

SURGERY ON CODIMENSION ONE IMMERSIONS IN \mathbf{R}^{n+1} : REMOVING n -TUPLE POINTS¹

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ABSTRACT. The self-intersection sets of immersed n -manifolds in $(n + 1)$ -space provide invariants of the n th stable stem and the $(n + 1)$ st stable homotopy of infinite real projective space. Theorems of Eccles [5] and others [1, 8, 14, 19] relate these invariants to classically defined homotopy theoretic invariants.

In this paper a surgery theory of immersions is developed; the given surgeries affect the self-intersection sets in specific ways. Using such operations a given immersion may be surgered to remove $(n + 1)$ -tuple and n -tuple points, provided the $\mathbf{Z}/2$ -valued $(n + 1)$ -tuple point invariant vanishes ($n \geq 5$). This invariant agrees with the Kervaire invariant for $n = 4k + 1$.

These results first appeared in my dissertation [2]; a summary was presented in [3]. Some results and methods have been improved since these works were written. In particular, the proof of Theorem 14 has been simplified.

I. Introduction. Two immersions $i_0: M_0^n \looparrowright \mathbf{R}^{n+k}$ and $i_1: M_1^n \looparrowright \mathbf{R}^{n+k}$ are *bordant* when there is an $(n + 1)$ -manifold W that has boundary the disjoint union of M_0 and M_1 , and there is a proper immersion

$$f: (W, \partial W) \looparrowright (\mathbf{R}^{n+k} \times [0, 1], \mathbf{R}^{n+k} \times (\{0\} \cup \{1\}))$$

which when restricted to the boundary agrees with i_0 disjoint union i_1 . The set of bordism classes of immersions forms an Abelian group (under disjoint union) which is identified with the stable homotopy group $\pi_{n+k}^s(MO(k))$, of the Thom space $MO(k)$, of the canonical k -plane bundle $\gamma_k \rightarrow BO(k)$.

In general, suppose the normal bundle, $\nu(i)$, of an immersion $i: M^n \looparrowright N^{n+k}$ is classified by a vector bundle ζ . This is called a ζ -structure on $\nu(i)$. The bordism group of immersions with a ζ -structure on the normal bundles is isomorphic to the stable homotopy group $\pi_{n+k}^s(M\zeta)$ where M denotes the Thom space. The isomorphism is given by the Pontryagin-Thom construction; see [15 or 20]. The cases with which we are chiefly concerned are the bordism groups of codimension one immersions in \mathbf{R}^{n+1} . If ζ is the universal oriented line bundle, $M\zeta \cong S^1$, and so the bordism group is isomorphic to π_n^s ; if ζ is the universal unoriented line bundle, the bordism group is isomorphic to $\pi_{n+1}^s(P^\infty)$.

Consider the group $\pi_2^s(P^\infty)$. Elements are represented by immersed one-manifolds in the plane. In Figure 8, an immersion of the circle with one double point that represents a generator of $\pi_2^s(P^\infty)$ is shown. Any immersion in the plane with an

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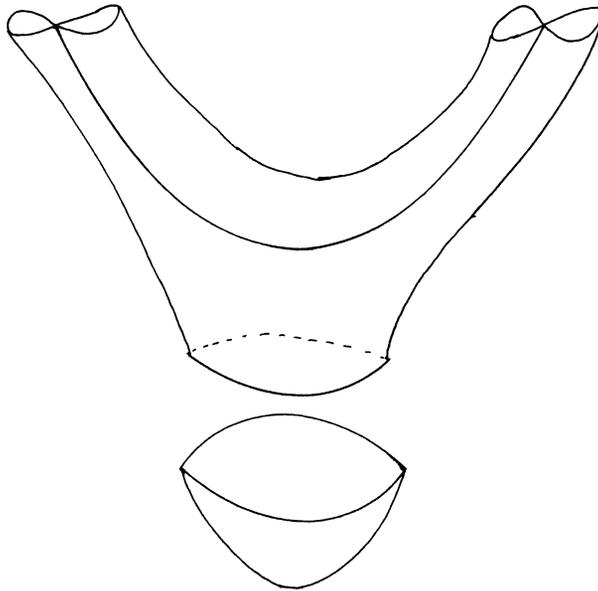


FIGURE 1

even number of double points is null bordant (Figure 1). Thus the parity of double point sets defines the isomorphism $\pi_2^s(P^\infty) \cong \mathbf{Z}/2$.

This example suggests that the self-intersection sets of immersions contain quite a bit of information about stable homotopy groups; indeed this is the case [6, 7, 14, 15] as we shall see.

The organization of this paper is as follows. In §II, the definition of the self-intersection invariants is given. The one-dimensional self-intersection set is discussed in some detail. In §III the n -tuple invariant c_n is computed in terms of the $(n + 1)$ -tuple invariant ψ_{n+1} . Specifically,

THEOREM 6. I. *If n is even and greater than 2, then $c_n(x) = \psi_{n+1}(x)$ as integers modulo 2 for every $x \in \pi_{n+1}^s(X)$ where $X = P^\infty$ or S^1 .*

II. *If n is odd and greater than 1, then $c_n(x) = 0$ for every $x \in \pi_{n+1}^s(X)$.*

This result is used to prove the main result:

THEOREM 13. *Let $n \geq 5$. Let $i: M^n \looparrowright \mathbf{R}^{n+1}$ be a general position immersion of a closed smooth n -manifold with an even number of $(n + 1)$ -tuple points. Then (i, M) is bordant to an immersion (i_1, M_1) with no n -tuple points. If M is orientable, then so are M_1 and the bordism.*

The corresponding theorem for $n = 2$ was proved by Banchoff, and the proof given here is a generalization of his technique, namely surgery on immersions. The proof occupies §V and §VI. In §IV, the general notion of surgery is developed.

The main theorem is interesting in light of the Eccles [5], Lannes [16], Koschorke [12] result:

THEOREM 5. *The number of $(n + 1)$ -tuple points of an immersion $i: M^n \looparrowright \mathbf{R}^{n+1}$ may be odd if and only if $n = 0, 2, 3, 6$, or $n + 3$ is a power of 2 and there is a framed $(n + 1)$ -manifold with Kervaire invariant 1. If M is oriented the number of $(n + 1)$ -tuple points may be odd if and only if $n = 0, 1$, and 3.*

Thus except for these dimensions every homotopy class has a representative immersion without n -tuple points.

II. Self-intersection invariants. The approach to the self-intersection invariants given below is based on Banchoff's proof [1] of the folk theorem:

The number of triple points of an immersed surface in 3-space is congruent modulo 2 to the Euler characteristic of that surface.

Proposition 1 shows that the definition given below agrees with the definition given in [6, 19, and 15].

Let H_r denote the hyperoctahedral group [4] of signed permutation $(r \times r)$ matrices. This group is a semidirect product

$$0 \rightarrow (\mathbf{Z}/2)^r \xrightarrow{\Delta} H_r \xrightarrow[\pi]{s} \Sigma_r \rightarrow 1.$$

Here we consider $\mathbf{Z}/2 = \{\pm 1\}$, the embedding Δ is along the diagonal, and the projection π takes a matrix $a = a_{jk}$ to the matrix $\pi(a)_{jk} = |a_{jk}|$. That is, π removes the minus signs from the entries of a . There is a natural r -plane bundle, $\theta_r \rightarrow BH_r$, over the classifying space of H_r induced by the given representation. The bordism group $\Omega_{n+1-r}(BH_r, \theta_r)$ is the group of bordism classes of triples (N, g, \bar{g}) , where N is a closed $(n + 1 - r)$ -manifold, $g: N \rightarrow BH_r$ is a continuous map, and $\bar{g}: TN \oplus g^*(\theta_r) \simeq \varepsilon^{n+1}$ is a vector bundle isomorphism onto the trivial $(n + 1)$ -plane bundle $\varepsilon^{n+1} \equiv N \times \mathbf{R}^{n+1}$. That is, \bar{g} is some trivialization or framing of $TN \oplus g^*(\theta_r)$. The Pontryagin-Thom construction yields a group isomorphism $\Omega_{n+1-r}(BH_r, \theta_r) \simeq \pi_{n+1}^s(M(\theta_r))$.

Let $i: M^n \looparrowright \mathbf{R}^{n+1}$ denote a general position immersion of a closed n -manifold in Euclidean $(n + 1)$ -space (all maps and manifolds are smooth). Let r be an integer between 1 and $n + 1$ inclusive. Let $M(r)$ denote the r -tuple manifold, let $\overline{M}(r)$ denote its principal Σ_r cover, and let $\overline{\overline{M}}(r)$ denote the associated r -fold cover. Let $i_r: M(r) \rightarrow \mathbf{R}^{n+1}$ denote the immersion of $M(r)$ induced from i . The construction of these manifolds and the immersion i_r is given in [6, p. 214]. The r -fold cover, $\overline{\overline{M}}(r)$, is immersed in M . The unit sphere bundle of $\nu(i)$ when restricted to $\overline{\overline{M}}(r)$ is a $\mathbf{Z}/2$ covering. Therefore, there is a $(2r)$ -sheeted covering over $M(r)$,

$$S(\nu(i))|_{\overline{\overline{M}}(r)} \xrightarrow{\mathbf{Z}/2} \overline{\overline{M}}(r) \xrightarrow{\{1, \dots, r\}} M(r).$$

This cover has coordinate transformations which land in H_r ; thus a map $g: M(r) \rightarrow BH_r$ is constructed. Clearly, g classifies $\nu(i_r)$. Let X denote either S^1 or P^∞ . The r -tuple point invariant

$$\psi_r: \pi_{n+1}^s(X) \rightarrow \Omega_{n+1-r}(BH_r, \theta_r) \simeq \pi_{n+1}^s(M(\theta_r))$$

assigns to each homotopy class in $\pi_{n+1}^s(X)$ the bordism class of the associated r -tuple data $(M(r), g, \bar{g})$ of a representative immersion.

PROPOSITION 1. *The definition of the self-intersection invariant ψ_r agrees with that given in [6].*

This result will follow from

LEMMA 2. *Let Y be an Eilenberg-Mac Lane space of type $K(G, 1)$ where G is a discrete group. Let*

$$\tilde{X} = \{((y_1, v_1), \dots, (y_r, v_r)): y_j \in Y, v_j \in \mathbf{R}^\infty, v_s \neq v_t \text{ when } s \neq t\}.$$

Then \tilde{X} is an Eilenberg-Mac Lane space of type $K(G^r, 1)$ on which Σ_r acts freely by permuting coordinates. The quotient space $X = \tilde{X}/G$ is an Eilenberg-Mac Lane space of type $K(G^r \rtimes \Sigma_r, 1)$.

PROOF. The space \tilde{X} is essentially the product of Y^r with $(\mathbf{R}^\infty)^r \setminus$ (the ‘‘multiagonal’’). The latter space has no homotopy since the multiagonal has infinite codimension. Therefore \tilde{X} is a $K(G^r, 1)$.

Clearly Σ_r acts freely, so there is a fibration

$$(1) \quad \Sigma_r \subset \tilde{X} \xrightarrow{\pi} X.$$

The homotopy sequence of the fibration (1) reduces to a short exact sequence

$$(2) \quad 1 \rightarrow \pi_1(\tilde{X}) \xrightarrow{\pi_*} \pi_1(X) \xrightarrow{\partial} \pi_0(\Sigma_r) \rightarrow 1.$$

Construct a splitting map $s: \pi_0(\Sigma_r) \rightarrow \pi_1(X)$ as follows. Let $*$ $\in Y$ denote the base point and choose $\tilde{x} = ((*, e_1), \dots, (*, e_r))$ to be the base point of \tilde{X} where $e_k = (0, \dots, 0, 1, 0, \dots)$ is the k th unit vector in \mathbf{R}^∞ for $k = 1, \dots, r$. The orbit space of the point \tilde{x} is the set

$$\Sigma_r \tilde{x} = \{((*, e_{\sigma_1}), \dots, (*, e_{\sigma_r})): \sigma \in \Sigma_r\}.$$

Let $\sigma \in \Sigma_r$ be a permutation that is not a transposition. Define a path $\tilde{\sigma}: I \rightarrow \tilde{X}$,

$$\tilde{\sigma}(t) = ((*, te_1 + (1-t)e_{\sigma_1}), \dots, (*, te_r + (1-t)e_{\sigma_r})).$$

In case σ is a transposition, let $\tilde{\sigma}$ be a path that is homotopic to the above path and that misses the multiagonal. So $\pi \circ \tilde{\sigma}$ is a loop in X . Define $s(\sigma) = [\pi \circ \tilde{\sigma}]$. By the homotopy lifting property, s is an injection, and it is clearly a splitting homomorphism. Thus the lemma is completed.

PROOF OF PROPOSITION 1. The proof now follows easily. For if $G = \mathbf{Z}/2$, then $X = BH_r$. With this model, the canonical r -plane bundle θ_r over BH_r is isomorphic to the bundle $D_r(\gamma_1)$ of [6, p. 215], and the definitions of the homomorphism ψ_r correspond.

Let $\alpha: S^1 \rightarrow P^\infty$ denote the inclusion of the Thom space over a point into the Thom space of the universal line bundle.

COROLLARY 3. *The self-intersection invariants are natural with respect to α . That is, the diagram*

$$\begin{array}{ccc} \pi_{n+1}^s(S^1) & \xrightarrow{\psi_r} & \pi_{n+1}^s(D_r S^1) \\ \downarrow \alpha_* & \searrow \psi_r & \downarrow (D_r \alpha)_* \\ \pi_{n+1}^s(P^\infty) & \xrightarrow{\psi_r} & \pi_{n+1}^s(D_r(P^\infty)) \end{array}$$

commutes.

As above, $D_r S^1$ is homotopy equivalent to $M(\rho_r)$ where ρ_r is the universal Σ_r r -plane bundle. The map $(D_r \alpha)$ is induced from the splitting map $s: \Sigma_r \rightarrow H_r$. The upper horizontal arrow is defined by Eccles [6, Proposition 2.2].

The n -tuple point invariant. The n -tuple manifold $M(n)$ of an immersion $i: M^n \hookrightarrow \mathbf{R}^{n+1}$ is one-dimensional. Therefore, the normal bundle $\nu(i_n)$ of the induced immersion is orientable, and we may reduce the structural group to be the group

$$SH_n = \{ a_{jk} \in SO(n) : a_{jk} \text{ is a signed permutation matrix} \}.$$

(For example, $SH_2 = \mathbf{Z}/4$.) Let $f: S^1 \rightarrow \mathbf{R}^{n+1}$ be a parametrization of some component of the n -tuple point curve. That is, $f(t) \in i_n(M(n))$ for $t \in [0, 1]$, and $f(0) = f(1)$. For each $t \in [0, 1]$ let $p_1(t), \dots, p_n(t) \in i^{-1}(f(t))$ be the inverse images under i of $f(t)$. Choose the indices so that $t \mapsto p_j(t)$ is a smooth arc in M for $j = 1, \dots, n$. For each $t \in [0, 1]$ there are vectors $l_1(t), \dots, l_n(t)$ depending continuously on $t \in [0, 1]$ with $l_j(t)$ normal to (i, M) at $p_j(t)$; these vectors span $\nu(i_n)_t$. The initial and final choices of bases need not agree. Hence, there is a matrix $\sigma \in SH_n$ defined by the equation

$$(l_1(0), \dots, l_n(0)) = \sigma(l_1(1), \dots, l_n(1)).$$

Such an n -tuple matrix is assigned to each component of the n -tuple manifold, and the collection of n -tuple matrices correspond to the classifying map $g: M(n) \rightarrow BSH_n$ of the n -tuple data. Let

$$\theta_n = ESH_n \times_{SH_n} \mathbf{R}^n;$$

the n -tuple invariant is a homomorphism $c_n: \pi_{n+1}^s(X) \rightarrow \Omega_1(BSH_n, \theta_n)$ defined for $X = S^1$ or P^∞ . Notice that the n -tuple matrix is defined up to conjugacy on each component of the n -tuple manifold. As such it is an invariant of the given immersion. In the sequel some one-dimensional bordism groups are computed. The n -tuple invariant is computed.

III. Computing the n -tuple invariants. Using the results 9.3 of 9.21 of [13] and elementary techniques, the following isomorphisms are obtained.

PROPOSITION 4.

$$\Omega_1(B\Sigma_n, \rho) \cong \begin{cases} \mathbf{Z}/3 \oplus \mathbf{Z}/2 & \text{if } n = 3, \\ \mathbf{Z}/3 & \text{if } n = 4, \\ 0 & \text{if } n > 4. \end{cases}$$

$$\Omega_1(BSH_n, \theta) \cong \begin{cases} \mathbf{Z}/8 & \text{if } n = 2, \\ \mathbf{Z}/2 & \text{if } n > 2. \end{cases}$$

For example, the isomorphism $\Omega_1(BSH_n, \theta) \rightarrow \mathbf{Z}/2$ identifies this bordism group with SH_n modulo its commutator. when $n \geq 3$, the commutator is the collection of matrices which when signs are removed lie in the alternating group.

Recall that the $(n + 1)$ -tuple point invariant ψ_{n+1} counts the number of $(n + 1)$ -tuple points modulo 2 of a representative immersion. This invariant has been studied by Banchoff [1], Freedman [8], Koschorke [12], Eccles [6, 7], and Lannes [6]. Their work may be summarized in

THEOREM 5. *The number of $(n + 1)$ -tuple points of an immersion $i: M^n \looparrowright \mathbf{R}^{n+1}$ may be odd if and only if $n = 0, 2, 3, 6$ or $n + 3$ is a power of 2 and there is a framed $(n + 1)$ -manifold with Kervaire invariant 1.*

Let X denote either S^1 or P^∞ . The n -tuple point invariant $c_n: \pi_{n+1}^s(X) \rightarrow \Omega_1(BSH_n, \theta)$ is related to the $(n + 1)$ -tuple point invariant as follows:

THEOREM 6. I. *If n is even and greater than 2, then $c_n(x) = \psi_{n+1}(x)$ as integers modulo 2 for every $x \in \pi_{n+1}^s(X)$.*

II. *If n is odd and greater than 1, then $c_n(x) = 0$ for every $x \in \pi_{n+1}^s(X)$.*

III. *Therefore, if $\psi_{n+1}(x) = 0$, then $c_n(x) = 0$ if $n > 2$.*

PROOF. Let $i: M^n \looparrowright \mathbf{R}^{n+1}$ denote an immersion representing the homotopy class x . Let $f: [0, 1] \rightarrow \mathbf{R}^{n+1}$ be a parametrization of one component of the immersed n -tuple point manifold; so $f(0) = f(1)$. Let $l_1(t), \dots, l_n(t)$ denote the vectors normal to the n -sheets of (i, M) which intersect at $f(t)$. Choose the indices so that $l_j(t)$ depends continuously on $t \in [0, 1]$. The n -tuple matrix σ for this component of the n -tuple manifold is defined by $(l_1(0), \dots, l_n(0)) = \sigma(l_1(1), \dots, l_n(1))$. Push the n -tuple curve to an arc

$$\alpha(t) = f(t) + \sum_{j=1}^n l_j(t) \quad \text{for } t \in [0, 1].$$

Now join the endpoints $\alpha(0)$ and $\alpha(1)$ by a sequence of arcs each passing once through the immersion (i, M) as in Figure 2. The number of such arcs is equal to the parity of the matrix σ modulo 2. (The parity of σ is defined to be the parity (odd or even) of the matrix $|\sigma|$ whose entries are the absolute value of the entries of σ .) Let $\bar{\alpha}$ denote this closed curve.

The intersection number of the curve $\bar{\alpha}$ and the immersion (i, M) must be trivial modulo 2. On the other hand, this intersection number is

$$0 = [i(M)] \circ [\bar{\alpha}(S^1)] = \text{parity of } \sigma + (n + 1)(\# \text{ of } (n + 1)\text{-tuples}).$$

If the n -tuple manifold has one component, then this completes the proof, for the n -tuple invariant is trivial, in this case, if and only if the matrix $|\sigma|$ is alternating. Otherwise, we may use the preceding argument on each component of (i, M) . The proof is complete.

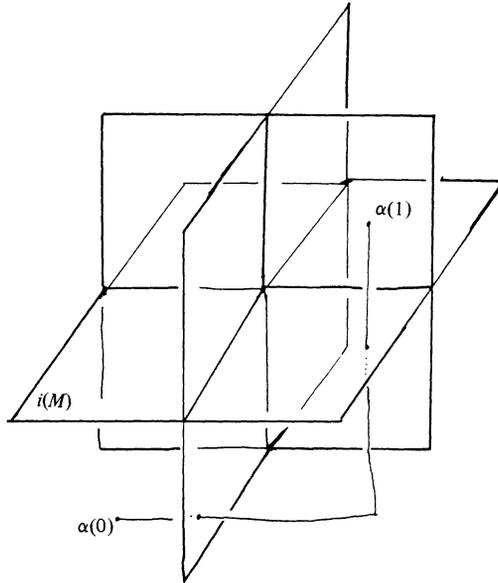


FIGURE 2

The ideas of the above proof apply to prove the isomorphism $\mathbf{Z}/8 \cong \pi_3^s(P^\infty)$. Boy's surface [9] is an immersion of $\mathbf{R}P^2$ in \mathbf{R}^3 with one triple point and with double-point matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The absolute value matrix

$$|\sigma| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a transposition.

The main theorem (Theorem 13, §VI) states that a given representative of a class $x \in \pi_{n+1}^s(X)$ can be surgered to an immersion without n -tuple points provided the invariants c_n and ψ_{n+1} vanish ($n \neq 2, 3$, or 4). In case $n = 4$, the invariant ψ_n is trivial. In case $n = 3$ the n -tuple invariant ψ_3 detects the 3-torsion of $\pi_3^s \cong \mathbf{Z}/24$. The simplification of the self-intersection sets of immersed 3-manifolds is a more subtle problem than we care to consider here. Either [21 or 22] contains further information.

IV. The technique of surgery. To surger an immersion $i: M^n \hookrightarrow \mathbf{R}^{n+1}$ a hollow k -handle ($D^k \times S^{n-k}$) is attached to the manifold M , and this handle is embedded in Euclidean space via a map H . If the core disk of the handle is an embedded disk in the r -fold self-intersection set of (i, M) , then this is called (k, r) -surgery. Such a surgery affects the self-intersection sets in a prescribed fashion (Lemma 7). Specifically, let $h: D^k \hookrightarrow \mathbf{R}^{n+1}$ denote an embedded disk in \mathbf{R}^{n+1} such that $h(\text{int } D^k)$ is contained in the r -fold multiple set and $h(\partial D^k)$ is contained in the $(r + 1)$ -tuple set.

A framing for this disk will be given momentarily. Using this framing, an embedding $\bar{H}: D^k \times D^{n-k+1} \hookrightarrow \mathbf{R}^{n+1}$ is defined. Let $H = \bar{H}|_{D^k \times S^{n-k}}$. Let

$$\bar{M} = M \setminus (S^{k-1} \times D^{n-k+1}) \cup (D^k \times S^{n-k})$$

where the union takes place among their common boundary, and define $\bar{i}: \bar{M} \hookrightarrow \mathbf{R}^{n+1}$ by

$$\bar{i}(x) = \begin{cases} i(x) & \text{if } x \in M, \\ H(x) & \text{if } x \in D^k \times S^{n-k}. \end{cases}$$

LEMMA 7. A. *If $0 < j \leq r + 1$ and $y \in \mathbf{R}^{n+1}$ is a j -tuple point of (i, M) in the neighborhood $\bar{H}(S^{k-1} \times \text{int } D^{n-k+1})$ of $h(S^{k-1})$, then y is a $(j - 1)$ -tuple point of (\bar{i}, \bar{M}) .*

B. *Given $0 \leq j \leq r$ and a subset $\{l_1, l_2, \dots, l_j\} \subset \{1, \dots, r\}$, then the points $H(x, y)$ for which $y_i = 0$ if and only if $i = l_m$ for some $m = 1, 2, \dots, j$ are j -tuple points of (i, M) and $(j + 1)$ -tuple points of (\bar{i}, \bar{M}) . The point $y \in S^{n-k}$ has coordinates $y = (y_1, \dots, y_r, \dots, y_{n+1-k})$.*

C. *A surgery of type $(n - r + 1, r)$ eliminates the sphere $h(S^{n-r})$ of $(r + 1)$ -tuple points.*

D. *Therefore, a type (k, r) -surgery adds $\binom{r}{j}$ handles of dimension k to the $(j + 1)$ -tuple point set.*

Figure 3 is a facsimile of a figure on page 409 of [1]. It illustrates that a $(1, n)$ -surgery eliminates a pair of $(n + 1)$ -tuple points. The notion of (k, r) -surgery is the natural generalization of the concept given by Banchoff.

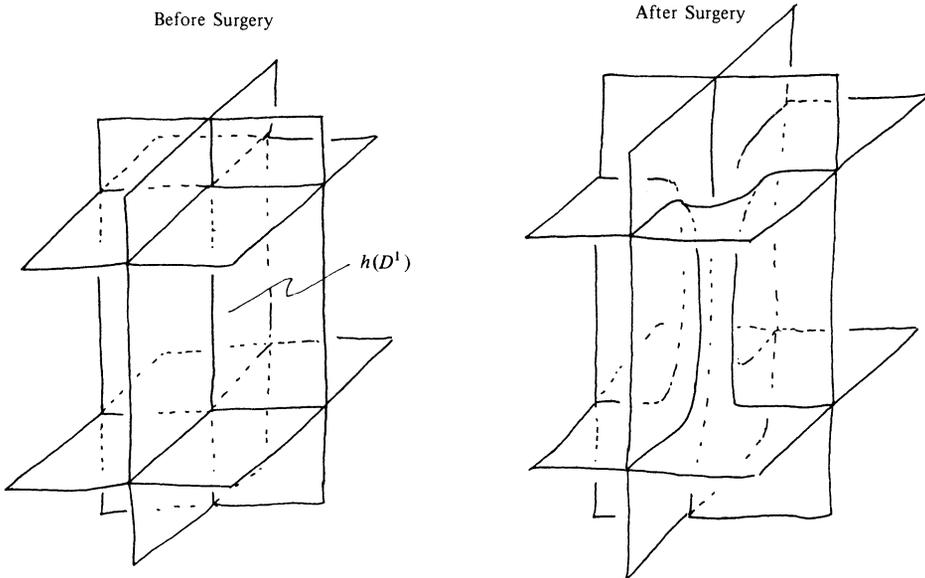


FIGURE 3

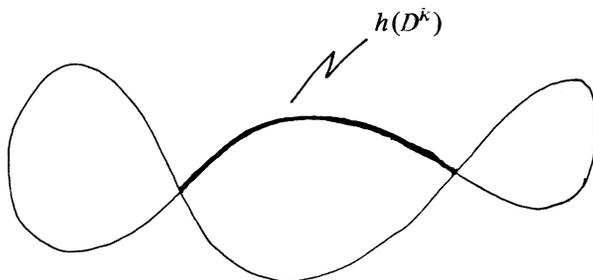


FIGURE 4

PROOF OF LEMMA 7. The proof depends on a rigorous definition of (k, r) -surgery. In particular, the promised framing of the core disk is given below.

Let $i: M^n \hookrightarrow \mathbb{R}^{n+1}$ be fixed and let r be an integer between 0 and $(n + 1)$ inclusive; the r -tuple point set

$$T_r(i) = \{y \in \mathbb{R}^{n+1}: \#i^{-1}(y) = r\}$$

has dimension $n + 1 - r$. The closure of $T_r(i)$ is the image of the r -tuple manifold under the induced immersion $i_r: M(r) \rightarrow \mathbb{R}^{n+1}$. Let k be an integer with $1 \leq k \leq n + 1 - r$ and let $h: D^k \hookrightarrow \mathbb{R}^{n+1}$ be an embedding with $h(\text{int } D^k)$ contained in $T_r(i)$ and $h(\partial D^k) \subset T_{r+1}(i)$ (see Figure 4). The disk $h(D^k)$ will be the core of the “hollow” k -handle which is to be attached to (i, M) .

For $x \in D^k$ let $z_1(x), \dots, z_r(x)$ be unit vectors normal to the immersion (i, M) at the points $p_j(x) \in i^{-1}(h(x))$, $j = 1, \dots, r$. If $k < n + 1 - r$, let $z_{r+1}(x), \dots, z_{n+1-k}(x)$ be unit vectors normal to the embedding of $h(D^k)$ in the r -tuple set such that $\{z_1(x), \dots, z_{n+1-k}(x)\}$ is an orthonormal basis for $\nu(h)_x$ as in Figure 5. Each vector $z_j(x)$ should depend smoothly on $x \in D^k$.

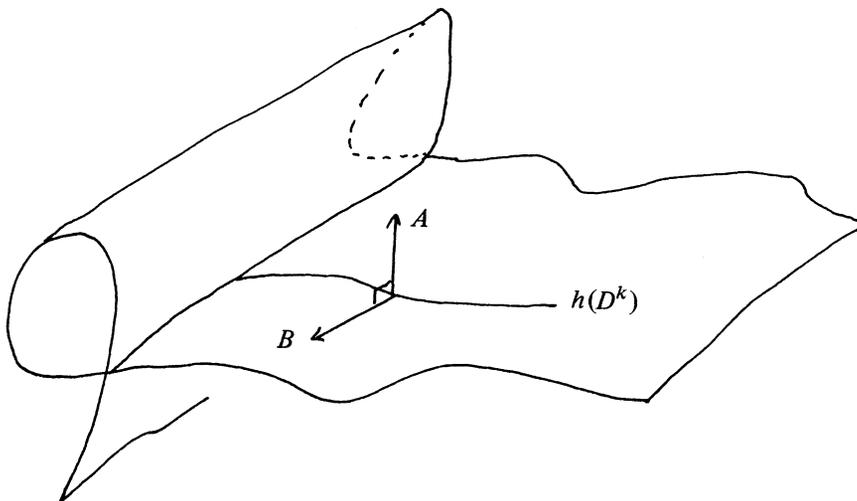


FIGURE 5

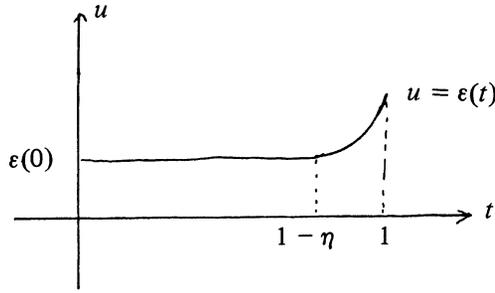


FIGURE 6

When $x \in \partial D^k$, the inward pointing normal of $S^{k-1} \subset D^k$ at x maps to the $(r + 1)$ st vector normal to (i, M) at $i^{-1}(h(x))$. The collection of these vectors as x ranges through S^{k-1} determines a lift of S^{k-1} into M . Furthermore, this lift has a trivial normal bundle in M determined by $z_1(x), \dots, z_{n+1-k}(x)$. The (k, r) -surgery to be defined will replace a sufficiently small copy of $S^{k-1} \times D^{n-k+1}$ in M with $D^k \times S^{n-k}$. Let \bar{M} be as above.

The tubular neighborhood $\bar{H}: D^k \times D^{n-k+1} \rightarrow \mathbf{R}^{n+1}$ of $h(D^k)$ is defined by the equation

$$(1) \quad \bar{H}(x, (y_1, \dots, y_{n+1-k})) = h(x) + \epsilon(|x|) \sum_{j=1}^{n+1-k} y_j z_j(x).$$

The function $\epsilon: [0, 1] \rightarrow \mathbf{R}$ has graph depicted in Figure 6; when $t \in [1 - \eta, 1]$, the graph looks like the fourth quadrant of a circle, otherwise $\epsilon(x) > 0$ is sufficiently small. Such a normalization of the belt direction has the effect of ensuring the immersion formed by surgery along H is C^1 . (A similar device is used in Lemmata 9 through 11; see Figure 9.) A surgery of type (k, r) to the immersion (i, M) is defined by forming the immersion $\bar{i}: \bar{M} \rightarrow \mathbf{R}^{n+1}$ as above. Now the lemma follows from the definition of the framing.

In most cases (Lemmata 8–12) (k, r) -surgery is sufficient, but the hypothesis that $h(\text{int } D^k) \subset T_r(i)$ is very restrictive.

The definition of transverse (k, r) -surgery is similar to that of (k, r) -surgery; however, the following weaker hypotheses are employed.

- (i) $h(\text{int } D^k) \subset \{y \in \mathbf{R}^{n+1} \mid r \leq \#i^{-1}(y) \leq r + k\}$,
 - (ii) $h(\partial D^k) \subset \{y \in \mathbf{R}^{n+1} \mid r + 1 \leq i^{-1}(y) \leq r + k\}$,
 - (iii) $h(\text{int } D^k)$ intersects the self-intersection strata $T_{r+1}(i), \dots, T_{r+k}(i)$ transversely, and
 - (iv) $h(\partial D^k)$ intersects the self-intersection strata $T_{r+2}(i), \dots, T_{r+k}(i)$ transversely.
- As in Lemma 7C, it is clear that transverse (k, r) -surgery reduces the multiplicity of the points $h(S^{k-1})$ by 1.

In general, a $(k - 1)$ -sphere in the $(r + 1)$ -tuple set may not bound a k -disk in the r -tuple point set. However, that situation may be constructed by an inductive

process. The inductive step roughly states: If it is possible to perform (k, r) -surgery, then it is possible to perform $(k, r + 1)$ -surgery. Rigorously, we need the following hypotheses:

Let $i: M^n \looparrowright \mathbf{R}^{n+1}$ be a general position immersion, and let k and r be integers with $1 \leq k \leq n - r$. Suppose there is an embedded “annulus” $A: I \times S^{k-1} \hookrightarrow \mathbf{R}^{n+1}$ which satisfies the conditions:

1. The sphere $a_0 = A(0, S^{k-1})$ is contained in the $(r + 2)$ -tuple point set.
2. The half-open annulus $A((0, 1] \times S^{k-1})$ is in the $(r + 1)$ -tuple point set.
3. For $t \in I$ and $\theta \in S^{k-1}$ there are orthonormal vectors $z_1(t, \theta), \dots, z_{n+1-k}(t, \theta)$ (depending smoothly on t and θ) such that $z_1(t, \theta), \dots, z_{r+1}(t, \theta)$ are normal to (i, M) at $A(t, \theta)$ and $z_{r+2}(t, \theta), \dots, z_{n+1-k}(t, \theta)$ are normal to A in the $(r + 1)$ -tuple manifold.
4. The sphere $a_1 = A(1, -): S^{k-1} \hookrightarrow \mathbf{R}^{n+1}$ extends to an embedded disk $\bar{a}_1: D^k \hookrightarrow \mathbf{R}^{n+1}$ whose interior is contained in the r -tuple point set, and it is possible to surger (i, M) along a hollow neighborhood $H: D^k \times S^{n-k} \hookrightarrow \mathbf{R}^{n+1}$ of \bar{a}_1 . Let (\bar{i}, \bar{M}) denote the result of this (k, r) -surgery.

LEMMA 8. Under these hypotheses it is possible to perform $(k, r + 1)$ -surgery to (\bar{i}, \bar{M}) along a hollow tubular neighborhood of a disk $h_0: D^k \rightarrow \mathbf{R}^{n+1}$, where $h_0(\theta) = a_0(\theta)$ for $\theta \in S^{k-1} = \partial D^k$.

PROOF. The idea of the proof is to reparametrize the handle H so that the trivialization of $\nu(\bar{a}_1)$ is compatible with the trivialization of $\nu(A)$. The notational details of this reparametrization are straightforward and left to the reader. Then the annulus $A([0, 1 - \epsilon] \times S^{k-1})$ is joined to the disk $H(D^k, (0, \dots, 0, -1, 0, \dots, 0))$ where -1 occurs in the correct entry; doing this yields h_0 . (See Figure 7.) This completes the proof.

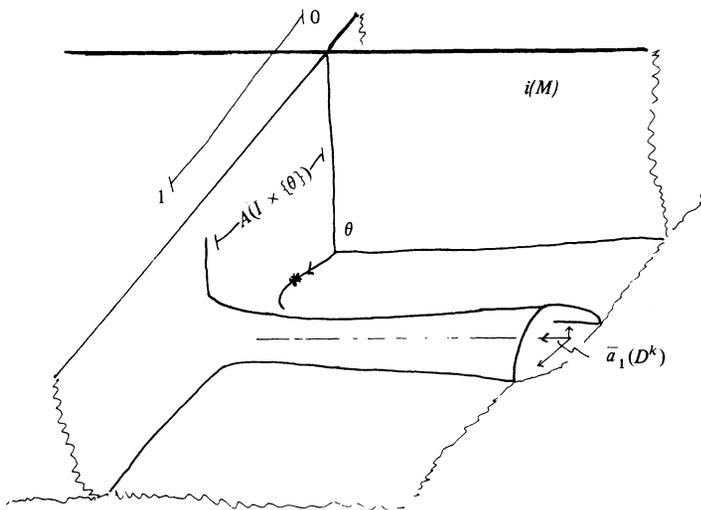


FIGURE 7

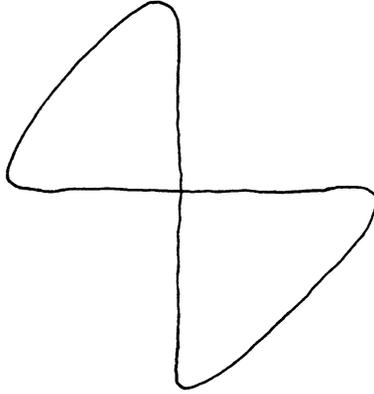


FIGURE 8

Lemma 8 will be used tacitly throughout §V. Figure 8 illustrates a generator of $\pi_1^s \cong \pi_2^s(P^\infty)$. The author wishes to call an eight: an eight.

Given a k -sphere $\bar{h}: S^k \rightarrow \mathbf{R}^{n+1}$ embedded in the $(n - k + 1)$ -tuple point set of an immersion that sphere may be removed by performing $(k + 1, n - k)$ -surgery provided \bar{h} extends to an embedded disk $h: D^{k+1} \rightarrow \mathbf{R}^{n+1}$ whose interior lies in the $(n - k)$ -tuple set. For in this case the trivialization of the bundle $\nu(\bar{h}) \oplus TS^k$ which is induced from (i, M) extends across $h(D^{k+1})$. The proviso is seldom satisfied even when the self-intersection invariants vanish. For example, the n -tuple matrix of a given immersion may be a nontrivial commutator. The following section contains the techniques for dealing with this problem.

V. Examples of surgery. There are two objectives to this section. The first is to illustrate the techniques of surgery developed above. The second is to produce examples of bordantly trivial immersions with ostensibly complicated n -tuple behavior. The existence of such examples is a key element in the proof of Theorem 13.

Preliminaries. Consider in \mathbf{R}^{n+1} the first k -hyperplanes

$$\pi_j = \{(x_1, \dots, x_j, \dots, x_{n+1}) \in \mathbf{R}^{n+1}: x_j = 0\}, \quad j = 1, \dots, k.$$

These extend (in $\mathbf{R}^{n+1} \cup \{\infty\}$) to produce an immersion $\cup_{j=1}^k S_j^n \hookrightarrow \mathbf{R}^{n+1} \cup \{\infty\}$ whose k -tuple manifold consists of a $(n + 1 - k)$ dimensional sphere. By removing a point in S^{n+1} not in the image of this immersion, an immersion $f: \cup_{j=1}^k S_j^n \hookrightarrow \mathbf{R}^{n+1}$ is obtained. This immersion (f, \mathbf{US}^n) represents the trivial element of π_n^s .

Consider the case $k = n$. The n -tuple curve of f is a simple closed curve. In the sequel it will be convenient to work with the hyperplanes; the n -tuple curve then is identified with the extended x_{n+1} -axis. The framing on this curve (and consequently the n -tuple matrix) is trivial. To alter the n -tuple data, a sequence of 1-handles is attached.

DEFINITIONS. Let a and b denote points in the image of f which are not multiple points of (f, \mathbf{US}^n) . These points are chosen sufficiently close to the n -tuple curve. An

initial core is an embedded arc $h: [0, 1] \rightarrow \mathbf{R}^{n+1}$ so that $h(0) = a$, $h(1) = b$ and these are the only points of the arc which intersect the immersion. In practice the initial core is similar to one of the arcs depicted in Figure 10 or 11(a) (cf. Lemma 12; the initial core is a composite arc in this case). In any case $h'(t)$ is normal to the immersion for $t = 0, 1$.

An *initial framing curve for the core* $h[0, 1]$ is a map $\alpha: [0, 1] \rightarrow SO(n + 1)$ such that the first column $\alpha_1(t)$ of the matrix $\alpha(t)$ is the unit tangent to the initial core at $h(t)$. The framing matrix $\alpha(t)$ and the velocity $h'(t)$ are held constant for $t \in [0, \eta] \cup [1 - \eta, 1]$ for $\eta > 0$ appropriately small.

Using these initial data it is possible to perform a $(1, 0)$ -surgery to $(f, \cup S^n)$, by attaching the handle $H: I \times S^{n-1} \rightarrow \mathbf{R}^{n+1}$,

$$(1) \quad H(t, (x_1, \dots, x_n)) = h(t) + \delta(t) \sum_{i=1}^n \alpha_{i+1}(t)x_i$$

where the path $\delta: [0, 1] \rightarrow \mathbf{R}$ is depicted in Figure 9. (Multiplication by δ has the effect of rounding some corners).

This handle is to be used to define subsequent surgeries that will alter the n -tuple data in a prescribed fashion. By specifying the coordinates of $h(0)$ and $h(1)$, and specifying the matrices $\alpha(0)$ and $\alpha(1)$ concurrently, this information can be used to determine the surgeries. The prescribed n -tuple behavior is given provided certain conventions are adopted. The following lemma exemplifies these conventions.

LEMMA 9 (CONSTRUCTION). *There exists an immersed closed oriented 4-manifold in \mathbf{R}^5 with quadruple-point matrix $\sigma = (e_4, e_3, e_2, e_1)$ (where e_j is the standard j th unit column vector).*

For $n = 4$ choose an initial core h with $h(0) = (1, 0, -1, -1, 1)$, $h'(0) = -e_2$, $h(1) = (1, -1, 0, -1, -1)$, and $h'(1) = e_3$. The arc h is to lie in the three-dimensional cross section $x_4 = -1$, $x_1 = 1$. Such an arc is illustrated in Figure 10 where the x -axis points in the negative e_3 -direction, the y -axis points in the negative e_2 direction, and the z -axis is parallel to the quadruple point curve of the immersion $(f, \cup S^n)$.

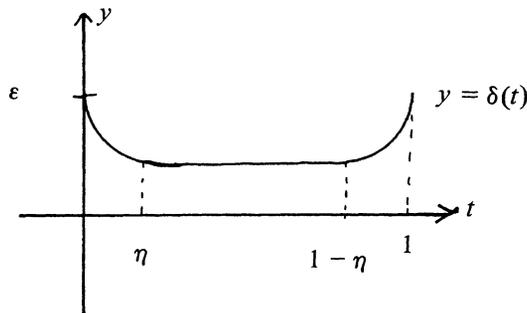


FIGURE 9

Set

$$\alpha(0) = (-e_2, -e_3, -e_4, e_1, e_5), \quad \alpha(1) = (e_3, e_1, -e_2, -e_4, e_5),$$

$$\|h'(t)\|_{\alpha_1(t)} = h'(t);$$

$\alpha(t) \in SO(5)$ is otherwise arbitrary for $t \in [0, 1]$.

Define a tubular neighborhood of h by equation (1) and perform a (1, 0)-surgery along h .

Define an arc $h_1: [\frac{1}{3}, \frac{2}{3}] \rightarrow \mathbf{R}^5$ parallel to $h(t)$ by the equation

$$h_1(t) = h(3t - 1) - \delta(3t - 1)\alpha_2(3t - 1).$$

Then join $h_1(\frac{1}{3})$ to the point $(1, 0, 0, -1, 1) = h_1(0)$ and $h_1(\frac{2}{3})$ to the point $(0, -1, 0, -1, -1) = h_1(1)$ by straight-line arcs with unit tangent vectors $\alpha_2(0)$, $-\alpha_2(1)$, respectively. Thus $h_1: [0, 1] \rightarrow \mathbf{R}^5$ is the resulting composite arc; the curve δ insures h_1 is C^1 and can be used to define a framing for h_1 in terms of the framing α . Specifically, let $\beta: [0, 1] \rightarrow SO(5)$ be defined with

$$\beta(0) = (-e_3, e_2, -e_4, e_1, e_5) \quad \text{and} \quad \beta(1) = (-e_1, e_3, -e_2, -e_4, e_5);$$

β is constant for $t \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and when $t \in [\frac{1}{3}, \frac{2}{3}]$ β rotates 90° up to $\alpha(0)$, and then rotates 90° down to $\alpha(1)$. Define a hollow tubular neighborhood of radius $\delta/2$ of h_1 in terms of β and perform a (1, 1)-surgery.

The arc $\bar{h}_2(t) = h_1(t) - (\delta(t)/2)\beta_3(t)$ is an arc of double points. The endpoints $\bar{h}_2(0)$ and $\bar{h}_2(1)$ are joined to the triple points $h_2(0) = (1, 0, 0, 0, 1)$ and $h_2(1) = (0, 0, 0, -1, -1)$ by straight-line arcs with unit tangent vectors $\beta_3(0)$ and $-\beta_3(1)$, respectively. Let $h_2: [0, 1] \rightarrow \mathbf{R}^5$ denote this composite arc. The arc h_2 is framed by a curve $\gamma: [0, 1] \rightarrow SO(5)$ with

$$\gamma(0) = (-e_4, e_2, e_3, e_1, e_5) \quad \text{and} \quad \gamma(1) = (e_2, e_3, -e_1, -e_4, e_5).$$

A type (1, 2)-surgery is then performed along a hollow tube of radius $\delta/4$.

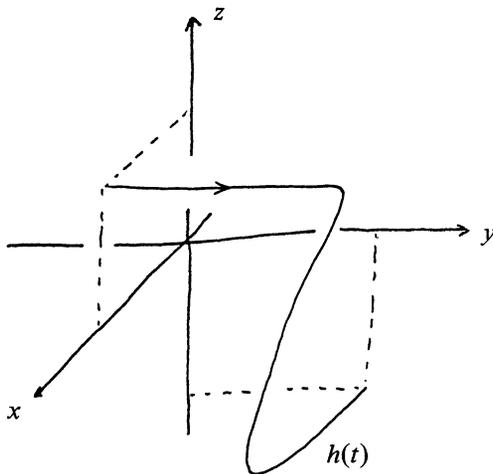


FIGURE 10

Finally, the arc $\bar{h}_3(t) = h_2(t) - (\delta(t)/4)\gamma_4(t)$ is an arc of triple points. The endpoints $\bar{h}_3(0)$ and $\bar{h}_3(1)$ are joined to the quadruple points $h_3(0) = (0, 0, 0, 0, 1)$ and $h_3(1) = (0, 0, \dots, 0, -1)$ by straight-line arcs, with unit tangent vectors $\gamma_4(0)$ and $-\gamma_4(1)$, respectively. The resulting arc $h_3: [0, 1] \rightarrow \mathbf{R}^5$ of triple points is framed by a path $\zeta: [0, 1] \rightarrow SO(5)$ with

$$\zeta(0) = (e_1, \dots, e_5) \quad \text{and} \quad \zeta(1) = (e_4, e_3, -e_1, e_2, e_5).$$

After (1, 3)-surgery the resulting immersion has n -tuple matrix $\sigma = (e_4, e_3, e_2, e_1)$. This completes the construction.

Notice that the resulting n -tuple matrix depended only on the original choice of framing and the points $h(0)$ and $h(1)$. Specifically, the middle columns of $\alpha(0)$ and $\alpha(1)$ determined the unit tangents of straight-line arcs used to connect subsequent self-intersection points; thus endpoints $h_j(0)$ and $h_j(1)$ were determined by the $(1 + j)$ th columns of $\alpha(0)$ and $\alpha(1)$, respectively. The framing of the arc $h_j(t)$ was determined by the framing of the arc $h_{j-1}(t)$ and by 90° rotations determined by $\delta(t)$. Holding $\alpha_5(0) = \alpha_5(1) = e_5$ ensured that the resulting n -tuple curve was connected. It is easy to write down the conventions for subsequent surgeries explicitly and thus prove the following lemmata, each of which is a construction.

LEMMA 10. *For $n \geq 5$ there is an immersion representing the identity of π_n^s with n -tuple matrix an arbitrary (unsigned) three cycle.*

LEMMA 11. *For $n \geq 4$ there is an immersion which represents the identity of π_n^s whose n -tuple data include a trivial n -tuple matrix and twisted normal framing.*

LEMMA 12. *For $n \geq 3$ there is an immersion representing the identity of $\pi_{n+1}^s(P^\infty)$ whose n -tuple matrix is diagonal with two of the diagonal entries equal to -1 .*

PROOF OF LEMMA 10. Let $h: [0, 1] \rightarrow \mathbf{R}^{n+1}$ be an arc similar to that pictured in Figure 10 with endpoints $h(0) = (1, 0, -1, \dots, -1, 1)$ and $h(1) = (1, -1, 0, -1, \dots, -1)$. So in Figure 10 the x -axis points in the negative e_3 -direction, the y -axis points in the negative e_2 -direction, and the z -axis points in the positive e_{n+1} -direction. This arc is the initial core. The initial framing $\alpha: I \rightarrow SO(n + 1)$ is defined with

$$\alpha(0) = (-e_2, -e_3, \dots, -e_n, e_1, e_{n+1})$$

and

$$\alpha(1) = (e_3, -e_2, -e_4, -e_5, \dots, -e_{n-2}, e_1, -e_{n-1}, -e_n, e_{n+1}).$$

An immersion with n -tuple matrix $\sigma = (e_1, \dots, e_{n-3}, e_{n-1}, e_n, e_{n-2})$ arises by performing the canonical sequence of surgeries outlined above. Since the n -tuple matrix depends on the initial choice of basis, any other 3-cycle can be obtained by a change of basis.

PROOF OF LEMMA 11. Let $h: [0, 1] \rightarrow \mathbf{R}^{n+1}$ be as in the proof of Lemma 9. Let $\alpha: I \rightarrow SO(n + 1)$ be a framing for h with

$$\alpha(0) = (-e_2, -e_3, \dots, -e_n, e_1, e_{n+1}),$$

and

$$\alpha(1) = (e_3, -e_2, -e_4, -e_5, \dots, -e_n, e_1, e_{n+1}).$$

Perform a sequence of surgeries as above. Then the resulting n -tuple matrix is trivial.

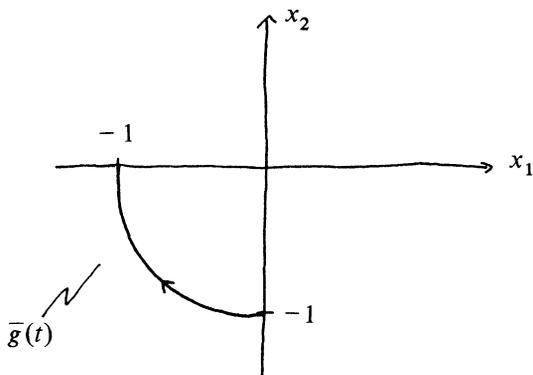


FIGURE 11(a)

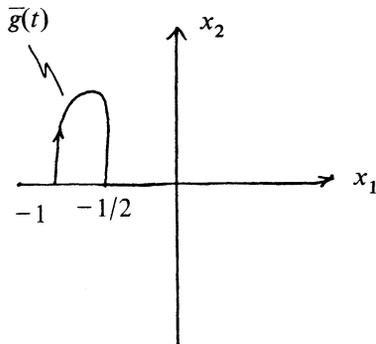


FIGURE 11(b)

That the normal frame is twisted can be seen as follows. The n -tuple curve has two parallel segments found on the last 1-handle. The normal frame of a particle inherits a one half-twist as it travels along either of these segments. The torques of these two twists are the same. Thus a twisted framing results.

PROOF OF LEMMA 12. This last construction requires slightly more work. The first step is to “dig a tunnel” between two hyperplanes along the path $g(t) = (-\sin(\pi/2)t, -\cos(\pi/2)t, -1, \dots, -1)$ depicted in Figure 11(a). That is, perform a $(1, 0)$ -surgery along any trivialization of that arc.

The initial core now is the arc $h: [0, 1] \rightarrow \mathbf{R}^{n+1}$ with $h(0) = (1, 0, -1, \dots, -1, 1)$ and $h(1) = (-\frac{1}{2}, 0, -1, \dots, -1)$ defined by

$$h(t) = \begin{cases} \bar{h}(3t) & \text{if } t \in [0, \frac{1}{3}], \\ g(3t - 1) & \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \bar{g}(3t - 2) & \text{if } t \in [\frac{2}{3}, 1], \end{cases}$$

The arc $\bar{h}(t)$ is similar to that depicted in Figure 10; $g(t)$ and $\bar{g}(t)$ are seen in Figure 11.

The framing path for the core h is a path $\alpha: [0, 1] \rightarrow SO(n + 1)$ with

$$\alpha(0) = (-e_2, \dots, -e_n, e_1, e_{n+1}) \quad \text{and} \quad \alpha(1) = (-e_2, \dots, -e_{n-1}, -e_1, -e_n, e_{n+1}).$$

By proceeding as above, an n -tuple matrix $\sigma = (e_1, \dots, -e_{n-1}, -e_n)$ is obtained.

VI. Removing spheres of self-intersection. The above machinery will be used to prove the main theorem.

THEOREM 13. *Let $n \geq 5$. Let $i: M^n \hookrightarrow \mathbf{R}^{n+1}$ be an immersion on which the $(n + 1)$ -tuple point invariant vanishes. Then there is an immersion (i_1, M_1) obtained from (i, M) by surgery so that (i_1, M_1) has no n -tuple points. If $[i, M] \in \pi_n^s$, then $[i_1, M_1] = [i, M] \in \pi_n^s$.*

REMARK. If $n = 4$, this question is moot since $\pi_4^s = \pi_5^s(P^\infty) = 0$; see e.g. [18 or 23]. Moreover, the techniques above break down for this case; see Lemma 10.

A sketch of the proof is as follows. In Proposition 14, Lemmata 10–12 are used to simplify the n -tuple set. Then the general question of removing a k -sphere of $(n + 1 - k)$ -tuple points from the self-intersection set is addressed. Lemma 15

reduces the question to a question on surgering the equatorial immersion to an immersion without k -dimensional multiple points. Lemma 16 achieves this for $k = 1$.

PROPOSITION 14. *Let $n \geq 5$. Let (i, M^n) be a codimension one immersion on which the $(n + 1)$ -tuple invariant vanishes. There is an immersion (i_1, M_1) obtained from (i, M) by surgery that has n -tuple data consisting of a simple closed curve with trivial n -tuple matrix and untwisted framing. If (i, M) is oriented, then oriented surgery suffices.*

Step 1. Remove pairs of $(n + 1)$ -tuple points by surgeries of type $(1, 0), \dots, (1, n)$. It may be necessary to perform several surgeries of type $(1, 0)$ if a pair of $(n + 1)$ -tuple points links the nonmultiple points in the image of the immersion.

Step 2. Connect the components of the n -tuple curve by using surgeries of type $(1, 0), \dots, (1, n - 1)$. Theorem 6. III implies that the n -tuple invariant is trivial. Therefore, the n -tuple matrix σ of this immersion is the commutator subgroup of SH_n . Therefore, $\pi(\sigma)$ is alternating (see Proposition 4 and the exact sequence describing the structure of H_r).

Step 3. The commutator subgroup of SH_n is generated by 3-cycles and diagonal matrices. The models of n -tuple behavior given in Lemma 10 and Lemma 12 can be inserted in small neighborhoods of the n -tuple curve to cancel the n -tuple matrix. A twisted framing is cancelled by means of Lemma 11.

Step 4. In case (i, M) is oriented, there is a choice of basis for the normal framing of the n -tuple curve so that the n -tuple matrix has no minus signs. Proceed as in Step 3 using only Lemmata 10 and 11.

This completes the proof.

Suppose $i: M^n \looparrowright \mathbf{R}^{n+1}$ is an immersion with $T_j(i) = \emptyset$ for $j > n + 1 - k$ and $T_{n+1-k}(i) = S^k$. Let $f: S^k \rightarrow \mathbf{R}^{n+1}$ denote this embedding. The sphere is said to be *framed for surgery* if for $j = 1, \dots, n + 1 - k$ there are unit vectors $\gamma_j(\theta) \in \mathbf{R}^{n+1}$ which are normal to (i, M) at $p_j(\theta) \in i^{-1}(f(\theta))$ such that one of the annuli $f(\theta) \pm t\gamma_j(\theta)$ extends to an embedded disk, and the remaining $\gamma_r(\theta)$ extend to a framing over that disk. Here $\theta \in S^k$ and each γ_j and p_j is a smooth function of θ . The conclusion of Proposition 14 may be rephrased to say “the n -tuple curve of (i_1, M_1) is framed for surgery.”

THEOREM 15. *Let $i: M \looparrowright \mathbf{R}^{n+1}$ be an immersion for which $T_j(i) = \emptyset$ for $j > n + 1 - k$ and $T_{n+1-k}(i) = S^k$. Suppose S^k is framed for surgery. Then there is an immersion bordant to (i, M) with no $(n + 1 - k)$ -tuple points provided one of the following conditions holds.*

(A) *Some push off of S^k bounds a disk in the $(n + 1 - r - k)$ -tuple set.*

(B) *It is possible to remove the k -dimensional stratum from the immersion $\alpha: \bigcup_{j=1}^{m+1} S_j^{m-1} \rightarrow S^m$ defined by*

$$\alpha(x_1, \dots, \hat{x}_j, \dots, x_{m+1}) = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{m+1})$$

via a bordism without $(m + 1)$ -tuple points. Here $m = n - k$.

PROOF. First assume that some push off of this k -sphere bounds a disk in the $(n + 1 - r - k)$ -tuple set where $r > 0$. Then Lemma 8 shows that this sphere may be removed, for since S^k is framed for surgery, hypotheses 3 and 4 are satisfied.

If no push off bounds a disk in the next lower stratum, then the sphere links the image $i(M)$. In this case there is an embedding $\tilde{f}: D^{k+1} \rightarrow \mathbf{R}^{n+1}$ so that the intersection of $\tilde{f}(\text{int } D^{k+1})$ with (i, M) is a closed k -manifold immersed in the $(k + 1)$ -disk. The extension exists since $f(S^k)$ is framed for surgery. Furthermore in a sufficiently small neighborhood of $\tilde{f}(\text{int } D^{k+1})$ the immersion (i, M) looks like a product of some $g: N^k \hookrightarrow \mathbf{R}^{k+1}$ and an $(n - k)$ -disk; (g, N^k) is the immersion found in $\tilde{f}(D^{k+1})$.

Let $m = n - k$. Perform transverse $(k + 1, 0), \dots, (k + 1, m)$ -surgeries to remove $f(S^k)$ from the intersection set of (i, M) . Let (i, M) again denote the result. Then in a neighborhood of $\tilde{f}(D^{k+1})$, the immersion (i, M) looks like the product of $g(N^k)$ and the immersion $(\alpha, \bigcup_{j=1}^{m+1} S_j^{m-1})$. The surgeries used to remove the k -dimensional set of α extend in \mathbf{R}^{n+1} to surgeries on (i, M) of the same type. Perform these surgeries in such a way that the diameter of the cocore of any handle is larger than the diameter of $g(N^k)$. Then the k -dimensional stratum of (i, M) will be removed. If the bordism of α has $(m + 1)$ -tuple points, the linking phenomenon will reappear. This completes the proof of 15.

LEMMA 16. *It is possible to remove the 1-dimensional self-intersections from the immersion $(\alpha, \bigcup_{j=1}^{m+1} S_j^{m-1})$ for all m .*

PROOF. In [21] the 0- and 1-dimensional strata are removed from the immersions $\alpha: S^1 \cup S^1 \cup S^1 \rightarrow S^2$ and $\alpha: \bigcup_{j=1}^4 S_j^2 \rightarrow S^3$. The null bordisms of these removals have no 0-dimensional multiple points. Use these null bordisms to remove the 0- and 1-dimensional strata from the immersion α in S^m as follows.

The set $\{1, 2, \dots, m + 1\}$ will be partitioned according to the residue of $m + 1$ modulo 4. If $m + 1 = 0 \pmod{4}$, then the partition is $\{1, 2, 3, 4\}, \dots, \{m - 2, m - 1, m, m + 1\}$. If $m + 1 = 2 \pmod{4}$ and $m > 2$, then the partition is $\{1, 2, 3, 4\}, \dots, \{m - 4, m - 3, m - 2\}, \{m - 1, m, m + 1\}$. If $m + 1 = 3 \pmod{4}$, then the partition is $\{1, 2, 3, 4\}, \dots, \{m - 5, m - 4, m - 3, m - 2\}, \{m - 1, m, m + 1\}$. In case $m + 1 = 1 \pmod{4}$ and $m + 1 > 5$, then use the partition $\{1, 2, 3, 4\}, \dots, \{m - 7, m - 6, m - 5\}, \{m - 4, m - 3, m - 2\}, \{m - 1, m, m + 1\}$. The cases when $m + 1 = 2$ or 5 will be handled separately.

In each of the 3-dimensional subspheres

$$S(\{j, j + 1, j + 2, j + 3\}) = \{(0, x_j, x_{j+1}, x_{j+2}, x_{j+3}, 0) : \sum x_i^2 = 1\},$$

where $\{j, j + 1, j + 2, j + 3\}$ is in the partition of $\{1, \dots, m + 1\}$, remove the 0- and 1-dimensional strata as in §III of [21]. In the similarly defined 2-dimensional subspheres, remove the 1-dimensional stratum as in §II of [21]. The surgeries used to remove these self-intersections extend in S^m to surgeries of the same dimension. Thus the 0- and 1-dimensional strata can be removed from α . In case $m + 1 = 2$, then the 0-dimensional stratum can easily be removed (see Figure 8). In case $m + 1 = 5$, then use the partition $\{1, 2, 3, 4\}, \{5\}$. The 0- and 1-dimensional strata in

the arctic and antarctic regions can also be removed (see [2, pp. 130–140] for further details).

This completes the proof of 16 and 13.

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