}$ and $\tilde{f}(\tilde{C} \setminus \tilde{D}) \cap A = \varnothing$. Also, $\tilde{f}(\tilde{C} \setminus \tilde{D}) \subset \phi^{-1}([0, \frac{3}{4}))$ from the definition of $\mathscr{V}$. Now, let f be the extension of $\tilde{f}$ over C via the identity. Then, f is an embedding that is $\mathscr{U}$-close to i_C such that $f|D = i_D$, and $f(C \setminus D) \cap A = \varnothing$. Therefore, $(\mathscr{D})$ holds true for A. On the other hand, the proof of the $(\mathscr{D}^*)$-case is similarly straightforward.

The proof of the proposition is now complete. □

Similar to Proposition 1.4 [$\mathbf{Sa}_2$], we can prove the following proposition by using Theorem 5.5 [$\mathbf{M}_1$] and Proposition 2.2 above.

PROPOSITION 2.3. *Let A be a closed subset of a paracompact total space X. Then, A is an f.p. D^*-set if and only if A is a local f.p. D^*-set.* □

Moreover, the following lemma is an f.p. version of Lemma 2.2 [$\mathbf{Sa}_2$].

LEMMA 2.4. *Let M, N be E^∞-manifolds, C an f.p. D-set in $\Lambda \times M$, and $A \subset B$ compactly proper G_δ-sets in $\Lambda \times N$. If $\Lambda \in \mathscr{CW}(\mathscr{M})$, assume in addition that $B \subset \Lambda \times K$ for some compact subset K of N. Let g: $B \to \Lambda \times M$ be an f.p. map with $g|A$ an embedding. Then, for each open cover $\mathscr{U}$ of $\Lambda \times M$, there is an f.p. embedding f: $B \to \Lambda \times M$ such that $f(B \setminus A) \cap C = \varnothing$ and $g \overset{\mathscr{U}}{\simeq} f$ (f.p.) rel A.*

PROOF. Let $\mathscr{V}$ be an open cover of $\Lambda \times M$ such that $\mathscr{V} \overset{*}{<} \mathscr{U}$. By Lemma 1.2, there is an f.p. embedding f': $B \to \Lambda \times M$ such that $f' \overset{\mathscr{V}}{\simeq} g$ (f.p.) rel A. Now, since C is an f.p. D-set and $(f'(B), f'(A))$ is a pair of compactly proper G_δ-sets in $\Lambda \times M$ (by use of Lemma 0.1(4) and Corollary 0.4(1) or Lemma 1.3), there is an f.p. embedding h: $f'(B) \to \Lambda \times M$ so that $h \overset{\mathscr{V}}{\simeq} i_{f'(B)}$ (f.p.) rel $f'(A)$ and $h(f'(B) \setminus f'(A)) \cap C = \varnothing$. Then $f = hf' \overset{\mathrm{st}(\mathscr{V})}{\simeq} g$ (f.p.) rel A such that $f(B \setminus A) \cap C = \varnothing$; and so the proof is complete. □

It is known that every compact subset of an E^∞-manifold M is a D-set in M. The following proposition is its f.p. version.

PROPOSITION 2.5. *Let M be an E^∞-manifold and A a closed G_δ-set in $\Lambda \times M$. Then, A is an f.p. D-set in $\Lambda \times M$ if either* (a) *$\Lambda \in \mathscr{CW}(\mathscr{C})$ and A is compactly proper, or* (b) *$\Lambda \in \mathscr{CW}(\mathscr{M})$ and A is homeomorphic to a closed subset of $\Lambda \times K$ for some compact metric space K.*

PROOF. Let (C, D) be a pair of compactly proper G_δ-sets in $\Lambda \times M$. We will reembed C to verify the f.p. D-set property of A.

(a) Let ϕ: $C \to I$ such that $\phi^{-1}(0) = D$ and let τ_ϕ: $C \to \Lambda \times M \times I$ be the embedding defined by $\tau_\phi(x) = (x, \phi(x))$. Then, $\tau_\phi(C)$ is a G_δ-set in $C \times I$ and $C \times I$ is a G_δ-set in $\Lambda \times M \times I$; hence $\tau_\phi(C)$ is a G_δ-set in $\Lambda \times M \times I$ by [E, Exercise 2.1.B(a)]. Define $\tilde{C} = (A \times \{0\}) \cup \tau_\phi(C)$ and $\tilde{D} = (A \cup D) \times \{0\}$. Then, $(\tilde{C}, \tilde{D})$ is a pair of compactly proper G_δ-sets in $\Lambda \times M \times I$ by Lemma 0.1. Let $g = \pi_{\Lambda \times M}|\tilde{C}$. Then $g\tau_\phi = i_C$: $C \to \Lambda \times M$. Given an open cover $\mathscr{U}$ of $\Lambda \times M$, by Lemma 1.2, there is an embedding $\tilde{h}$: $\tilde{C} \to \Lambda \times M$ such that $g \overset{\mathscr{U}}{\simeq} \tilde{h}$ (f.p.) rel $\tilde{D}$. Observe that $i_C = g\tau_\phi \overset{\mathscr{U}}{\simeq} \tilde{h}\tau_\phi \equiv h$ (f.p.) rel D and that $h(C \setminus D) = \tilde{h}\tau_\phi(C \setminus D) = \tilde{h}(\tilde{C} \setminus \tilde{D})$ misses $\tilde{h}(\tilde{D}) = A \cup D$. Therefore, $h|D = i_D$ and $h(C \setminus D) \cap A = \varnothing$; so, A is an f.p. D-set.

(b) Assume that $\Lambda \in \mathscr{M}$. Identify M with an open subset of E^∞. Similar to the first part of the proof of Case 2 of Lemma 1.2, we can show that for each $\lambda \in \Lambda$ there is a neighborhood ω of λ and an integer n such that $A_\omega \subset \omega \times (M \cap (E^n \times 0))$. Therefore, A is a locally f.p. D^*-set in $\Lambda \times M$. It follows from Proposition 2.3 that A is an f.p. D^*-set (so, an f.p. D-set) in $\Lambda \times M$. (Consequently, we can reembed C by keeping D fixed as we wanted.)

In general, let $\Lambda = \operatorname{dir\,lim}\{\Lambda_0 \subset \Lambda_1 \subset \cdots\} \in \mathscr{CW}(\mathscr{M})$. Let (C, D) be a pair of compactly proper sets in $\Lambda \times M$. Recall that $\Lambda \times M$ is perfectly normal by Corollary 0.4. The proof of Lemma 1.2 for $\Lambda \in \mathscr{CW}(\mathscr{M})$ can be translated word for word into this context where the special case $\Lambda \in \mathscr{M}$ above and the pairs $(C_{\Lambda_n}, D_{\Lambda_n})$ take the roles of Case 2 of Lemma 1.2 and the pairs (B_n, A_n), respectively. □

3. Fibered stability theorem. In this section, we will prove a strong fibered deformation theorem (Theorem 3.4) and a relative fibered stability theorem. The latter, a consequence of Theorem 3.6, will be used to prove the fibered α-approximation theorem in §6.

Given a $\delta > 0$, we write $J_\delta = [-\delta, \delta]$. Let $J = [-1, 1]$ and $\mathring{J} = (-1, 1)$. For given subsets $\Delta \subset \Sigma$ of a space Λ and a given sequence of maps $\{\alpha_n: \Sigma \to I | n = 1, 2, \ldots\}$, we define

$$(\Delta \times J^n;\ \alpha_1, \ldots, \alpha_n) = \bigcup\left\{\lambda \times J_{\alpha_1(\lambda)} \times \cdots \times J_{\alpha_n(\lambda)} \middle| \lambda \in \Delta\right\} \subset \Lambda \times R^n$$

and

$$(\Delta;\ \alpha_1, \alpha_2, \ldots) = \operatorname{dir\,lim}\{(\Delta \times J^n;\ \alpha_1, \ldots, \alpha_n)\}.$$

Similarly,

$$(\Delta;\ \alpha_1, \alpha_2, \ldots)^0 = \operatorname{dir\,lim}\{(\Delta \times \mathring{J}^n;\ \alpha_1, \ldots, \alpha_n)\}.$$

Recall that Λ will be in $\mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$ if it is not specified otherwise.

LEMMA 3.1. *Let $\Lambda \in \mathscr{CG}$ be a paracompact space and N an E^∞-manifold. Given an open cover $\mathscr{U}$ of $\Lambda \times N$, there are a locally convex structure on N and an open refinement $\mathscr{U}'$ of $\mathscr{U}$ such that, for each $U' \in \mathscr{U}'$ and $\lambda \in \pi_\Lambda(U')$, $\pi_N(U'_\lambda)$ is a convex open subset of N.*

PROOF. We will first prove the lemma for the case where N is an R^∞-manifold, then for the other case by a modification. Think of N as an open subset of R^∞ [**He**$_2$] (a locally convex space) and $U \subset \Lambda \times N$ ($U \in \mathscr{U}$) as an open subset of $\Lambda \times R^\infty$. Given a point $(\lambda, x) \in \Lambda \times N$, let $U \in \mathscr{U}$ that contains (λ, x). Write $\mathscr{O} = \pi_\Lambda(U \cap (\Lambda \times x))$, then $\mathscr{O}$ is an open neighborhood of λ in Λ. Since Λ is regular, there is an open neighborhood ω of λ in $\mathscr{O}$ such that its closure $\Omega \subset \mathscr{O}$. Recall that $(R^\infty, x) \cong (R^\infty, 0)$ through a translation. Therefore, without loss of generality, we can assume that $(N, x) = (R^\infty, 0)$. We will define by induction a sequence of maps $\{\varepsilon_n\colon \Omega \to \mathring{I} \mid n = 1, 2, \dots\}$ such that $U'(\lambda, x) = (\omega;\ \varepsilon_1, \varepsilon_2, \dots)^0 \subset U$. Consequently, $\mathscr{U}' = \{U'(\lambda, x) \mid (\lambda, x) \in \Lambda \times N\}$ will be an open refinement of $\mathscr{U}$ that we wanted. For each positive integer k, we write $U_k = U \cap (\mathscr{O} \times J^k)$ and $\mathring{U}_k = U \cap (\mathscr{O} \times \mathring{J}^k)$.

First, since $\mathring{U}_1$ is an open neighborhood of $\Omega \times 0$ in $\mathscr{O} \times J$ and since Ω is paracompact [**D**, p. 165], there is a map $\varepsilon_1\colon \Omega \to \mathring{I}$ such that $(\Omega \times J;\ \varepsilon_1) \subset \mathring{U}_1$. Next, assume that we have already defined n maps $\varepsilon_1, \dots, \varepsilon_n\colon \Omega \to \mathring{I}$ such that $(\Omega \times J^n;\ \varepsilon_1, \dots, \varepsilon_n) \subset \mathring{U}_n$. We will define a suitable map $\varepsilon_{n+1}\colon \Omega \to \mathring{I}$ as follows. Given a $\mu \in \Omega$, since $(\mu \times J^n;\ \varepsilon_1, \dots, \varepsilon_n) = \mu \times J_{\varepsilon_1(\mu)} \times \cdots \times J_{\varepsilon_n(\mu)}$ is compact, there are a neighborhood $\Omega(\mu)$ of μ in Ω and a positive number $\delta(\mu)$ such that $(\Omega(\mu) \times J^n;\ \varepsilon_1, \dots, \varepsilon_n) \times J_{\delta(\mu)} \subset \mathring{U}_{n+1}$. Let $\{\Sigma_a \mid a \in A\}$ be an nbd-finite open refinement of $\{\Omega(\mu) \mid \mu \in \Omega\}$ (refer to [**D**]). Observe that if $\Sigma_a \subset \Omega(\mu)$, then

$$(*) \qquad (\Sigma_a \times J^n;\ \varepsilon_1, \dots, \varepsilon_n) \times J_{\delta(\mu)} \subset (\Omega(\mu) \times J^n;\ \varepsilon_1, \dots, \varepsilon_n) \times J_{\delta(\mu)} \subset \mathring{U}_{n+1}.$$

Let $\psi\colon A \to \Omega$ be a set-theoretical map such that $\Sigma_a \subset \Omega(\psi(a))$. Then, from [**D**, Ex. 4.1, p. 179], there is a map $\varepsilon_{n+1}\colon \Omega \to \mathring{I}$ such that $\varepsilon_{n+1}(\lambda) \leqslant \sup\{\delta(\psi(a)) \mid \lambda \in \Sigma_a\}$. Observe that $\varepsilon_{n+1}(\lambda) \leqslant \delta(\psi(a(\lambda)))$ for some $a(\lambda)$ such that $\lambda \in \Sigma_{a(\lambda)}$ since $\{\Sigma_a \mid a \in A\}$ is nbd-finite. Consequently, $(\Omega \times J^{n+1};\ \varepsilon_1, \dots, \varepsilon_{n+1}) \subset \mathring{U}_{n+1}$ by $(*)$. Therefore,

$$U'(\lambda, x) = (\omega;\ \varepsilon_1, \varepsilon_2, \dots)^0 \subset (\omega;\ \varepsilon_1, \varepsilon_2, \dots) \subset U,$$

and $U'(\lambda, x)$ is an open subset of $\Lambda \times N$ since

$$U'(\lambda, x) \cap (\Lambda \times J^n) = (\omega \times \mathring{J}^n;\ \varepsilon_1, \dots, \varepsilon_n)$$

is an open subset of $\Lambda \times J^n$. Moreover, for each $\mu \in \Lambda$, observe that

$$[U'(\lambda, x)]_\mu = \operatorname{dir\,lim}\left\{\mu \times \mathring{J}_{\varepsilon_1(\mu)} \times \cdots \times \mathring{J}_{\varepsilon_n(\mu)}\right\}$$

is open and convex in $\mu \times N$.

Now, for the case of Q^∞-manifolds, recall that $Q^\infty \cong Q \times R^\infty$. Therefore, we can write $N \cong \tilde{N} \times R^\infty$ where $\tilde{N}$ is an open subset of Q [**Ch**, Corollary 16.3]. Then, from [**Gr**, Exercise 1, p. 60], $\Lambda \times N \cong (\Lambda \times \tilde{N}) \times R^\infty$. Since $\Lambda \times \tilde{N} \cong \Lambda \times_c \tilde{N}$ and $\tilde{N}$ is a locally convex subspace of Q, there is from the paracompactness of Ω a map $\varepsilon\colon \Omega \to \mathring{I}$ such that $\tilde{\Omega} = \bigcup\{\lambda \times B(x_1;\ \varepsilon(\lambda)) \mid \lambda \in \Omega\}$, where $x = (x_1, x_2) \in \tilde{N} \times R^\infty$ and $B(x_1;\ \varepsilon(\lambda))$ is the closed convex $\varepsilon(\lambda)$-neighborhood of x_1 in $\tilde{N} \subset Q$, is a closed neighborhood of $\Omega \times x_1$ in $U \cap (\Omega \times \tilde{N})$. Now, repeating the construction of $U'(\lambda, x)$ in the R^∞-manifold case above by use of $\tilde{\Omega}$ (paracompact) in place of Ω, we can similarly define a desired open refinement $\mathscr{U}'$ of $\mathscr{U}$. The proof of Lemma 3.1 is complete. $\square$

LEMMA 3.2. *Let $\Lambda \in \mathscr{CW}(\mathscr{C}) \cup \mathscr{CW}(\mathscr{M})$. Let M, N be E^∞-manifolds and A a closed subset of $\Lambda \times M$. Let $f\colon \Lambda \times M \to \Lambda \times N$ be an f.p. map such that $f|A$ is a closed embedding.*

(1) *If f is an f.p. homotopy equivalence, then f has an f.p.* rel A *homotopy inverse.*

(2) *Given an open cover $\mathscr{U}$ of $\Lambda \times N$, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that if f is an f.p. $\mathscr{V}$-homotopy equivalence, then f has an f.p.* rel A *$\mathscr{U}$-homotopy inverse.*

PROOF. We will give a proof of (2), then observe that (1) will follow. Let $\mathscr{U}'$ be an open-cover refinement of $\mathscr{U}$ obtained from Lemma 3.1. Let $\mathscr{V}$ be an open cover of $\Lambda \times N$ such that $\mathrm{st}^2(\mathscr{V}) < \mathscr{U}'$, and let $g'\colon \Lambda \times N \to \Lambda \times M$ be an f.p. $\mathscr{V}$-homotopy inverse of f. In the rest of this proof, all maps and homotopies will be Λ-fiber preserving and we will suppress the term f.p. from our notation.

Let h be an $f^{-1}(\mathscr{V})$-homotopy from $g'f$ to $\mathrm{id}_{\Lambda\times M}$. Then, $h(f^{-1} \times \mathrm{id}_I)|(f(A) \times I)$ is an $f^{-1}(\mathscr{V})$-homotopy from $g'|f(A)$ to $f^{-1}|f(A)$. Since $f(A)$ is a closed subset of $\Lambda \times N$, there is by Lemma 1.1 an $f^{-1}(\mathscr{V})$-homotopy $G\colon g' \simeq g$ such that G and g are extensions of $h(f^{-1} \times \mathrm{id}_I)|(f(A) \times I)$ and $f^{-1}|f(A)$, respectively. We will verify that g is a rel A $\mathscr{U}$-homotopy inverse of f.

First, observe that $fg \overset{\mathrm{st}(\mathscr{V})}{\simeq} \mathrm{id}$ since $g \overset{f^{-1}(\mathscr{V})}{\simeq} g'$ and $fg' \overset{\mathscr{V}}{\simeq} \mathrm{id}$. Hence, fg is $\mathscr{U}'$-close to id; consequently, $fg \overset{\mathscr{U}'}{\simeq} \mathrm{id}$ rel $f(A)$ via the straight-line homotopy by the choice of $\mathscr{U}'$.

Second, we now prove that $gf \overset{f^{-1}(\mathscr{U})}{\simeq} \mathrm{id}$ rel A. Let $H\colon \Lambda \times M \times I \to \Lambda \times M$ be the $\mathrm{st}(f^{-1}(\mathscr{V}))$-homotopy defined by $H_t = h_{1-2t}$ $(0 \leqslant t \leqslant \frac{1}{2})$ and $H_t = G_{2t-1}f$ $(\frac{1}{2} \leqslant t \leqslant 1)$. Note that $H\colon \mathrm{id} \simeq g'f \simeq gf$ and that for each $a \in A$ the paths $H|(a \times [0, \frac{1}{2}])$ and $H|(a \times [\frac{1}{2}, 1])$ are $f^{-1}(\mathscr{V})$-limited and opposite each other by the definition of G. Observe that the proof of Theorem 2.1 [**G**] is so canonical that we can define an $f^{-1}(\mathscr{V})$-homotopy

$$\tilde{H}\colon A \times I \times I \to \Lambda \times M \,\mathrm{rel}\, A \times \partial I,$$

from $H|(A \times I)$ to $\pi_A\colon A \times I \to A \subset \Lambda \times M$ such that $\tilde{H}(a \times I \times I) = H(a \times I)$ for each $a \in A$. Let

$$F\colon [((A \times I) \cup (\Lambda \times M \times \partial I)) \times I] \cup [\Lambda \times M \times I \times 0] \to A \times M$$

be the map defined by

$$F(z) = \begin{cases} \tilde{H}(z) & \text{if } z = (v, s, t) \in A \times I \times I, \\ v & \text{if } z = (v, 0, t) \in \Lambda \times M \times 0 \times I, \\ gf(v) & \text{if } z = (v, 1, t) \in \Lambda \times M \times 1 \times I, \\ H(v, s) & \text{if } z = (v, s, 0) \in \Lambda \times M \times I \times 0, \end{cases}$$

where $(s, t) \in I \times I$. Note that $F|(\{(A \times I) \cup (\Lambda \times M \times \partial I)\} \times I)$ is an $f^{-1}(\mathscr{V})$-homotopy. Therefore, by Lemma 1.1, F extends to an $f^{-1}(\mathscr{V})$-homotopy $\tilde{F}\colon (\Lambda \times M \times I) \times I \to \Lambda \times M$. Combining this with the fact that $\tilde{F}_0 = H$ is a $\mathrm{st}(f^{-1}(\mathscr{V}))$-homotopy, we can verify that $\tilde{F}_1\colon \Lambda \times M \times I \to \Lambda \times M$ is a $\mathrm{st}^2(f^{-1}(\mathscr{V}))$-homotopy rel A from $\mathrm{id}_{\Lambda\times M}$ to gf. Therefore, the proof of (2) is complete.

To prove (1), observe that the existence of the uncontrolled version of g and F does not depend on that of $\mathscr{V}$. Therefore, from the above paragraph, we can infer that each f.p. homotopy equivalence f as in hypothesis has a homotopy inverse g: $\Lambda \times N \to \Lambda \times M$ such that $g|f(A) = f^{-1}|f(A)$ and $gf \simeq \mathrm{id}_{\Lambda \times M}$ (f.p.) rel A. Similarly, g will have a homotopy inverse f': $\Lambda \times M \to \Lambda \times N$ such that $f'|g(f(A)) = g^{-1}|g(f(A))$ (so, $f'|A = g^{-1}|A = f|A$) and $f'g \simeq \mathrm{id}_{\Lambda \times N}$ (f.p.) rel $f(A)$. Consequently, $f \simeq f'gf \simeq f'$ (f.p.) rel A; hence, $fg \simeq f'g \simeq \mathrm{id}_{\Lambda \times N}$ rel $f(A)$. Therefore, (1) is proved and the proof of the lemma is complete. □

In the following, A^* denotes the G_δ-set $(\Gamma \times M) \cup A$ where Γ is a closed G_δ-set in Λ and A is an f.p. (G_δ, D)-set in $\Lambda \times M$.

LEMMA 3.3. *Let M, N be E^∞-manifolds, Γ a closed G_δ-set in Λ, and A an f.p. (G_δ, D)-set in $\Lambda \times M$. If f: $\Lambda \times M \to \Lambda \times N$ is an f.p. homotopy equivalence such that $f|$: $\Gamma \times M \to \Gamma \times N$ is an f.p. homeomorphism and $f|A$ is an f.p. (G_δ, D)-embedding, then f is f.p. homotopic* rel A^* *to an f.p. homeomorphism.*

PROOF. Write $\Lambda \times M = \operatorname{dir\,lim}\{X_n\}$, $\Lambda \times N = \operatorname{dir\,lim}\{Y_n\}$ where $X_n = \Lambda \times M_n$ and $Y_n = \Lambda \times N_n$ (Lemma 0.3). All maps and homotopies will be Λ-fiber preserving and the term f.p. will be suppressed from our notation if it is not specified otherwise.

Let $X_1^* = X_1$ and $A_1^* = A^* \cap X_1^*$. Since $f(A)$ is an f.p. D-set, we can apply Lemma 2.4 to the pair (X_1^*, A_1^*) of compactly proper G_δ-set in $\Lambda \times M$ (by Lemma 0.1(3)) to obtain an embedding f_1: $X_1^* \to \Lambda \times N$ such that $f_1 \simeq f|X_1^*$ rel A_1^* and $f_1(X_1^* \setminus A_1^*) \cap f(A) = \varnothing$. Let $\bar{f}_1$ be the extension of f_1 over $A^* \cup X_1^*$ via f. Then $\bar{f}_1 \simeq f|A^* \cup X_1^*$ rel A^*. So, there is from Lemma 1.1 an extension $\tilde{f}_1$ of $\bar{f}_1$ over $\Lambda \times M$ such that $\tilde{f}_1 \simeq f$ rel A^*.

Let g be a homotopy inverse of f. Then, g is also a homotopy inverse of $\tilde{f}_1$. By Lemma 3.2(1), we can choose g so that

(a_1) $g|\tilde{f}_1(A^* \cup X_1^*) = \tilde{f}_1^{-1}|\tilde{f}_1(A^* \cup X_1^*)$,

(b_1) $g\tilde{f}_1 \simeq \mathrm{id}_{\Lambda \times M}$ rel $A^* \cup X_1^*$, and

(c_1) $\tilde{f}_1 g \simeq \mathrm{id}_{\Lambda \times N}$ rel $\tilde{f}_1(A^* \cup X_1^*)$.

Set $Y_1^* = Y_1 \cup \tilde{f}_1(X_1^*)$ and $B_1^* = Y_1^* \cap \tilde{f}_1(A^* \cup X_1^*) \supset \tilde{f}_1(X_1^*) = f_1(X_1^*)$. Since $\tilde{f}_1^*(X_1^*)$ is a G_δ-set in $\Lambda \times N$ by Lemma 1.3, B_1^* is a G_δ-set in $\Lambda \times N$ by [**E**, p. 45]. Now, since X_1^* is an f.p. D-set in $\Lambda \times M$ by Lemma 2.5, for the pair $(Y_1, Y_1 \cap B_1^*)$, there is by Lemma 2.4 an f.p. embedding $\hat{g}_1$: $Y_1 \to \Lambda \times M$ such that $\hat{g}_1(Y_1 \setminus B_1^*) \cap X_1^* = \varnothing$ and $\hat{g}_1 \simeq g|Y_1$ rel $Y_1 \cap B_1^*$. Let h: $Y_1^* \to \Lambda \times M$ be the extension over Y_1^* of $\hat{g}_1$ via $\tilde{f}_1^{-1}|\tilde{f}_1(X_1^*)$. Observe that h is a well-defined embedding and $h \simeq g|Y_1^*$ rel B_1^*. Then, since A is an f.p. D-set in $\Lambda \times M$ and $(h(Y_1^*), h(B_1^*))$ is a pair of compactly proper G_δ-sets in $\Lambda \times M$ by Lemma 1.3, there is an embedding k: $h(Y_1^*) \to \Lambda \times M$ such that $k(h(Y_1^*) \setminus h(B_1^*)) \cap A = \varnothing$ and $k \simeq i_{h(Y_1^*)}$ rel $h(B_1^*)$. Define $g_1 = kh$: $Y_1^* \to \Lambda \times M$. Then, $g_1 \simeq g|Y_1^*$ rel B_1^* and $g_1(Y_1^* \setminus B_1^*) \cap A = \varnothing$. Define $\bar{g}$: $Y_1^* \cup \tilde{f}_1(A^*) \to \Lambda \times M$ to be the extension of g_1 via g. Then, $\bar{g}_1 \simeq g|Y_1^* \cup \tilde{f}_1(A^*)$ rel $\tilde{f}_1(A^* \cup X_1^*)$. By Lemma 1.1, there is an extension $\tilde{g}_1$ of $\bar{g}_1$ over $\Lambda \times N$ such that

$$(*_1) \qquad \tilde{g}_1 \simeq g \operatorname{rel} \tilde{f}_1(A^* \cup X_1^*).$$

Write $X_2^* \equiv X_2 \cup \tilde{g}_1(Y_1^*) = X_2 \cup \tilde{g}_1(Y_1)$ and $A_2^* = X_2^* \cap (A^* \cup \tilde{g}_1(Y_1^*)) \supset \tilde{g}_1(Y_1^*) = g_1(Y_1^*) \supset X_1^*$. Observe that

$$\begin{aligned}\tilde{g}_1^{-1}|\tilde{g}_1(Y_1^*) &\simeq \tilde{f}_1 g \tilde{g}_1^{-1}|\tilde{g}_1(Y_1^*) \operatorname{rel}(A^* \cup X_1^*) \cap \tilde{g}_1(Y_1^*) \quad \text{by } (c_1)\\ &\simeq \tilde{f}_1 \tilde{g}_1 \tilde{g}_1^{-1}|\tilde{g}_1(Y_1^*) \operatorname{rel}(A^* \cup X_1^*) \cap \tilde{g}_1(Y_1^*) \quad \text{by } (*_1)\\ &= \tilde{f}_1|\tilde{g}_1(Y_1^*).\end{aligned}$$

Moreover, since $\tilde{g}_1^{-1}|A^* \cup X_1^* = \tilde{f}_1|A^* \cup X_1^*$ by (a_1), it follows that

$$\tilde{g}_1^{-1}|A^* \cup \tilde{g}_1(Y_1^*) \simeq \tilde{f}_1|A^* \cup \tilde{g}_1(Y_1^*) \operatorname{rel}(A^* \cup X_1^*).$$

By Lemma 1.1, there is a map f_1': $\Lambda \times M \to \Lambda \times N$ such that $f_1' \simeq \tilde{f}_1$ rel $A^* \cup X_1^*$ and $f_1'|A^* \cup \tilde{g}_1(Y_1^*) = \tilde{g}_1^{-1}|A^* \cup \tilde{g}_1(Y_1^*)$. Apply Lemma 2.4 to the pair $(X_2, X_2 \cap A_2^*)$ to obtain an embedding $\hat{f}_2$: $X_2 \to \Lambda \times N$ such that $\hat{f}_2(X_2 \setminus A_2^*) \cap Y_1 = \varnothing$ and $\hat{f}_2 \simeq f_1'|X_2$ rel $X_2 \cap A_2^*$. Extend $\hat{f}_2$ over $\tilde{g}_1(Y_1)$ via $\tilde{g}_1^{-1}|\tilde{g}_1(Y_1)$ to obtain an embedding h: $X_2^* \to \Lambda \times N$, then reembed $h(X_2^*)$ to obtain an embedding f_2: $X_2^* \to \Lambda \times N$ such that $f_2 \simeq f_1'|X_2^*$ rel A_2^* and $f_2(X_2^* \setminus A_2^*) \cap f(A) = \varnothing$. Extend f_2 to $\tilde{f}_2$ over $A^* \cup X_2^*$ via f_1', then $\tilde{f}_2$ over $\Lambda \times M$ by use of Lemma 1.1 such that $\tilde{f}_2 \simeq f_1' \simeq \tilde{f}_1$ rel $A^* \cup X_1^*$.

Observe that $\tilde{g}_1$ is also a homotopy inverse of $\tilde{f}_2$. By Lemma 3.2(1), we can assume that $\tilde{g}_1$ is a rel $A^* \cup X_2^*$ homotopy inverse of $\tilde{f}_2$. In a similar manner, we write

$$Y_2^* = Y_2 \cup \tilde{f}_2(X_2^*) = Y_2 \cup \tilde{f}_2(X_2)$$

and

$$B_2^* = Y_2^* \cap \tilde{f}_2(A^* \cup X_2^*) \supset \tilde{f}_2(X_2^*) = f_2(X_2^*) \supset Y_1^*,$$

and extend $\tilde{f}_2^{-1}|B_2^*$ to an embedding g_2: $Y_2^* \to \Lambda \times M$ such that $g_2 \simeq \tilde{g}_1|Y_2^*$ rel B_2^* and $g_2(Y_2^* \setminus B_2^*) \cap A = \varnothing$, and to a map $\tilde{g}_2$: $\Lambda \times M \to \Lambda \times N$ such that $(*_2)$ $\tilde{g}_2 \simeq \tilde{g}_1$ rel $\tilde{f}_2(A^* \cup X_2^*)$, and so on.

Thus, inductively we can obtain the following commutative diagram:

$$\begin{array}{ccccccc} X_1^* & \hookrightarrow & X_2^* & \hookrightarrow & X_3^* & \hookrightarrow & \cdots \\ f_1 \downarrow & g_1 \nearrow & \downarrow f_2 & g_2 \nearrow & \downarrow f_3 & \nearrow g_3 & \\ Y_1^* & \hookrightarrow & Y_2^* & \hookrightarrow & Y_3^* & \hookrightarrow & \cdots \end{array}$$

with $f \simeq \tilde{f}_1$ rel A^* and $\tilde{f}_k \simeq \tilde{f}_{k+1}$ rel $A^* \cup X_k^*$ for each $k \geqslant 1$. Therefore, f is homotopic rel A^* to a homeomorphism that is induced from $\{f_n | n = 1, 2, \ldots\}$. □

THEOREM 3.4. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles. Let Γ be a closed G_δ-set in Λ and A an f.p. (G_δ, D)-set in X. If f: $X \to Y$ is an f.p. homotopy equivalence such that $f|A$ is an f.p. (G_δ, D)-embedding and $f|$: $X_\Gamma \to Y_\Gamma$ is an f.p. homeomorphism, then f is f.p. homotopic* rel A^* *to an f.p. homeomorphism.*

PROOF. We will follow [$\mathbf{M}_1$, Proposition 5.3]. Since Λ is paracompact there is from [**K**, Theorem 5.28] an nbd-finite, σ-discrete open cover $\mathscr{V} = \bigcup_{n=1}^\infty \mathscr{V}_n$ of Λ such that each $\mathscr{V}_n$ is a family of pairwise disjoint open sets and that, for each $V \in \mathscr{V}$, $(X_V, \pi|, V)$ and $(Y_V, \pi'|, V)$ are trivial bundles. Let $\{B_\beta\}$ be a precise refinement of $\mathscr{V}$ such that each B_β is a closed G_δ-set in Λ. For each $n = 1, 2, \ldots$, let $\mathscr{B}_n$ denote

the indexing set for $\mathscr{V}_n$. Then, each $\{B_\beta | \beta \in \mathscr{B}_n\}$ is a discrete family in Λ whose union will be denoted by $B(n)$. For $n = 1, 2, \ldots,$ write $B^*(n) = \Gamma \cup B(1) \cup \cdots \cup B(n)$.

Applying first Lemma 3.3 to each $f|B_\beta$ with $\beta \in \mathscr{B}_1$, then Lemma 1.1, we can obtain an f.p. map f_1: $X \to Y$ such that

$(1)_1$ $f_1 \simeq f$ rel $A \cup X_\Gamma$, and

$(2)_2$ $f_1|X_{B^*(1)}$: $X_{B^*(1)} \to Y_{B^*(1)}$ is an f.p. homeomorphism.

Similarly, applying first Lemma 3.3 to each $f_1|B_\beta$ with $\beta \in \mathscr{B}_2$ (here, $A \cup X_{B^*(1)}$ plays the role of A^*), then Lemma 1.1, we can obtain an f.p. map f_2: $X \to Y$ such that

$(1)_2$ $f_2 \simeq f_1$ rel $A \cup X_{B^*(1)}$, and

$(2)_2$ $f_2|X_{B^*(2)}$: $X_{B^*(2)} \to Y_{B^*(2)}$ is an f.p. homeomorphism.

Finally, we can construct by induction a sequence of f.p. maps $\{f_n\}$ satisfying similar properties $(1)_n$ and $(2)_n$; and define $f_\infty = \lim_{n\to\infty} f_n$. Then, f_∞ is a well-defined f.p. homeomorphism (by the nbd-finite property of $\{B_\beta\}$) that is f.p. homotopic rel A^* to f. □

The following is a weak form of Theorem 5.3. However, it is an ingredient for the proofs of Theorems 4.1 and 4.3, also Theorem 5.3.

COROLLARY 3.5 (FIBERED UNKNOTTING THEOREM FOR D-EMBEDDINGS). *Let (X, π, Λ) be an E^∞-manifold bundle, Γ a closed G_δ-set in Λ, A an f.p. (G_δ, D)-set in X and $\mathscr{U}$, $\mathscr{V}$ open covers of X. Let f: $A \cup X_\Gamma \to X$ be an f.p. map such that $f|A$ is an f.p. (G_δ, D)-embedding and $f|X_\Gamma$: $X_\Gamma \to X_\Gamma$ is an f.p. homeomorphism. If f is f.p. $\mathscr{U}$-homotopic to $i_{A\cup X_\Gamma}$, say by Φ, then there is an f.p. homeomorphism $\tilde{f}$ of X that extends f and is* $\mathrm{st}(\mathscr{U}, \mathscr{V})$*-close to* id_X. *Moreover, if Σ is a closed G_δ-set in Λ such that Φ is stationary on $A \cap X_\Sigma$, then $\tilde{f}$ can be chosen to be identity on X_Σ.*

PROOF. For the trivial-bundle case, it is a controlled version of Lemma 3.3 and its proof is parallel with that of Theorem 2.1 [**Sa**$_2$]. For the general case, the proof is similar to that of Theorem 3.4; however, we need an additional control in the construction of f_n's. Let $\mathscr{V} \overset{*}{>} \mathscr{V}_1 \overset{*}{>} \mathscr{V}_2 \overset{*}{>} \cdots$ be a sequence of open covers of X. Define $\mathscr{W}_1 = \mathrm{st}(\mathscr{U}, \mathscr{V}_1)$ and $\mathscr{W}_n = \mathrm{st}(\mathscr{W}_{n-1}, \mathscr{V}_n)$, inductively. It is straightforward to verify that $\mathscr{W}_n < \mathrm{st}(\mathscr{U}, \mathscr{V})$ for all n. From the trivial-bundle case, by induction we can assume additionally that, for each n, f_n is $\mathscr{W}_n$-close to id_X. This will imply that f_∞ is $\mathrm{st}(\mathscr{U}, \mathscr{V})$-close to id_X. □

THEOREM 3.6. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles with fibers M and N, respectively. Let Γ be a closed G_δ-set in Λ and A an f.p. (G_δ, D)-set in X. If f: $X \to Y$ is an f.p. fine homotopy equivalence such that $f|X_\Gamma$: $X_\Gamma \to Y_\Gamma$ is an f.p. homeomorphism and $f|A$ is an f.p. (G_δ, D)-embedding, then f is an f.p. near homeomorphism* rel A^*.

PROOF. Similar to the proof of Corollary 3.5, the general case follows from the trivial-bundle case. Therefore, we will only prove for the case $X = \Lambda \times M$ and $Y = \Lambda \times N$. Let $\mathscr{U}$ be an open cover of $\Lambda \times N$. We will show that f is f.p. rel A^* $\mathscr{U}$-homotopic to an f.p. homeomorphism. We will follow the proof of Theorem 2.3

[**Sa**$_1$], and use Lemma 2.4 and the controlled version of Lemma 1.1. We include here an outline of the proof which is similar to that of the proof of Lemma 3.3 above. For more details, the reader may refer to [**Sa**$_1$] with the controlled version in mind. As in [**Sa**$_1$] we define inductively two sequences of open covers, $\mathscr{U}_n$'s and $\mathscr{V}_n$'s of $\Lambda \times N$: First, $\mathscr{U} \overset{*}{>} \mathscr{U}' \overset{*}{>} \mathscr{U}_1 \overset{*}{>} \mathscr{U}_2 \overset{*}{>} \cdots$ where $\mathscr{U}'$ is an open cover of $\Lambda \times N$ given by Lemma 3.1; then, $\mathscr{V}_1 = \mathscr{U}_1$ and $\mathscr{V}_{n+1} = \mathrm{st}(\mathscr{V}_n, \mathscr{U}_{n+1})$ inductively.

Observe that $\mathscr{V}_n < \mathscr{U}$ for each [**Sa**$_1$]. All maps and homotopies are Λ-fiber preserving, and X_n^*'s, Y_n^*'s, A_n^*'s and B_n^*'s are defined by induction as in the proof of Lemma 3.3 above.

By Lemma 2.4, we can obtain an embedding f_1: $X_1^* \to \Lambda \times N$ such that $f_1 \overset{\mathscr{V}_2}{\simeq} f|X_1^*$ rel A_1^* and $f_1(X_1^* \setminus A_1^*) \cap f(A) = \varnothing$.

Let g be a rel A^* $\mathscr{U}_3$-homotopy inverse of f given from Lemma 3.2(2). By use of Lemmas 1.1, 1.3, 2.4, and 2.5, we can obtain from g an embedding g_1: $Y_1^* \to \Lambda \times M$ that is an extension of f_1^{-1} such that $g_1^{-1} \overset{\mathrm{st}(\mathscr{V}_1, \mathscr{U}_2)}{\simeq} f|g_1(Y_1^*)$ rel B_1^* and $g_1(Y_1^* \setminus B_1^*) \cap A = \varnothing$. By use of Lemmas 1.1, 1.3, 2.4 and 2.5, we can extend g_1^{-1} to an embedding f_2: $X_2^* \to \Lambda \times N$ such that $f_2 \overset{\mathscr{V}_2}{\simeq} f|X_2^*$ rel A_2^* and $f_2(X_2^* \setminus A_2^*) \cap f(A) = \varnothing$. Then, we can extend f_2^{-1} to an embedding g_2: $Y_2^* \to \Lambda \times M$ such that $g_2^{-1} \overset{\mathrm{st}(\mathscr{V}_2, \mathscr{U}_3)}{\simeq} f|g_2(Y_2^*)$ rel B_2^* and $g_2(Y_2^* \setminus B_2^*) \cap A = \varnothing$.

Therefore, we will obtain a similar diagram as above such that, for each $n = 1, 2, \ldots$, $f_n \overset{\mathrm{st}(\mathscr{V}_n, \mathscr{U}_{n+1})}{\simeq} f|X_n^*$ rel A_n^* and $f_n(X_n^* \setminus A_n^*) \cap f(A) = \varnothing$. Note that $A^* = \bigcup A_n^*$. Then, $\{f_n | n = 1, 2, \ldots\}$ induces a homeomorphism h such that h is $\mathscr{U}'$-close to f and $h|A^* = f|A^*$. Finally, by the choice of $\mathscr{U}'$, it follows that $f \overset{\mathscr{U}'}{\simeq} h$ rel A^* via the straight-line homotopy. □

COROLLARY 3.7 (FIBERED STABILITY THEOREM). *Let (X, π, Λ) be an E^∞-manifold bundle, $A \subset X$ an f.p. (G_δ, D)-set and $\mathscr{U}$ an open cover of X. Then π_X: $X \times E^\infty \to X$ is f.p. $\mathscr{U}$-homotopic* (rel $A \times 0$) *to an f.p. homeomorphism.*

PROOF. Since E is contractible, the projection map π_X: $X \times E^\infty \to X$ is an f.p. fine homotopy equivalence. Therefore, the corollary follows from Theorem 3.6. □

REMARK 3.8. For Q^∞-manifolds, the absolute fibered stability theorem holds true over paracompact spaces Λ. The proof uses Theorem A2.9 [**TW**] and the homeomorphisms $(Q^\infty \times Q^\infty, Q^\infty \times 0) \cong (Q^\infty \times Q, Q^\infty \times 0)$, $M \cong M \times Q^\infty$.

4. Equivalent forms of fibered infinite deficiencies. Let $(\mathscr{F}, 0)$ denote either $(E^\infty, 0)$ or $(I, 0)$. Let (X, π, Γ) be an E^∞-manifold bundle. A closed subset A of X is said to be f.p. *$\mathscr{F}$-deficient* in X if there is an f.p. homeomorphism f: $X \to X \times \mathscr{F}$ such that $f(A) \subset X \times 0$.

THEOREM 4.1. *Let (X, π, Λ) be an E^∞-manifold bundle and A a closed G_δ-set in X. The following are equivalent.*

(1) *A is f.p. E^∞-deficient in X.*

(2) *A is f.p. I-deficient in X.*

(3) *A is contained in Y where Y is a collared, closed E^∞-manifold subbundle of X.*

(4) *A is contained in an f.p. collared subset of X.*

(5) *A is an f.p. D^*-set in X.*

(6) *A is an f.p. D-set in X.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. (4) $\Rightarrow$ (5) is Proposition 2.1. (5) $\Rightarrow$ (6) is trivial. To prove (6) $\Rightarrow$ (1), we follow the proof of ((vi) $\Rightarrow$ (i)) of Theorem 3.1 [**Sa**$_2$] by use of Corollary 3.7 (Fibered stability theorem) and Corollary 3.5 (Fibered unknotting theorem for D-embeddings). □

REMARK 4.1.1. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) hold true without requiring A to be a G_δ-set in X.

REMARK 4.1.2. Any f.p. E^∞-deficient set in X is contained in an f.p. E^∞-deficient G_δ-set in X. Therefore, Corollary 3.7 also holds true if A is an arbitrary f.p. E^∞-deficient set in X.

As in [**L**$_1$ and **L**$_2$], we can prove the following corollary to Theorem 3.6 and Theorem 4.1.

COROLLARY 4.2. *Let (X, π, Λ) be an E^∞-manifold bundle. If A is an f.p. (G_δ, D)-set in X, then A is an f.p. negligible in X; i.e., the inclusion $X \setminus A \to X$ is an f.p. near homeomorphism.* □

THEOREM 4.3. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles, and let f: $Y \to X$ be an f.p. closed embedding. Then, f is an f.p. (G_δ, D)-embedding (or an f.p. E^∞-deficient embedding) if and only if there is an f.p. open embedding F: $Y \times [0,1) \to X$ such that $F(x, 0) = f(x)$ for all $x \in Y$. F is called an f.p. collaring of f.*

PROOF. The idea is to show that $f(Y)$ is locally f.p. collared in X by use of Theorems 3.3 and 3.4 [**L**$_3$] (or Theorem 7.3 [**Sa**$_2$]) and Corollary 3.5 above, and to apply the f.p. version of Brown's collaring theorem for paracompact spaces (the proof of local collaring implying global collaring in [**B**, pp. 224–229] works well for the f.p. version for the paracompact space X). The details are similar to those in the proof of Proposition 2.5 [**Ch-F**]. □

The following proposition follows from Propositions 2.2, 2.3, Theorem 4.1 and Remark 4.1.2.

PROPOSITION 4.4. *Let (X, π, Λ) be an E^∞-manifold bundle.*

(1) *The union of two f.p. E^∞-deficient sets (f.p. (G_δ, D)-sets, resp.) in X is an f.p. E^∞-deficient set (f.p. (G_δ, D)-set, resp.) in X.*

(2) *Let A be a closed G_δ-set in X. If A is locally f.p. E^∞-deficient in X, then A is f.p. E^∞-deficient in X.*

(3) *Consequently, if A is a locally compact closed G_δ-set in X, then A is f.p. E^∞-deficient in X.*

We conclude this section by proving the following proposition that will be used to prove Theorem 5.2.

PROPOSITION 4.5. *Let M, N be E^∞-manifolds, and $A \subset Z$ closed G_δ-sets in $\Lambda \times N$. Let $f\colon Z \to \Lambda \times M$ be an f.p. map such that $f|A$ is an f.p. (G_δ, D)-embedding. Then, for each open cover $\mathscr{V}$ of $\Lambda \times M$, there is an f.p. continuous injection $\phi\colon Z \to \Lambda \times M$ such that $\phi \overset{\mathscr{V}}{\simeq} f$ (f.p.) rel A.*

PROOF. Define inductively a sequence $\{\mathscr{V}_n\}$ of open covers of $\Lambda \times M$ such that $\mathscr{V} \overset{*}{>} \mathscr{V}_1 \overset{*}{>} \mathscr{V}_2 \overset{*}{>} \cdots$. Write $N = \operatorname{dir\,lim}\{N_n\}$ and $Z_n = Z \cap (\Lambda \times N_n)$, a compactly proper set by Lemma 0.1(3). From Lemma 1.2, we have an embedding $\tilde{\phi}_1\colon Z_1 \to \Lambda \times M$ such that $\tilde{\phi}_1(Z_1 \setminus A) \cap f(A) = \varnothing$ and $\tilde{\phi}_1 \overset{\mathscr{V}_1}{\simeq} f|Z_1$ (f.p.) rel $A \cap Z_1$ (this is a G_δ-set in Z_1). Then, from Lemma 1.1, we can obtain an extension $\phi_1\colon Z \to \Lambda \times M$ of $\tilde{\phi}_1$ such that $\phi_1 \overset{\mathscr{V}_1}{\simeq} f$ (f.p.) rel A. Since $\phi_1(Z_1)$ is an f.p. G_δ-set in $\Lambda \times M$ by Lemma 1.3, it follows from [**E**, p. 45], Propositions 2.5 and 4.4(1) that $\phi_1|Z_1 \cup A$ is an f.p. (G_δ, D)-embedding. Therefore, we can obtain similarly a map $\phi_2\colon Z \to \Lambda \times M$ such that

$(1)_2$ $\phi_2|(Z_2 \cup A)$ is an f.p. (G_δ, D)-embedding, and

$(2)_2$ $\phi_2 \overset{\mathscr{V}_2}{\simeq} \phi_1$ (f.p.) rel $Z_1 \cup A$.

In a similar manner, we can define by induction on n a sequence of maps, $\phi_n\colon Z \to \Lambda \times M$, such that

$(1)_n$ $\phi_n|(Z_n \cup A)$ is an f.p. (G_δ, D)-embedding, and

$(2)_n$ $\phi_n \overset{\mathscr{V}_n}{\simeq} \phi_{n-1}$ (f.p.) rel $Z_{n-1} \cup A$.

Define $\phi = \lim \phi_n$. Then, ϕ is an f.p. continuous injection that is f.p. $\mathscr{V}$-homotopic (rel A) to f. □

5. Strong fibered unknotting theorem. In this section, we also prove a fibered D-embedding approximation theorem (Theorem 5.2) and a strong fibered unknotting theorem for D-embeddings (Theorem 5.3).

LEMMA 5.1. *Let $\Lambda \in \mathscr{CG}$, M and N E^∞-manifolds and X a closed subset of $\Lambda \times N$. If $f\colon X \to \Lambda \times M$ is an f.p. continuous injection such that $f^{-1}(\Lambda \times M_n)$ is a compactly proper subset of X for each n, then f is a closed embedding.*

PROOF. We only need to show that f is a closed map. Let C be a closed subset of X. Observe that $C \cap f^{-1}(\Lambda \times M_n)$ is compactly proper by Lemma 0.1(3) since it is a closed subset of $f^{-1}(\Lambda \times M_n)$. By Lemma 0.1(4), its continuous image $f(C) \cap (\Lambda \times M_n)$ is a closed subset of $\Lambda \times M_n$. So, $f(C)$ is a closed subset of $\Lambda \times M = \operatorname{dir\,lim}\{\Lambda \times M_n\}$.

THEOREM 5.2. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles and $A \subset Z$ closed G_δ-sets in Y. Let $f\colon Z \to X$ be an f.p. map such that $f|A$ is an f.p. (G_δ, D)-embedding. Then, for each open cover $\mathscr{U}$ of X, there is an f.p. (G_δ, D)-embedding $g\colon Z \to X$ such that $g \overset{\mathscr{U}}{\simeq} f$ (f.p.) rel A.*

PROOF. Similar to the proof of Corollary 3.5, the general case follows from the trivial-bundle case with control by use of a sequence of open covers of X with $\mathscr{U} \overset{*}{>} \mathscr{U}_1 \overset{*}{>} \mathscr{U}_2 \overset{*}{>} \cdots$. Therefore, we can assume that $X = \Lambda \times M$ and $A \subset Z$ closed G_δ-sets in $Y = \Lambda \times N$, where M and N are E^∞-manifolds. Note that $E^\infty \cong E^\infty \times R^\infty$. Let $\mathscr{U} \overset{*}{>} \mathscr{W} \overset{*}{>} \mathscr{W}'$ be open covers of $\Lambda \times M$. By use of Corollary 3.7, we can obtain an f.p. homeomorphism $\psi\colon \Lambda \times M \times R^\infty \to \Lambda \times M$ such that

$$\psi \overset{\mathscr{W}'}{\simeq} \pi_{\Lambda \times M} \text{ (f.p.) rel } f(A) \times 0.$$

Then, the map $\tilde{f}\colon Z \to \Lambda \times M$, defined by $\tilde{f}(x) = \psi(f(x), 0)$ for $x \in Z$, is f.p. $\mathscr{W}'$-homotopic (rel A) to f. Consequently,

$$\psi^{-1}\tilde{f} \overset{\psi^{-1}(\mathscr{W}')}{\simeq} \psi^{-1}f \text{ (f.p.) rel } A.$$

Write $\mathscr{V} = \psi^{-1}(\mathscr{W}') \overset{*}{>} \mathscr{V}_1 \overset{*}{>} \mathscr{V}_2 \overset{*}{>} \cdots$ a sequence of open covers of $\Lambda \times M \times R^\infty$.

First, from Proposition 4.5, we have an f.p. continuous injection $\phi\colon Z \to \Lambda \times M \times 0 \subset \Lambda \times M \times R^\infty$ such that $\phi \overset{\mathscr{V}}{\simeq} \Psi^{-1}\tilde{f}$ (f.p.) rel A. So, $\phi \overset{\mathrm{st}(\mathscr{V})}{\simeq} \psi^{-1}f$ (f.p.) rel A.

Second, we define an f.p. closed embedding $h\colon Z \to \Lambda \times M \times R^\infty$ such that $h \overset{\mathscr{V}}{\simeq} \phi$ (f.p.) rel A. For each integer j, write $Z_j' = \phi^{-1}(\Lambda \times M_j \times 0) \cap (\Lambda \times N_j)$, where $M = \operatorname{dir\,lim}\{M_j\}$ and $N = \operatorname{dir\,lim}\{N_j\}$. Each Z_j' is a compactly proper set by Lemma 0.1(3), and Z_j' is a G_δ-set in Z [**E**, p. 45] since $\phi^{-1}(\Lambda \times M_j \times 0)$ and $Z \cap (\Lambda \times N_j)$ are G_δ-sets in Z; therefore, each $Z_j' \cup A$ is a closed G_δ-set in Z [**E**, p. 45]. It is clear that $Z = \bigcup Z_j'$. Now, there is a map $\varepsilon_1\colon Z \to I$ with $\varepsilon_1^{-1}(0) = Z_1' \cup A$ such that the map

$$h_1 = (\pi_{\Lambda \times M}\phi) \times \varepsilon_1 \times 0\colon Z \to \Lambda \times M \times R \times 0 \subset \Lambda \times M \times R^\infty$$

is f.p. $\mathscr{V}_1$-homotopic (rel $Z_1' \cup A$) to ϕ. Observe that h_1 is still a continuous injection and that $h_1^{-1}(\Lambda \times M \times 0) = Z_1' \cup A$. In the following, given an integer j, write $D^j = [-j, j]^j$. Assume by induction that we have already defined n continuous injections $\{h_k\colon Z \to \Lambda \times M \times R^k \times 0 \subset \Lambda \times M \times R^\infty | k = 1, \ldots, n\}$ such that, for each $k \leqslant n$,

(a)$_k$ $h_k \overset{\mathscr{V}_k}{\simeq} h_{k-1}$ (f.p.) rel $Z_k' \cup A$, and

(b)$_k$ $h_k^{-1}(\Lambda \times M \times D^{k-1} \times 0) = Z_k' \cup A$.

Then, we define h_{n+1}. Let $\varepsilon_{n+1}\colon Z \to I$ be a map such that $\varepsilon_{n+1}^{-1}(0) = Z_{n+1}' \cup A$. Define $h_{n+1} \equiv (\pi_{\Lambda \times M \times R^n} h_n) \times \varepsilon_{n+1} \times 0\colon Z \to \Lambda \times M \times R^{n+1} \times 0 \subset \Lambda \times M \times R^\infty$. Observe that we can choose ε_{n+1} so that $h_{n+1} \overset{\mathscr{V}_{n+1}}{\simeq} h_n$ (f.p.) rel $Z_{n+1}' \cup A$; hence, (a)$_{n+1}$ is satisfied. To verify (b)$_{n+1}$, since $\varepsilon_j(Z) \subset I$ $(1 \leqslant j \leqslant n+1)$, observe that

$$\begin{aligned} h_{n+1}^{-1}(\Lambda \times M \times D^n \times 0) &= h_{n+1}^{-1}(\Lambda \times M \times R^n \times 0) \\ &= h_n^{-1}(\Lambda \times M \times R^n \times 0) \cap \varepsilon_{n+1}^{-1}(0) \\ &= Z \cap \varepsilon_{n+1}^{-1}(0) = Z_{n+1}' \cup A. \end{aligned}$$

Define $h = \lim h_n$. Then, $\phi \overset{\mathscr{V}}{\simeq} h$ (f.p.) rel A by use of $(a)_n$, and h is a closed continuous injection by Lemma 5.1 as follows. Observe that

(i) $$\Lambda \times M \times R^\infty = \operatorname{dir\,lim}\{\Lambda \times M_n \times D^{n-1} \times 0\},$$

$$\begin{aligned} h^{-1}(\Lambda \times M_n \times D^{n-1} \times 0) &= (Z_n' \cup A) \cap h_n^{-1}(\Lambda \times M_n \times D^{n-1} \times 0) \\ &\qquad \text{(by } (b)_n \text{ and } (a)_k,\ k > n) \\ &= [Z_n' \cap h_n^{-1}(\Lambda \times M_n \times D^{n-1} \times 0)] \\ &\quad \cup [A \cap h_n^{-1}(\Lambda \times M_n \times D^{n-1} \times 0)] \\ &= Z_n' \cup [A \cap \phi^{-1}(\Lambda \times M_n \times 0)] \\ &\qquad \text{(by the definitions of } h_n \text{ and } Z_n') \\ &= Z_n' \cup [A \cap f^{-1}(\Lambda \times M_n)] \\ &\qquad \text{(from the definition of } \phi). \end{aligned}$$

(ii)

Being the union of two compactly proper sets Z_n' and $A \cap f^{-1}(\Lambda \times M_n)$, by Lemma 0.1(2) $h^{-1}(\Lambda \times M_n \times D^{n-1} \times 0)$ is a compactly proper set. (Note that $h \overset{\mathrm{st}^2(\mathscr{V})}{\simeq} \psi^{-1}f$ (f.p. rel A).)

Finally, write $R^\infty = (R^\infty)_1 \times (R^\infty)_2$ (by Stability Theorem). Without loss of generality, we can assume that $h(Z) \subset \Lambda \times M \times (R^\infty)_1 \times 0$. So, we can assume in addition that h is an f.p. D-embedding. Define $g = \psi h$. Then, g is an f.p. (G_δ, D)-embedding and $g \overset{\mathscr{U}}{\simeq} f$ (f.p.) rel A since $\psi(\mathrm{st}^2(\mathscr{V})) < \psi(\psi^{-1}(\mathscr{U})) = \mathscr{U}$ and $\psi^{-1}(g(Z))$ is a G_δ-set in $\Lambda \times M \times R^\infty \times 0$ by Lemmas 0.1, 1.3 and 0.2.

Now, we are ready to prove a strong fibered unknotting theorem for fibered D-embeddings or fibered E^∞-deficient embeddings.

THEOREM 5.3. *Let (X, π, Λ) be an E^∞-manifold bundle, A an f.p. (G_δ, D)-set in X and $\mathscr{U}$, $\mathscr{V}$ open covers of X. Let $f\colon A \to X$ be an f.p. (G_δ, D)-embedding such that $\Phi\colon f \overset{\mathscr{U}}{\simeq} i_A$ (f.p.). Then, there is an f.p. ambient invertible $\mathrm{st}(\mathscr{U}, \mathscr{V})$-isotopy ψ_t $(t \in I)$ of X such that $\psi_0 = \mathrm{id}_X$ and $\psi_1|A = f$. Moreover, if A_0 and Γ (possibly empty) are closed G_δ-sets in A and Λ, respectively, such that $A \cap X_\Gamma \subset A_0$ and Φ is stationary on A_0, and if $\mathrm{cl}(\Phi((A \setminus A_0) \times I))$ is contained in an open subset W of X, then the isotopy ψ can be chosen to be identity on $A_0 \cup (X \setminus W) \cup X_\Gamma$.*

PROOF. The proof of Theorem 5.1 [$\mathbf{Sa_2}$] can be translated word for word into this context. For the sake of completeness, we include here an outline of the proof.

Let $\mathscr{V}' \overset{*}{<} \mathscr{V}$, and W' an open subset of X such that $\mathrm{cl}(\Phi((A \setminus A_0) \times I)) \subset W' \subset \mathrm{cl}(W') \subset W$. Observe that the f.p. versions of Lemmas 1.1 and 5.2 [$\mathbf{Sa_2}$] hold true for the total space X. The proof of the theorem consists of three steps.

1. Construct an f.p. ambient invertible $\mathscr{V}'$-isotopy $\psi'\colon X \times I \to X$ stationary on $A_0 \cup (X \setminus W') \cup X_\Gamma$ such that $\psi_0' = \mathrm{id}$ and $\psi_1'(f(A \setminus A_0)) \cap A = \varnothing$. The ingredients of this proof are Proposition 4.4(1), Theorem 3.6 (in place of Proposition 1.3(2), Corollary 3.2 in [$\mathbf{Sa_2}$], resp.) and the f.p. versions of Lemmas 1.1 and 5.2 [$\mathbf{Sa_2}$].

2. Construct an f.p. ambient invertible $\mathrm{st}^2(\mathscr{U}, \mathscr{V}')$-isotopy ψ'': $X \times I \to X$ stationary on $A_0 \cup (X \setminus W) \cup X_\Gamma$ such that $\psi''_0 = \mathrm{id}$ and $\psi''_1|A = \psi'_1 f$. The ingredients of this proof are Proposition 4.4(1), Theorem 5.2, Corollary 3.5 (in place of Proposition 1.3(2), Theorem 4.3, Theorem 2.1 in [**Sa**$_2$], resp.) and the f.p. version of Lemma 5.2 [**Sa**$_2$].

3. Define ψ: $X \times I \to X$ by $\psi_t = \psi'^{-1}_t \psi''_t$ $(t \in I)$.

6. Fibered α-approximation theorem. We conclude this note by proving the following theorem.

THEOREM 6.1. *Let (X, π, Λ) and (Y, π', Λ) be E^∞-manifold bundles, $\mathscr{U}$ an open cover of $\Lambda \times M$. If f: $Y \to X$ is an f.p. $\mathscr{U}$-homotopy equivalence with $f|A$ being an f.p. (G_δ, D)-embedding, where A is an f.p. (G_δ, D)-set of Y, then f is $\mathrm{st}(\mathscr{U})$-close to an f.p. homeomorphism $\tilde{f}$ agreeing with f on A.*

PROOF. Let $\mathscr{V}$ be an open cover of X such that $\mathrm{st}(\mathscr{V}) < \mathscr{U}$. Translating the proof of Theorem 3.4 [**F**] word for word into this context by use of Theorems 4.3, 5.2, 5.3 and Corollary 3.7, we can obtain a Λ-f.p. homeomorphism F: $Y \times [0, \infty) \to X \times [0, \infty)$ such that

(1) $\pi_X F$ is $\mathrm{st}(\mathscr{U}, \mathscr{V})$-close to $\pi_X(f \times \mathrm{id}_{[0,\infty)})$,

(2) $d(\pi_{[0,\infty)}F, \pi_{[0,\infty)}) < 3$, and

(3) $F(x, 0) = (f(x), 0)$ for all for $x \in Y$. (Recall that we first assume that f is an f.p. (G_δ, D)-embedding.)

Now, by use of Corollary 3.7, we can obtain homeomorphisms ϕ: $Y \times [0, \infty) \to Y$ and ψ: $X \times [0, \infty) \to X$ such that ϕ is $f^{-1}(\mathscr{V})$-close to π_Y, ψ is $\mathscr{V}$-close to π_X, and $\phi(x, 0) = x$ and $\psi(f(x), 0) = f(x)$ for all $x \in A$. Define $\tilde{f} = \psi F \phi^{-1}$: $Y \to X$. Then, it is straightforward to verify that $\tilde{f}$ is $\mathrm{st}^3(\mathscr{U}, \mathscr{V})$-close to f and $\tilde{f}|A = f|A$. So, $\tilde{f}$ is $\mathscr{U}$-close to f since $\mathrm{st}^3(\mathscr{U}, \mathscr{V}) = \mathrm{st}(\mathscr{U}, \mathrm{st}(\mathscr{V})) < \mathrm{st}(\mathscr{U})$. □

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