POLAR CLASSES AND SEGRE CLASSES
ON SINGULAR PROJECTIVE VARIETIES

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ABSTRACT. We investigate the relation between polar classes of complex varieties
and the Segre class of K. Johnson [Jo]. Results are obtained for hypersurfaces of
projective spaces and for certain varieties with isolated singularities.

0. Introduction. The relation between polar classes and Chern classes (Chern-
Mather and Chern-MacPherson classes) of singular complex varieties has been
studied by several authors (Dubson, Lê, Teissier, Piene, etc.). This paper is moti-
vated by trying to understand the relation between Chern classes and Segre classes
[Jo] of singular varieties, which has not been clarified yet. As one of the steps for this
we have tried to capture the relation between Segre classes $S^*_X$ defined by K.
Johnson [Jo] and our Segre-Mather classes $S^M_*(X)$ defined in a similar manner to
that of Chern-Mather classes $C^M_*(X)$.

Our first main theorem is

THEOREM A. Let $X^n \subseteq \mathbb{P}^{n+1}$ be a reduced hypersurface with $S$ denoting the singular
subvariety of $X$. Then we have

$$S^i_*(X) = S^M_i(X) + S^{i-1}(\mathbb{P}^{n+1}) \cap \left( \sum_j e_j [S_{n-1,j}] \right),$$

where $S_{n-1,j}$ are irreducible components of dimension $n - 1$ of singular subvariety $S$ of
$X$, $e_j$ is the multiplicity of the Jacobian ideal in the local ring of $X$ at the generic point
of $S_{n-1,j}$ (e.g., see [F1, §4.3]), and $S^{i-1}(\mathbb{P}^{n+1})$ is the $(i - 1)$st usual Segre class of
$\mathbb{P}^{n+1}$.

Our second main theorem is for $X^n \subseteq \mathbb{P}^{2n}$ with isolated singularities. Let $x$ be a
singular point of $X$, let $P$ be a generic point off $X$, and let $H$ be a hyperplane not
containing $x$. Consider the affine variety $X^a = X - H \subseteq \mathbb{P}^{2n} - H = \mathbb{C}^{2n}$. Let $\epsilon$ be
a sufficiently small positive number and let $t$ be a complex number such that
$|t| \ll \epsilon$. Shift $X^a$ towards the point $P$ by the length $t$. Let $S_{P,t}(X^a)$ denote such a
shifted $X^a$. Then count the intersection points of $X^a$ and the shifted $S_{P,t}(X^a)$ within
the $\epsilon$-ball $B(x)$ around the singular point $x$. (For a generic point $P$, $S_{P,t}(X^a)$ and

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$X^a$ are transverse to each other within $B_i(x)$ for sufficiently small $\varepsilon$ and $t.$ This number is denoted $n(x, P, t, \varepsilon)$ and called the shift multiplicity of $x.$ With this new multiplicity, our second main theorem is

**Theorem B.** Let $X^a \subseteq \mathbf{P}^2n$ be a reduced singular variety with isolated singularities $x_1, x_2, \ldots, x_r.$ Then we have

$$S_i(X) = S_i^M(X) \quad \text{for} \quad i < n,$$

and

$$S_n(X) = S_n^M(X) + \sum_{i=1}^r n_i[x_i]$$

where $n_i$ is the shift multiplicity of each singularity $x_i.$

In §1 we discuss Chern-Mather and Segre-Mather classes and polar classes corresponding to them. §2 is a quick review of K. Johnson’s thesis [Jo]. §3 deals with the decomposition of the class $[P(X)],$ which appears in the definition of Johnson’s Segre classes $S_a(X).$ §§4 and 5 contain our main results, for hypersurfaces and for $X^a \subseteq \mathbf{P}^2n$ with isolated singularities, respectively.

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1. Chern-Mather, Segre-Mather, and polar classes. Let $X^a$ be a projective variety of pure dimension $n$ in the complex projective space $\mathbf{P}^N.$ A variety is understood to be a reduced scheme (possibly reducible). Let $\text{Gr}(T\mathbf{P}^N, n)$ be the Grassmannian bundle over $\mathbf{P}^N,$ its fiber over a point $x \in \mathbf{P}^N$ is the Grassmannian $\text{Gr}(T_x\mathbf{P}^N, n)$ of $n$-planes in the tangent space $T_x\mathbf{P}^N$ at $x.$ Let $X_{sm}$ denote the open dense subvariety of nonsingular points. We consider the canonical embedding

$$g: X_{sm} \hookrightarrow \text{Gr}(T\mathbf{P}^N, n), \quad x \mapsto T_xX_{sm}.$$ 

The closure of the image $g(X_{sm})$ is called the Nash blowup of $X$ and is denoted by $\hat{X};$ the Nash blowup map $\nu: \hat{X} \to X$ is the restriction of the projection map $\pi: \text{Gr}(T\mathbf{P}^N, n) \to \mathbf{P}^N.$ It is well known (e.g., [Du or Go]) that (i) $\hat{X}$ is algebraic, (ii) $\nu: \hat{X} \to X$ is algebraic proper, and (iii) over $X_{sm}$ $\nu$ is an (algebraic) isomorphism. The restriction to $\hat{X}$ of the tautological bundle on $\text{Gr}(T\mathbf{P}^N, n)$ is called the Nash tangent bundle of $\hat{X}$ and is denoted by $\hat{T}X.$

**Definition.** The $i$th Chern-Mather class $C_i^M(X)$ is defined by

$$C_i^M(X) = \nu_*\left(c_i(\hat{T}X) \cap [\hat{X}] \right),$$

where $c_i(\hat{T}X)$ is the $i$th Chern class of $\hat{T}X.$ The $i$th Segre-Mather class $S_i^M(X)$ is defined by

$$S_i^M(X) = \nu_*\left(s_i(\hat{T}X) \cap [\hat{X}] \right),$$

where $s_i(\hat{T}X)$ is the $i$th inverse Chern class of $\hat{T}X.
Let \( \mathbf{P}(\bar{TX}) \) be the projectivization of the Nash tangent bundle, \( \mathcal{O}_{\mathbf{P}(\bar{TX})}(1) \) the dual of the tautological line bundle over \( \mathbf{P}(\bar{TX}) \), \( t: \mathbf{P}(\bar{TX}) \to \hat{X} \) the projection map, \( p_1 = v \cdot t \), and \( \theta = \xi_1(\mathcal{O}_{\mathbf{P}(\bar{TX})}(1)) \). Then it is well known (e.g., [KI]) that \( S_i^M(X) = p_1^*(\theta^{n-1+i} \cap [\mathbf{P}(\bar{TX})]) \).

Let \( \text{Gr}(N, n) \) denote the Grassmannian of \( \mathbb{P}^n \)'s in \( \mathbb{P}^N \). We define the projective Gauss map \( \hat{\gamma}: \hat{X} \to \text{Gr}(N, n) \) by \( \hat{\gamma}(x) = \xi \), where \( \xi \) is the unique linear subspace of \( \mathbb{P}^N \) of dimension \( n \) whose tangent space at \( x \) is \( \xi \). Let \( \hat{E}^{n+1} \) be the tautological bundle of rank \( n + 1 \) over \( \text{Gr}(N, n) \). Let \( \bar{TX} \) denote the pull-back of \( \hat{E}^{n+1} \) to \( \hat{X} \) via \( \hat{\gamma} \). Define \( \gamma: \text{Gr}(TP^N, n) \to \text{Gr}(N, n) \) by \( \gamma(\xi) = \hat{\gamma}(\xi) \) just as above, so that \( \hat{\gamma} = \gamma \) composed with inclusion. Let \( L_{\hat{X}} \) be the pull-back (via \( \nu: \hat{X} \to X \)) of the line bundle \( \mathcal{O}_X(-1) \) on the projective variety \( X \). There is an exact sequence (the Euler sequence)

\[
0 \to \pi^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \gamma^*\hat{E}^{n+1} \to E^n \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(-1) \to 0.
\]

See, for instance, [GH, p. 409]. Pulling back this exact sequence via the inclusion map \( i: \hat{X} \to \text{Gr}(TP^N, n) \) yields the exact sequence (cf. [GH, Sh])

\[
0 \to L_{\hat{X}} \to \bar{TX} \to \bar{TX} \otimes L_{\hat{X}} \to 0.
\]

The Chern and Segre classes of \( \bar{TX} \) are closely related to "polar loci". Let \( A = A^{N-n+k-2} \) be a linear subspace of dimension \((N-n+k-2)\) of \( \mathbb{P}^N \). The polar locus \( P(A) \) of \( X \) with respect to \( A \) is defined to be the closure of the locus of points \( x \) of \( X_{sm} \) such that the projective tangent space \( T_x X_{sm} \) intersects \( A \) in a space of at least \((k-1)\) dimension. Piene [Pi1, Pi2] showed that for a generic \( A \), the polar locus is of codimension \( k \), and that the homology class \( P_k(X) \) represented by the polar locus is independent of the choice of a generic \( A \). Explicitly, if \( \text{Sch}_k(A) \) denote the Schubert variety

\[
\{ P \in \text{Gr}(N, n) : \dim(P \cap A) \geq k - 1 \},
\]

then for a generic \( A \),

\[
P(A) = \nu(\hat{\gamma}^{-1}(\text{Sch}_k(A))), \quad \text{and} \quad P_k(X) = \nu_*([\hat{\gamma}^{-1}(\text{Sch}_k(A))]).
\]
Recall the Gauss-Bonnet Theorem (cf. [GH or KL]): The Poincaré dual of the Schubert cycle \([\text{Sch}_k(A)]\) is equal to \((-1)^k c_k(\tilde{E})\), i.e., \([\text{Sch}_k(A)] = (-1)^k c_k(\tilde{E}) \cap [\text{Gr}(N, n)]\).

Thus for a generic \(A\),

\[ P_k(X) = (-1)^k \nu_*(c_k(\tilde{TX}) \cap [\tilde{X}]). \]

From the Euler exact sequence we get

\[ c_k(\tilde{TX}) = \sum_{i=0}^{k} \binom{n+1-i}{k-i} c_i(\tilde{TX}) \cup c_1(L_X^*)^{k-i} \]

and

\[ c_k(\tilde{TX}) = \sum_{i=0}^{k} (-1)^{k-1} \binom{n+1-i}{k-i} c_i(\tilde{TX}) \cup c_1(L_X^*)^{k-i}. \]

These formulas imply the following Todd formulas for polar classes and Chern-Mather classes [Pl2]:

\[ C_k^M(X) = \sum_{i=0}^{k} (-1)^i \binom{n+1-i}{k-i} U^{k-i} \cap P_i(X), \]

and

\[ P_k(X) = \sum_{i=0}^{k} (-1)^i \binom{n+1-i}{k-i} U^{k-i} \cap C_i^M(X), \]

where \(U = c_1(\mathcal{O}_X(1))\).

Now let \(A = A^{N-m-1} (m \geq n)\) be a linear subspace of dimension \((N - m - 1)\) of \(\mathbb{P}^N\). The polar locus \(\overline{P}(A)\) of \(X\) with respect to \(A\) is defined to be the closure of the locus of points \(x\) of \(X_{sm}\) such that the projective tangent space \(T_x X_{sm}\) intersects \(A\). In [Jo and Pi1] it is shown that for a generic \(A\), \(\overline{P}(A)\) has the “expected” codimension \(k = m - n + 1\) and that its homology class \(\overline{P}_k(X)\) is independent of \(A\). We call this class the polar class.

Explicitly, if \(\text{Sch}'_1(A)\) denotes the special Schubert variety

\[ \{ P \in \text{Gr}(N, n) : P \cap A \neq \emptyset \}, \]

then for a generic \(A\),

\[ \overline{P}(A) = \nu(\gamma^{-1}(\text{Sch}'_1(A))) \quad \text{and} \quad \overline{P}_k(X) = \nu_*(\gamma^{-1}(\text{Sch}'_1(A))). \]

By the Gauss-Bonnet Theorem, the Poincaré dual of the homology class \([\text{Sch}'_1(A)]\) is equal to the \((m - n + 1)\)st Chern class \(c_{m-n+1}(Q)\) of the tautological rank \((N - n)\) quotient bundle \(Q\) of the trivial bundle \(\mathcal{O}^{N+1}\) by \(\tilde{E}^{n+1}:\)

\[ 0 \to \tilde{E}^{n+1} \to \mathcal{O}^{N+1} \to Q^{N-n} \to 0. \]

Thus

\[ \overline{P}_k(X) = \nu_*(c_k(\gamma Q) \cap [\tilde{X}]). \]
From the previous exact sequence we get the exact sequence over $\hat{X}$

$$0 \to \overline{TX} \to \hat{\varphi}^*\mathcal{E}^{N+1} \to \hat{\varphi}^*Q^{N-n} \to 0.$$  

Hence by the Whitney product formula

$$\overline{P}_k(X) = \nu_*\left( s_k(\overline{TX}) \cap [\hat{X}] \right).$$  

The Euler exact sequence again yields Todd formulas, this time involving Segre classes:

$$\overline{P}_k(X) = \sum_{i=0}^{k} \binom{n+k}{i} U^i \cap S^M_{k-i}(X),$$

and

$$S^M_k(X) = \sum_{i=0}^{k} (-1)^i \binom{n+k}{i} U^i \cap \overline{P}_{k-i}(X).$$

2. A quick review of K. W. Johnson's thesis [Jo]. Johnson's Segre class $S_\ast(X)$ is defined as the relative Segre class $S(d(X), X \times X)$ of $X \times X$ with respect to the diagonal $d(X)$. We recall the details here. Let $\overline{X \times X} \to X \times X$ be the blowup of $X \times X$ along the diagonal, which is defined by some ideal sheaf $I$. Let $P(X)$ be the exceptional divisor of this blowup; i.e.,

$$P(X) = \text{Proj} \left( \bigoplus_{j \geq 0} I^j/I^{j+1} \right),$$

which is the projectivization of the normal cone [F3]

$$C_{d(X)}(X \times X) = \text{Spec} \left( \bigoplus_{j \geq 0} I^j/I^{j+1} \right).$$

Note that if $X$ is of equidimension $n$, then $P(X)$ is of equidimension $2n - 1$. If we restrict $\pi$ to the exceptional divisor $P(X)$ and identify $d(X)$ with $X$, we have the projection map $p: P(X) \to X$. Let $\xi = c_1(\mathcal{O}_{P(X)}(1))$. Then the $i$th Segre-Johnson class $S_i(X)$ is defined by

$$S_i(X) = p_*\left( \xi^{n-i} \cap [P(X)] \right) \in H_{2(n-i)}(X).$$

(For the general notion of relative Segre classes, see [F3, FL, FM1].)

Johnson studied the scheme-theoretical fiber over $x$,

$$\text{Spec} \left( \bigoplus I^j/I^{j+1} \right) \times X \text{ Spec}(k(x)),$$

(where $k(x)$ is the residue field $\mathcal{O}_{X,x}/M_x$) and called it the tangent star to $X$ at $x$. We denote this by $T_x X$. We call the normal cone $C_{d(X)}(X \times X)$ the tangent star-bundle (not a bundle in the usual sense), and denote it by $TX$. Thus, $P(X) = \text{Proj}(TX)$. Note that if $\Theta(X) = \text{Spec}(\text{Sym}_{\mathcal{O}_X}(I/I^2))$, then $\Theta X$ is the Zariski tangent space to $X$ at $x$. The surjection $\text{Sym}_{\mathcal{O}_X}(I/I^2) \to \bigoplus_{j \geq 0} I^j/I^{j+1}$ induces a scheme-theoretical inclusion $TX \hookrightarrow \Theta(X)$, so the tangent star is a subscheme of the Zariski tangent space. Johnson gave a geometric description of the tangent star (as a set) as follows: If $X$ is a subvariety of $C^N$, then $^*T_x X$ is the union of all lines $L$
through \( x \) for which there are sequences \( \{ y_i \}, \{ y'_i \} \) of points in \( X \) converging to \( x \) such that the sequence of lines \( y_i y'_i \) converges to \( L \). In fact, this tangent star \( *T_x X \) was already introduced by Whitney as \( C_s(X, x) \) [Wh, Chapter 7].

Let \( A^{N-m-1} \) be a generic linear subspace of \( \mathbb{P}^N \). Then the projection map \( p_A : X \to \mathbb{P}^m \), with \( A \) as the center of the projection map, induces a linear map \( d(P_A)_*: \Theta_X(x) \to \Theta_{P_A(x)} \mathbb{P}^m \) for each point \( x \in X \). Johnson defined a ramification locus of \( p_A \) as follows: \( p_A \) ramifies at \( x \) if the induced map \( p_A|_{T_x X} : *T_x X \to \Theta_{P_A(x)} \mathbb{P}^m \) is not finite-to-one. He showed that [Jo, Lemma 2.1 and §2.2] \( p_A \) ramifies at \( x \) if and only if \( *T_x X \cap A = \emptyset \), where \( *T_x X \) is the projective closure of the tangent star. The ramification locus is not necessarily equidimensional, but its largest components are of dimension \( (2n - m - 1) \) and their union is the support of a scheme \( R^d(X) \) described below.

First we give another description of the ramification locus \( R^d(X) \). The variety \( \overline{X \times X} \) is the closure of the image of the map

\[
X \times X - d(X) \to (X \times X) \times G, \quad (x, y) \mapsto (x, y, \overline{xy}),
\]

where \( G = \text{Gr}(N, 1) \) is the Grassmannian of lines in \( \mathbb{P}^N \), and \( \overline{xy} \) is the secant line through \( x \) and \( y \). By definition \( P(X) \) is a subscheme of \( \overline{X \times X} \). Let \( g: P(X) \to G \) and \( p: P(X) \to d(X) = X \) be the projection maps. Let \( W(H) = \{ L \in G \mid L \cap A \neq \emptyset \} \). Then

\[
R^d(X) = p \cdot g^{-1}W(A).
\]

The scheme-theoretical analog of this equation gives the desired ramification scheme

\[
R^d(X) = p \cdot g^{-1}\mathfrak{M}(A).
\]

Johnson's ramification class is defined by

\[
R_k(X) = p_*([W(A)] \cap [P(X)]), \quad k = m - n + 1.
\]

Its support is the union of the largest components of \( R^d(X) \), and its homology class is independent of a generic \( A \). Johnson [Jo, §5.2] showed the Todd formula

\[
R_k(X) = \sum_{i=0}^k \binom{n+k}{i+n} U^{k-i} \cap S_i(X).
\]

He also defined a double point class \( D_{k-1} \) \( (k = m - n + 1) \) associated with a generic center \( A^{N-m-1} \), and obtained the double point formula

\[
D_{k-1}(X) = U^{k-1} \cap d[ X] - \sum_{i=0}^{k-1} \binom{n+k}{i+n+1} U^{k-1-i} \cap S_i(X)
\]

where \( d \) is the degree of \( X^n \subseteq \mathbb{P}^N \). From the Todd formula and the double point formula, he obtained Johnson's connecting formula:

\[
\text{JCF}(k) : U \cdot D_{k-1} - D_k = R_k, \quad (k = m - n + 1).
\]

This connecting formula implies a quite surprising result: Let \( X \) be an \( n \)-dimensional subvariety of \( \mathbb{P}^N \), \( N \leq 2n \). If \( X \) can be immersed in a lower dimensional projective space by projection, then it can be so embedded.
One of the intriguing questions about Johnson’s Segre classes is their invariance properties in a flat family. Using Johnson’s Todd formula one can show that for reduced hypersurfaces \( X^n \) and \( Y^n \) of the same degree \( d \) in \( \mathbb{P}^{n+1} \), \( \deg S_i(X) = \deg S_i(Y) \) for each \( i \). Explicitly, for \( i > 0 \)

\[
\deg S_i(X) = (-1)^{i-1} \frac{n+i}{i-1} d^2 + (-1)^{i-1} \left( \frac{n+i}{i} - (n+2) \frac{n+i}{i-1} \right) d.
\]

3. Decompositions of \( P(X) \). In this section we compare the projectivizations \( P(X) \) and \( P(TX) \) of, respectively, the tangent star-bundle and the Nash tangent bundle of a projective variety \( X \).

**Proposition 3.1.** For any irreducible projective variety \( X \), there exists a canonical morphism \( q: \mathbb{P}(TX) \to P(X) \) such that the image \( q(P(TX)) \) is an irreducible component of \( P(X) \) and also that \( q|_{P(TX_{sm})} \) is an isomorphism from \( P(TX_{sm}) \) into \( P(X_{sm}) \).

**Proof.** Let \( \pi: \text{Gr}(TP^N, n) \to P^N \) be the projection map (i.e., Grassmannian bundle map) and \( \mathcal{E} \) be the pull-back of the tangent bundle \( TP^N \) via \( \pi \), i.e., \( \mathcal{E} = \pi^*TP^N \). Note that the exceptional divisor \( E \) of the blowup \( \overline{P^N} \times \mathbb{P}^N \) of \( P^N \times P^N \) is the projectivization of the tangent bundle \( TP^N \), i.e., \( E = P(TP^N) \), and also that the tautological rank \( n \) bundle \( E^n \) over \( \text{Gr}(TP^N, n) \) is a subbundle of \( \mathcal{E} = \pi^*TP^N \).

Now consider the following diagram, where \( C \) denotes the inclusion map, and also note that \( \nu = \pi|_{\mathbb{P}^N} \). Then we restrict the map \( P(\mathcal{E}) \to E \) to \( P(\mathbb{T}X) \) and denote this restriction map by \( q \). If we furthermore restrict \( q \) to \( P(\mathbb{T}X_{sm}) \), then we have the isomorphism \( \mathbb{P}(\mathbb{T}X_{sm}) \to P(X_{sm}) \). Hence, since \( P(\mathbb{T}X) \) is the closure of \( P(\mathbb{T}X_{sm}) \) and \( q \) is continuous, \( q(P(\mathbb{T}X)) \subset P(X) \). Since \( q(P(\mathbb{T}X)) \) and \( P(X_{sm}) \) are irreducible and of the same dimension (\( = 2n - 1 \)), it follows that \( q(P(\mathbb{T}X)) = P(X_{sm}) \), which is an irreducible component of \( P(X) \). Q.E.D.

**Corollary 3.2.** Let \( X \) be a reduced projective variety of equidimension and \( X = X_1 \cup \cdots \cup X_r \) be the irreducible decomposition of \( X \). Then there exists a canonical morphism \( q: \mathbb{P}(\mathbb{T}X) \to P(X) \) such that the image \( q(P(\mathbb{T}X_i)) \) is an irreducible
component of $P(X)$, and that $q(P(TX)) = q(P(TX_1)) \cup \cdots \cup q(P(TX_r))$ is the irreducible decomposition of $q(P(TX))$, and also $q(P(TX)) = P(X_{sm})$.

We call $q(P(TX_1))$ a typical component of $P(X)$. Note that since $P(TX)$ is reduced and $q$ is an isomorphism, $q(P(TX))$ is reduced, i.e., the multiplicity of each typical component $q(P(TX_1))$ in $P(X)$ is equal to one.

Now let $\{V_j\}$ be the extra components of $P(X)$ other than the typical components, supported on the singular part of $X$, with the multiplicity $m_j$ for each $V_j$, and $\pi_j: V_j \to X$ be the projection map. Then it is not hard to show the following naive formula between Johnson's Segre class $S_*(X)$ and our Segre-Mather class $S^M_*(X)$:

**Proposition 3.3.** Let $X$ be a reduced projective variety of equidimension $n$. Then we get

$$S_i(X) = S^M_i(X) + \sum_j m_j \cdot \pi_j^*(\psi_j^{n-1+i} \cap [V_j]),$$

where $\psi_j = c_1(\mathcal{O}_{V_j}(1))$.

**Proposition 3.4.** Let $X^n \subset P^N$ be a reduced projective variety of equidimension $n$ and the singular set of $X$ be of dimension $k$. If $N - n < n - k$, then $S_i(X) = S^M_i(X)$ for any $i$.

**Proof.** Since $^*T_xX \subseteq T_xP^N = C^N$, the “fiber” dimension of each extra component $V_j$ of $P(X)$ is at most $N - 1$. So the dimension of the extra component $V_j$ is at most $(N - 1) + k$. Since $N - n < n - k$, $(N - 1) + k < 2n - 1$.

Hence, in fact, there is no such extra component $V_j$ because $P(X)$ must be of equidimension $2n - 1$. Thus the above formula follows from Proposition 3.3. Q.E.D.

**Remark 3.5.** The multiplicity $m_j$ attached to each extra component $V_j$ is given by

$$m_j = \text{length}(\mathcal{O}_{P(X), V_j}).$$

Here $\mathcal{O}_{P(X), V_j}$ is the local ring of $P(X)$ at $V_j$. As in [Ho1, Proof of Lemma 8.1.1] A. Holme discussed $m_j$ a little, and this integral coefficient $m_j$ can be interpreted as the intersection multiplicity of the scheme-theoretical intersection $P(X) = E \cap \overline{X} \times X$. This multiplicity is given by Serre's Tor-formula (see [K1, p. 317]):

$$\sum_i (-1)^i \text{length} \left( \text{Tor}_i^{\mathcal{O}_{P(X), V_j}} \left( \mathcal{O}_{\overline{X} \times X, V_j}, \mathcal{O}_{E, V_j} \right) \right).$$

By some algebra this turns out to be equal to $\text{length}(\mathcal{O}_{P(X), V_j})$. It seems that it is hard to compute this multiplicity or even to identify the extra components of $P(X)$. In §§4 and 5 we will find such multiplicities in the cases of hypersurfaces and $X^n \subseteq P^2n$ with isolated singularities.

**4. Hypersurfaces.** In this section we will give an explicit formula between Johnson’s Segre class $S_*(X)$ and our Segre-Mather class $S^M_*(X)$ for hypersurfaces without any restrictions on singularities.

**Proposition 4.1.** Let $X^n$ be a reduced hypersurface of $P^{n+1}$. If $x$ is a singular point of $X$, then the tangent star $^*T_xX$ to $X$ at $x$ is isomorphic to $T_xP^{n+1} = C^{n+1}$ (even as a scheme).
Proof. Since this is a local problem, we can assume that \( x \in X \subset \mathbb{C}^{n+1} \), and also we can assume that \( x \) is the origin. Let \( f(X_1, X_2, \ldots, X_{n+1}) \) be a reduced polynomial defining \( X \). Since \( x \) is a singular point and \( x \) is the origin, \( f(X_1, X_2, \ldots, X_{n+1}) \) has no constant term and the degree of its initial part is \( \geq 2 \). Let \( X_1, \ldots, X_{n+1}, U_1, \ldots, U_{n+1} \) be the affine coordinates of \( \mathbb{C}^{n+1} \subset \mathbb{C}^{n+1} \). Let \( J \) be the ideal defining the diagonal \( d(\mathbb{C}^{n+1}) \) of \( \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \), generated by \( (X_1 - U_1, X_2 - U_2, \ldots, X_{n+1} - U_{n+1}) \), \( \mathcal{M} = (X_1, U_1, \ldots, X_{n+1}, U_{n+1}) \), and \( I \) be the ideal defining the diagonal \( d(X) \) of \( X \times X \) in the coordinate ring \( R = \mathbb{C}[X_1, \ldots, X_{n+1}, U_1, \ldots, U_{n+1}]/(f(X_1, \ldots, X_{n+1}) \cdot f(U_1, \ldots, U_{n+1})) \), \( \mathcal{M} = \mathcal{M}/(f(X_1, \ldots, X_{n+1}), f(U_1, \ldots, U_{n+1})) \).

Then by some standard algebra, we have the surjection \( \phi: \bigoplus_{j \geq 0} J^j/\mathcal{M} \cdot J^j \to \bigoplus_{j \geq 0} \mathcal{M}/\mathcal{M} \cdot \mathcal{M}^{j} \).

If \( \phi \) is not surjective, i.e. \( \text{Ker} \phi \) is not zero, then it follows by taking Proj that \( P(\ast_T X) = \text{Proj} \left( \bigoplus_{j \geq 0} \mathcal{M}/\mathcal{M} \cdot \mathcal{M}^{j} \right) \)

is a proper closed subscheme of \( \text{Proj}(\bigoplus_{j \geq 0} J^j/\mathcal{M} \cdot J^j) = P(\ast_T \mathbb{C}^{n+1}) = \mathbb{P}^{n} \). Hence \( P(\ast_T X) \) consists of at most a finite number of hypersurfaces. On the other hand, since the multiplicity of \( X \) at \( x \) is \( \geq 2 \) (in fact, is equal to the degree of the initial part of the defining polynomial \( f(X_1, \ldots, X_{n+1}) \)), any line \( L \) going through the point is the limit of the secant line \( x_jy_j \), where \( x_j \to x \), \( y_j \to x \), and \( x_j, y_j \) are smooth points. Thus any line going through the singular point \( x \) is in \( \ast_T X \) (as a set); i.e., \( \ast_T X = \mathbb{C}^{n+1} \) as a set, i.e., \( P(\ast_T X) = \mathbb{P}^{n} \) as a set. This is a contradiction. Thus \( \phi \) must be injective, so \( \phi \) is an isomorphism. Hence,

\[
\ast_T X = \text{Spec} \left( \bigoplus_{j \geq 0} \mathcal{M}/\mathcal{M} \cdot \mathcal{M}^{j} \right) = \text{Spec} \left( \bigoplus_{j \geq 0} J^j/\mathcal{M} \cdot J^j \right)
\]

\[
\text{def} \ast_T \mathbb{C}^{n+1} = T_X \mathbb{C}^{n+1}
\]

\[= \text{Spec}(\mathbb{C}[X_1 - U_1, \ldots, X_{n+1} - U_{n+1}]) = \mathbb{C}^{n+1}.
\]

Therefore, the tangent star \( \ast_T X \) to \( X \) at a singular point \( x \) is \( \mathbb{C}^{n+1} = T_X \mathbb{C}^{n+1} \) even as a scheme. Q.E.D.

Now we are ready to state our first main theorem.

Theorem 4.2 (A Formula for Segre Classes for Hypersurfaces). Let \( X^n \subseteq \mathbb{P}^{n+1} \) be a reduced hypersurface with \( S \) denoting the singular subvariety of \( X \). Then

\[
S_i(X) = S_i^M(X) + S^{i-1}(\mathbb{P}^{n+1}) \cap \left( \sum_j e_j \cdot [S_{n-1,j}] \right)
\]

\[
= S_i^M(X) + (-1)^{i-1} \binom{n+i}{i-1} U_{i-1} \cap \left( \sum_j e_j \cdot [S_{n-1,j}] \right)
\]
where $S_{n-1,j}$ are irreducible components of dimension $n - 1$ of the singular subvariety $S$, $e_j$ is the multiplicity of the Jacobian ideal in the local ring of $X$ at the generic point of $S_{n-1,j}$, $U = c_1(\mathcal{O}_X(1))$ and $S^{i-1}(\mathbf{P}^{n+1})$ is the usual $(i - 1)$st Segre (cohomology) class of $\mathbf{P}^{n+1}$. Hence, in particular, if $\dim S < n - 1$, then $S_{n-1,j} = \emptyset$, so $S_i(X) = S^M_i(X)$ for any $i$.

**Proof.** The second assertion is clear by Proposition 3.4. So we consider the case when $\dim S = n - 1$. In this case, by Proposition 4.1, $P(X)$ has extra components supported on each irreducible component, $S_{n-1,j}$ of dimension $n - 1$ of the singular subvariety $S$ of $X$, and they are $P(T\mathbf{P}^{n+1}|_{S_{n-1,j}}).$ We show the following formula (recalling the notations in §3):

$$\pi_*\left(\psi^{n-1+i}_j \cap \left[ P\left(T\mathbf{P}^{n+1}|_{S_{n-1,j}}\right) \right]\right) = S^{i-1}(\mathbf{P}^{n+1}) \cap \left[ S_{n-1,j} \right]$$

$$= (-1)^{i-1}\binom{n+i}{i-1} U^{i-1} \cap \left[ S_{n-1,j} \right],$$

where we understand $S^{i-1}(\mathbf{P}^{n+1}) = e^*S^{i-1}(\mathbf{P}^{n+1})$, $e : X \to \mathbf{P}^{n+1}$ the inclusion map, and $\left[ S_{n-1,j} \right] = i_*[S_{n-1,j}]$, $i_j : S_{n-1,j} \to X$ the inclusion map.

**Proof of the above formula.** For convenience, we omit the subscript $j$ from the notation. Then

$$T\mathbf{P}^{n+1}|_{S_{n-1}} = (e \circ i)^*T\mathbf{P}^{n+1} = i^*e^*T\mathbf{P}^{n+1}.$$

Since $i^*e^*TP^{n+1}$ is a rank $(n + 1)$ bundle over $S_{n-1}$, by considering $\pi = i \circ \pi^0$, where $\pi^0 : P(T\mathbf{P}^{n+1}|_{S_{n-1}}) \to S_{n-1}$ is the projection map, we have

$$\pi_*\left(\psi^{n-1+i} \cap \left[ P\left(T\mathbf{P}^{n+1}|_{S_{n-1}}\right) \right]\right) = i_*\pi^0_*\left(\psi^{(n+1)-1+(i-1)} \cap \left[ P\left(T\mathbf{P}^{n+1}|_{S_{n-1}}\right) \right]\right)$$

$$= i_*\left(S^{i-1}\left(T\mathbf{P}^{n+1}|_{S_{n-1}}\right) \cap \left[ S_{n-1} \right]\right) = i_*\left(S^{i-1}(i^*e^*TP^{n+1}) \cap \left[ S_{n-1} \right]\right)$$

$$= i_*\left(i^*e^*S^{i-1}(T\mathbf{P}^{n+1}) \cap \left[ S_{n-1} \right]\right) = e^*S^{i-1}(\mathbf{P}^{n+1}) \cap i_*[S_{n-1}].$$

by the projection formula and the definition of $e^*(\mathbf{P}^{n+1})$. Since

$$S(\mathbf{P}^{n+1}) = 1/C(\mathbf{P}^{n+1}) = (1 + c_1(\mathcal{O}_{\mathbf{P}^{n+1}}(1)))^{-(n+2)}$$

and $e^*\mathcal{O}_{\mathbf{P}^{n+1}}(1) = \mathcal{O}_X(1)$, by a well-known binomial formula

$$e^*S^{i-1}(\mathbf{P}^{n+1}) = (-1)^{i-1}\binom{n+i}{i-1} U^{i-1}.$$

Thus, by Proposition 3.3, we get the following formula:

$$S_i(X) = S^M_i(X) + S^{i-1}(\mathbf{P}^{n+1}) \cap \left( \sum m_j \cdot [S_{n-1,j}] \right).$$

So all we have to do is to find $\sum m_j \cdot [S_{n-1,j}]$ (or strongly $m_j$ for each $j$).

Now by Johnson's Todd formula and our Todd formula relating $\bar{P}_*(X)$ and $S^M_*(X)$ we easily get

$$R_i(X) = \bar{P}_1(X) + \sum m_j \cdot [S_{n-1,j}],$$

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Recalling the definitions of $P(X)$ and $\overline{P}(X)$ in §1 we notice that $P_1(X) = P(X)$. (In general, of course, $P_k(X) \neq P_k(X)$. By [Pl1, Corollary 2.2, p. 257]
\begin{equation}
(4.4) \quad P_1(X) = (d - 1)U \cap [X] - \sum_j e_j \cdot [S_{n-1,j}],
\end{equation}
where $d$ is the degree of the hypersurface $X$. Johnson's double point formula implies that $D_0(X) = d[X] - [X] = (d - 1)[X]$ and also Johnson's connecting formula implies that, since $D_1(X) = 0$ (because $X$ is already contained in $P^{n+1}$), $R_1(X) = U \cap D_0(X)$. Then we combine them to get
\begin{equation}
(4.5) \quad R_1(X) = (d - 1)U \cap [X].
\end{equation}
Thus (4.4) and (4.5) imply
\begin{equation}
(4.6) \quad R_1(X) = P_1(X) + \sum_j e_j \cdot [S_{n-1,j}].
\end{equation}
Thus (4.3) and (4.6) imply
\begin{equation}
(4.7) \quad \sum_j m_j \cdot [S_{n-1,j}] = \sum_j e_j \cdot [S_{n-1,j}].
\end{equation}
This completes the proof of the theorem.

Remark 4.8. One might be tempted to immediately conclude that for each $j$, $m_j = e_j$. But identity (4.7) is in the homology group (or Chow homology group), thus, as cycles, $\sum_j (m_j - e_j) \cdot [S_{n-1,j}]$ is homologous to zero (or rationally equivalent to zero). If the singular locus of $X$ has only one irreducible component of dimension $n - 1$, then $m = e$; i.e., $e = \text{length}(\mathcal{O}_{P(X), x_i})$. Also, both $m$ and $e$ are described locally, so we conjecture that $m_j = e_j$ for each $j$.

Remark 4.9. As an example of Theorem 4.2, let us consider a reduced plane curve $X$ with isolated singularities $x_1, \ldots, x_r$. Then we have
\[
S_1(X) = S_1^M(X) + \sum_j e_j[x_j],
\]
where $e_j$ is the Jacobian multiplicity of $x_j$. For instance the Jacobian multiplicity of a node singularity is 2 and that of a cusp singularity is 3 (see [Kl]). This formula and Dubson's formula for Chern-MacPherson classes $C_*(X)$ give an interesting formula between $S_1(X)$ and $C_1(X)$ involving Milnor numbers:
\[
S_1(X) = -C_1(X) + \sum_i \mu_i[x_i];
\]
where $\mu_i$ is the Milnor number of the singularity $x_i$. We have been unable to generalize this to other higher dimensional cases.

5. $X^n \subseteq \mathbb{P}^{2n}$ with isolated singularities. In §4 we considered the hypersurface case, a special case of which is the plane curve case. In this section we consider the case when $X^n \subseteq \mathbb{P}^{2n}$ with isolated singularities, a special case of which is also the plane curve case.

\[1\] R. Varley and the present author have recently proved this conjecture affirmatively. The proof will appear elsewhere.
If \( X^1 \subset \mathbf{P}^2 \) is a reduced plane curve of degree \( d \) with a single singularity \( x_0 \), then we have

\[
S_0(X) = S_0^M(X) = [X] \quad \text{and} \quad S_1(X) = S_1^M(X) + m \cdot [x_0],
\]

where \( m \) is the Jacobian multiplicity of \( x_0 \), \( m = d(d - 1) - \deg(\overline{P}_1(X)) \). This can be generalized to

**Theorem 5.1.** Let \( X^n \subset \mathbf{P}^{2n} \) be a reduced singular variety of degree \( d \) with a single isolated singularity \( x_0 \). Then

\[
S_i(X) = S_i^M(X), \quad i < n, \quad \text{and} \quad S_n(X) = S_n^M(X) + m \cdot [x_0],
\]

where

\[
m = d(d - 1) - \sum_{k=1}^{n} \deg(\overline{P}_{n-k+1}(X)).
\]

**Proof.** The first equality is clear (by the dimension reason). We note that

(5.1) \[ S_n(X) = S_n^M(X) + m \cdot [x_0], \]

where \( m \) is a nonnegative integer. By Johnson’s Todd formula and our Todd formula (see §1) we can easily see that

(5.2) \[ R_{n-k+1}(X) = \overline{P}_{n-k+1}(X), \quad 2 \leq k \leq n, \]

and

(5.3) \[ R_n(X) = \overline{P}_n(X) + m \cdot [x_0]. \]

Then by Johnson’s connecting formula, (5.2), and (5.3) we have

\[
\begin{align*}
\text{JCF}(1) & \quad U \cdot D_0 - D_1 = \overline{P}_1, \\
\text{JCF}(k) & \quad U \cdot D_{k-1} - D_k = \overline{P}_k, \\
\text{JCF}(n-1) & \quad U \cdot D_{n-2} - D_{n-1} = \overline{P}_{n-1}, \\
\text{JCF}(n) & \quad U \cdot D_{n-1} - D_n = \overline{P}_n + m[x_0].
\end{align*}
\]

Now by applying \( U^{n-k} \) on both sides of each JCF(\( k \)) and adding all of \( U^{n-k} \).

(5.4) \[ U^{n} \cdot D_0 - D_n = \sum_{k=1}^{n} U^{k-1} \cdot \overline{P}_{n-k+1}(X) + m \cdot [x_0]. \]

Since \( X^n \subset \mathbf{P}^{2n} \), \( D_n = 0 \), so (5.4) becomes

(5.5) \[ U^{n} \cdot D_0 - \sum_{k=1}^{n} U^{k-1} \cdot \overline{P}_{n-k+1}(X) = m \cdot [x_0]. \]

Since \( D_0 = (d - 1)[X] \), (5.5) becomes

(5.6) \[ (d - 1)U^{n} \cdot [X] - \sum_{k=1}^{n} U^{k-1} \overline{P}_{n-k+1}(X) = m \cdot [x_0]. \]

Since \( X^n \) is of dimension \( n \) and \( \overline{P}_{n-k+1}(X) \) is of dimension \( (k - 1) \), we obtain

(5.7) \[ d(d - 1) - \sum_{k=1}^{n} \deg(\overline{P}_{n-k+1}(X)) = m. \]

This completes the proof of the theorem.
Remark 5.2. In general, if $X^n \subset \mathbb{P}^N$ ($N \geq 2n$) with a single isolated singularity, then the multiplicity $m$ becomes

$$m = d(d - 1) - \deg(D_n(X)) - \sum_{k=1}^{n} \deg(\mathcal{P}_{n-k+1}(X)).$$

As for $\deg(D_n(X))$, refer to [HR, Theorem 3.3 and Ho2].

Theorem 5.1 is valid only for the single isolated singularity case. If a singular variety has many isolated singularities, say $x_1, x_2, \ldots, x_r$, then as a corollary of the proof of Theorem 5.1, we have

$$(5.7') \quad \sum_{i=1}^{r} m_i = d(d - 1) - \sum_{k=1}^{n} \deg(\mathcal{P}_{n-k+1}(X)),$$

where

$$m_i = \begin{cases} 
\text{length}(\Theta_{P(X),V_i}), & \text{where } V_i = \mathbb{P}(*)T_{x_i}X \text{ is supported on the singularity } x_i \text{ and } \dim(*)T_{x_i}X = 2n, \\
0, & \text{if } \dim(*)T_{x_i}X < 2n
\end{cases}$$

and

$$(5.1') \quad S_n(X) = S_n^M(X) + \sum_{i=1}^{r} m_i[x_i]$$

or, equivalently,

$$(5.3') \quad R_n(X) = \mathcal{P}_n(X) + \sum_{i=1}^{r} m_i[x_i].$$

By (#), if we use Fulton’s notation (see §6.1 of [F4] and a remark right after Lemma 7.1, p. 120), we can see

**Proposition 5.3.** Under the same hypothesis as above,

$$m_i[x_i] = (\mathbb{P}^{2n} \cdot (X \times X))^{\{(x_i, x_i)\}},$$

which is called the part of $\mathbb{P}^{2n} \cdot (X \times X)$ supported on $(x_i, x_i)$.

We may call this the localized self-intersection class of $X$ at $x_i$ and for simplicity denote it by $(X \cdot X)^{(x_i)}$. So we can express (5.1) as follows:

$$S_n(X) = S_n^M(X) + \sum_{i=1}^{r} (X \cdot X)^{\{x_i\}}.$$

Now, just as in the hypersurface case (cf. Remark 4.8) we will give the integer $m_i = \text{length}(\Theta_{P(X),V_i})$ (or $= 0$) a local (and “down-to-earth”) description with respect to the singular point $x_i$. For this we introduce the following definitions.

**Definition 5.4.** Let $P$ be a point in $\mathbb{P}^{2n}$ and let $H$ be a hyperplane not containing the point $P$. We identify $\mathbb{P}^{2n} - H$ with $\mathbb{C}^{2n}$. Let $t$ be a complex number. Then a shift $S_{P,t}(y)$ of a point $y (\neq P) \in \mathbb{C}^{2n}$ is defined to be (see Figure 1)

$$S_{P,t}(y) = y + t \cdot \frac{P - y}{\|P - y\|},$$

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where \( \| P - y \| \) is the norm of the vector \( P - y \), i.e.,

\[
P - y = \sqrt{|P_1 - y_1|^2 + \cdots + |P_{2n} - y_{2n}|^2}
\]

if we let \( P = (P_1, \ldots, P_{2n}) \) and \( y = (y_1, \ldots, y_{2n}) \).

**Definition 5.5.** Let \( X^n \subseteq \mathbb{P}^{2n} \) with isolated singularities. Let \( x \) be a singular point and \( P \) be a point off \( X \). Let \( H \) be hyperplane not containing \( x \) and consider the affine variety \( X^a = X - H \subseteq \mathbb{P}^{2n} - H = \mathbb{C}^{2n} \). We assume that \( P \) does not lie on \( H \). Let \( \varepsilon \) be a small positive number and \( t \) a complex number such that \( |t| < \varepsilon \), and \( B_\varepsilon(x) \) an \( \varepsilon \)-ball around \( x \). Then the number \( n(x, P, t, \varepsilon) \) associated to the singular point \( x \) is defined to be (see Figure 2) \( \#(S_{P,t}(X^a) \cap X^a \cap B_\varepsilon(x)) \), if \( S_{P,t}(X^a) \) is transverse to \( X^a \) within \( B_\varepsilon(x) \). (We will see later that \( S_{P,t}(X^a) \) and \( X^a \) are transverse to each other within \( B_\varepsilon(x) \) for a generic point \( P \) off \( X \) and sufficiently small \( \varepsilon \) and \( t \).)

We note that as long as we look at a sufficiently small neighborhood of \( x \), we may consider the shift \( S_{P,t} \) as the translation via the direction vector \( t \cdot (P - x)/\|P - x\| \).
Now we are ready to state the local description of the integer $m_i$ appearing in (5.1') or (5.3').

**Theorem 5.6.** Let $X^n \subseteq P^{2n}$ be a reduced singular variety with isolated singularities $x_1, x_2, \ldots, x_r$. Let $P, t, \varepsilon$ be as in Definition 5.5. Then

(i) For each singularity $x_i$, the number $n_i = n(x_i, P, t, \varepsilon)$ is well defined and constant for a generic point $P$,

(ii) The integer $m_i$ appearing in (5.1') is exactly equal to $n_i$, i.e.,

$$(5.1'') \quad S_n(X) = S_n^M(X) + \sum_{i=1}^{r} n_i[x_i].$$

**Proof.** Since (5.1') and (5.3') are equivalent and the objects of Johnson's ramification class $R_n(X)$ and our polar class $P_n(X)$ are more geometrical, we look at (5.3') instead of (5.1'). And the idea (or strategy) of our proof is as follows: First we take a closer and more careful look at Johnson's ramification cycle, whose homology class (or rational equivalence class) is Johnson's ramification class, and then step by step we recapture the multiplicity $m_i$ within a more tractable or "down-to-earth" set up. We prove both (i) and (ii) simultaneously.

First, for the sake of convenience, we set up the following notations:

$BL = P^N \times P^N$, the blowup of $P^N \times P^N$ along the diagonal,

$E$ = the exceptional divisor of $BL$,

$\pi: X \times X \to X \times X$, the blowup map,

$\gamma: X \times X \to G = \text{Gr}(N, 1)$, the Gauss map,

$W(p) = \{L \in G | L \cap P \neq \emptyset\}$, the first Schubert variety, where $P$ is a generic center (in our case, $P$ is a point),

$i: X \times X \to BL$, the inclusion map,

$j: D = D(X) \to X \times X$, the inclusion map, where $D = D(X) = \gamma^{-1}(W(P))$ is the double point scheme,

$\tilde{R} = \tilde{R}(X) = g^{-1}(W(P))$, the ramification scheme, where $g = \gamma|_{P(X)}$ is the restriction of $\gamma$ to $P(X)$,

$k: P(X) \to X \times X$, the inclusion map,

$p: P(X) \to X$, the projection map defined by $p = pr_1 \circ \pi \circ k$, where $pr_1: X \times X \to X$ is the projection map.

$$E \subset BL = P^N \times P^N$$

$$P(X) \subset \tilde{D}(x) \to X \times X \to G$$

$X \times X \to X$
Then, Johnson’s ramification cycle $R_n(X)$ is, by definition,

$$R_n(X) = p_*g^*[W(P)] \quad (\text{using Fulton's notation})$$

$$= p_*(g^*[W(P)] \cap [P(X)])$$

$$= pr_1 \cdot \pi_* \cdot k_* (g^*[W(P)] \cap [P(X)]) \quad (\text{because } p = pr_1 \circ \pi \circ k).$$

Since $\tilde{R} = g^{-1}(W(P))$,

$$[\tilde{R}] = g^*[W(P)] \quad (\text{using Fulton's notation})$$

$$= g^*[W(P)] \cap [P(X)].$$

Thus

$$R_n(X) = p_*[\tilde{R}] = pr_1 \cdot \pi_* \cdot k_* [\tilde{R}].$$

Let

$$[\tilde{R}] = \sum_{i=1}^{r} n_i [(x_i, x_i, L_{x_i})] + \sum_{j=r+1}^{s} n_j [(x_j, x_j, L_{x_j})],$$

where

1. Each $x_i \ (1 \leq i \leq r)$ is a singular point, and each $L_{x_i}$ is a line in the projective tangent star $*T_{x_i}X$ such that $L_{x_i}$ hits the generic center $P \cdot n_i$ attached to each $[(x_i, x_i, L_{x_i})]$ is a multiplicity of $(x_i, x_i, L_{x_i})$ in the ramification scheme $\tilde{R}$. Here if $\dim *T_{x_i}X < 2n$ for a singular point $x_i$, then by the genericity of the center $P$, $*T_{x_i}X$ misses the point $P$, hence $[(x_i, x_i, L_{x_i})]$ does not appear in $[\tilde{R}]$. For this case we set $n_i = 0$.

2. Each $x_j \ (r+1 \leq j < s)$ is a polar point and a smooth point (by the genericity of the center $P$) and each $L_{x_j}$ is in the projective tangent space $T_{x_j}X$.

Then we can show

**Lemma 5.7.** (1) $n_j \ (r+1 \leq j \leq s)$ is equal to 1,

(2) $n_i \ (1 \leq i \leq r)$ is equal to $m_i k_i$ for some positive integer $k_i$.

**Proof.** (1) If we restrict $g: P(X) \to \text{Gr}(N,1)$ to the reduced subscheme $q(P(*T_X))$ of $P(X)$ (see §3) and let $\hat{g}$ be the restriction map, then for a generic point $P$ $\hat{g}^{-1}(W(P))$ is the polar point (cf. [Jo, Proof of Theorem of §3.2]), i.e., as a cycle

$$\hat{g}^{-1}(W(P)) = \sum_{j=r+1}^{s} n_j [(x_j, x_j, L_{x_j})].$$

However, by [Pi1, Lemma (1.3)], $\hat{g}^{-1}(W(P))$ is reduced for a generic point $P$, hence each $n_j$ must be equal to 1.

(2) The extra components of $P(X)$ are $P(*T_x X)$ with the multiplicity $m_i$ (see (#) before Proposition 5.3) and if we restrict $g: P(X) \to \text{Gr}(N,1)$ to these components, then we obtain the ramification (but not polar) cycle

$$\sum_{i=1}^{r} n_i [(x_i, x_i, L_{x_i})].$$

Thus $n_i$ is a positive multiple of $m_i$, i.e.,

$$n_i = m_i \cdot k_i \quad \text{for some integer } k_i \geq 1.$$
Since $k$ is the inclusion map and, by Lemma 5.7,

\begin{equation}
[k(\tilde{R})] = k_\bullet[\tilde{R}] = \sum_{i=1}^{r} m_i \cdot k_i[(x_i, x_i, L_{x_i})] \\
+ \sum_{j=r+1}^{s} [(x_j, x_j, L_{x_j})].
\end{equation}

Now, since $\pi$ is a one-to-one map on the reduced scheme (i.e., variety), $k(\tilde{R})_{\text{red}} = \{(x_j, x_j, L_{x_j})\}_{j=1}^{s}$ and $\pi((x_i, x_i, L_{x_i})) = (x_i, x_i)$, by [F3, §3.1],

\[\pi_\bullet \cdot k_\bullet[\tilde{R}] = \sum_{i=1}^{r} m_i \cdot k_i[(x_i, x_i)] + \sum_{j=r+1}^{s} [(x_j, x_j)].\]

Similarly, since $\pi_1$ is a one-to-one map on the diagonal

\[R_n(X) = \pi_1_\bullet \cdot \pi_\bullet \cdot k_\bullet[\tilde{R}] = \sum_{i=1}^{r} m_i \cdot k_i[x_i] + \sum_{j=r+1}^{s} [x_j].\]

Since $\Sigma_{j=r+1}^{s} [x_j]$ is the polar part, i.e., $\tilde{P}_n(X) = \Sigma_{j=r+1}^{s} [x_j],

\begin{equation}
R_n(X) = \tilde{P}_n(X) + \sum_{i=1}^{r} m_i \cdot k_i[x_i].
\end{equation}

Then (5.9) and (5.3') imply

\begin{equation}
\sum_{i=1}^{r} m_i \cdot k_i[x_i] = \sum_{i=1}^{r} m_i[x_i],
\end{equation}

i.e.,

\begin{equation}
\sum_{i=1}^{r} m_i(k_i - 1)[x_i] = 0.
\end{equation}

Since $m_i$ is positive, $k_i = 1$, i.e., $n_i = m_i$. Hence, by (5.8), we have

**Observation 5.8.** The integer $m_i = \text{length}(E_{P(X)\cdot x_i})$ is nothing but the multiplicity $n_i$, appearing in the ramification scheme $k(\tilde{R})$ in $X \times X$.

**Lemma 5.9.** Let $h = i \circ j: \tilde{D} \to \text{BL}$. Then using Fulton's notation, we get

\begin{equation}
k_\bullet[R] = j_\bullet h^\bullet[E].
\end{equation}

**Proof.**

\[k_\bullet[R] = k_\bullet(g^\bullet[W(P)] \cap [P(X)]) = k_\bullet(k_\bullet \circ \gamma^\bullet[W(P)] \cap [P(X)]) \quad \text{because } g = \gamma \circ k,
\]

\[= \gamma^\bullet[W(P)] \cap k_\bullet[P(X)] \quad \text{by the projection formula}.
\]

Using the following identities (cf. [J0, Appendix B])

\[k_\bullet[P(X)] = i^\bullet(E) \cap [X \times X],
\]

\[j_\bullet[\tilde{D}] = \gamma^\bullet[W(P)] \cap [X \times X],
\]
we get
\[
k_*[\tilde{R}] = \gamma^*[W(P)] \cap (i^*[E] \cap [\overline{X \times X}]),
\]
\[
= i^*[E] \cap (\gamma^*[W(P)] \cap [\overline{X \times X}])
\]
\[
= i^*[E] \cap j_*(\tilde{D})
\]
\[
= j_*(j^*i^*[E] \cap [\tilde{D}]) \quad \text{by the projection formula}
\]
\[
= j_*(h^*[E] \cap [\tilde{D}]) \quad \text{because } h = i \circ j
\]
\[
= j_*h^*[E] \quad \text{using Fulton's notation.}
\]

Thus we get (5.11).

Since \(j\) is an embedding, the multiplicity \(n_i\) appearing in \(k_*[\tilde{R}]\) is nothing but the multiplicity appearing in \(h^*[E]\). Therefore we only have to analyze \(h^*[E]\), i.e., to analyze the scheme-theoretical intersection \(h(\tilde{D})\) and \(E\), because \(h\) is an embedding. (See Figure 3.)

For this analysis, first of all, we observe the following two things: (1) There is a projective line bundle over \(BL\), denoted by \(K\), which is the pull-back of the tautological bundle over \(G = \text{Gr}(N, 1)\) via the Gauss map \(BL \to G\). To be more precise, let

\[
q: \mathbb{P}^N \times \mathbb{P}^N - d(\mathbb{P}^N) \to \mathbb{P}^N \times \mathbb{P}^N \times G
\]

be defined by \(g(x, y) = (x, y, xy)\), where \(xy\) is the projective line going through the two points \(x\) and \(y\). Then \(BL\) is the closure of the image \(q(\mathbb{P}^N \times \mathbb{P}^N \setminus d(\mathbb{P}^N))\). Let \(p: BL \to G\) be the restriction of the canonical projection map \(p_3: \mathbb{P}^N \times \mathbb{P}^N \times G \to \text{Gr}(N, 1)\). Let \(\tilde{K}\) be the tautological \(\mathbb{P}^1\)-bundle over \(G\), which is a subbundle of the trivial \(\mathbb{P}^N\)-bundle over \(G\). Then the bundle \(\tilde{K}\) over \(BL\) is defined to be the pull-back \(p^*\tilde{K}\) of \(\tilde{K}\) via the map \(p: BL \to G\). So, \(K\) is a subbundle of the trivial \(\mathbb{P}^N\)-bundle over \(BL\), which is \(p^*(\mathbb{P}^N \times G)\).

(2) Then we can define the two canonical sections \(s_1, s_2: BL \to K\) as follows:

\[
s_1((x, y, \text{line})) = ((x, y, \text{line}), x), \quad s_2((x, y, \text{line})) = ((x, y, \text{line}), y).
\]

(Clearly \(s_1(BL) \cap s_2(BL) = s_1(E) = s_2(E)\).)

Now, our procedure of the analysis is as follows: (Step 1) Capture the scheme-theoretical intersection of \(h(\tilde{D})\) and \(E\) as the scheme-theoretical intersection of \(s_2h(\tilde{D})\) and \(s_1(BL)\), and (Step 2) finally, get the local description for the multiplicity \(n_i\) described in statement (i) in the theorem. (See Figure 4.)

![Figure 3](https://www.ams.org/journal-terms-of-use)
Step 1. It is not hard to show that $s_1$ and $s_2$ are transverse, i.e., that $s_1(\text{BL})$ and $s_2(\text{BL})$ are transverse. Then we get

**Lemma 5.10.** $h^*[E] = (s_2h)^*[s_1(\text{BL})]$, using Fulton’s notation.

**Proof.** Since $s_1$ and $s_2$ are transverse, and $\text{BL}$ is reduced,

$$s_2^{-1}s_1(\text{BL}) = s_2^{-1}s_1(\text{BL})_{\text{red}}.$$

It is clear that $s_2^{-1}s_1(\text{BL})_{\text{red}} = E_{\text{red}}$ (as a set). Since $E$ is also reduced, we get $s_2^{-1}s_1(\text{BL}) = E$, as schemes. Hence

$$h^*[E] = h^*[s_2^{-1}s_1(\text{BL})]$$

$$= h^*[s_1(\text{BL})]$$

using Fulton’s notation

$$= (s_2h)^*[s_1(\text{BL})].$$

Therefore, since $s_2h: \tilde{D} \to K$ is a scheme-theoretical embedding, $(s_2)^*[s_1(\text{BL})]$ is considered as the scheme-theoretical intersection of $s_2h(\tilde{D})$ and $s_1(\text{BL})$ (cf. Figure 4).

Step 2. Now we are going to show that the multiplicity $m_j$ of the point $(x_j, x_j, L_{x_j}, x_j)$ is the scheme-theoretical intersection of $s_2h(\tilde{D})$ and $s_1(\text{BL})$ is equal to $n(x_j, P, t, e)$ described in Definition 5.5.

Since in our case $\tilde{D}$ has dimension $1 = 2n - (2n - 1)$ (see [Jo]), $\dim BL = 2N$ and $\dim(s_2h(\tilde{D}) \cap s_1(\text{BL})) = 0$, $s_2h(\tilde{D})$ and $s_1(\text{BL})$ intersect properly. Also we note that $s_1h(\tilde{D})$ and $s_1(\text{BL})$ are reduced, i.e., that $\tilde{D}$ and $\text{BL}$ are reduced. The
reducedness of $BL$ is clear. As for the reducedness of $\tilde{B}$, we consider two cases: if $n > 1$, then $2n - 1 > n$; i.e., $X^n$ is projected to the projective space $\mathbf{P}^{2n-1}$ whose dimension is strictly greater than the dimension of $X^n$. In this case, it is known (see [K1, p. 383]) that $\text{pr}_{1,*}\pi_*[D] = [D]$ has no multiple components; hence $\tilde{D}$ must be reduced, or else $[D]$ has a multiple component. If $n = 1$, i.e., in the plane curve case, $[D] = \text{pr}_{1,*}\pi_*[D] = (d - 1)[X]$, where $d$ is the degree of $X$, and $\tilde{D}$ is a $(d - 1)$-sheeted cover of $X$, so $\tilde{D}$ must be reduced, otherwise $\text{pr}_{1,*}\pi_*[D] = (d - 1)[X]$ does not hold. Thus in any case, $D$ is a reduced scheme, so $s_2 h(\tilde{D})$ is reduced. Then it is known (see [FM2, §1 and §4]) that this scheme-theoretical intersection multiplicity $m_i$ is nothing but the topological intersection multiplicity of $s_2 h(\tilde{D})$ and $s_1 (BL)$ at the point $(x_i, x_i, L_{x_i}, x_i)$. For this we move $s_1 (BL)$ slightly along the fiber so that the moved $s_1 (BL)$ intersects $s_2 h(\tilde{D})$ transversely in a small neighborhood of $(x_i, x_i, L_{x_i}, x_i)$ and count the intersection points in this neighborhood. Since this is a local problem, we can analyze it as follows. Take an $\epsilon$-neighborhood $BL_\epsilon$ of $(x_i, x_i, L_{x_i})$ in $BL$ and an $\epsilon$-neighborhood $\tilde{D}_\epsilon$ of $(x_i, x_i, L_{x_i})$ in $\tilde{D}$. Then move $s_1 (BL_\epsilon)$ slightly along the fiber and count the intersection points of the moved $s_1 (BL_\epsilon)$ and $s_2 h(\tilde{D}_\epsilon)$. To be more precise, let $H$ be a hyperplane not containing $x_i$ and consider the affine variety $X^a = X - H \subset \mathbf{P}^N - H = \mathbf{C}^N$. Here we can assume that a generic point $P$ does not lie on $H$, so $P \in \mathbf{C}^N$. Let $\pi: \mathbf{B}L \to \mathbf{C}^N \times \mathbf{C}^N$ be the blowup (we use the same symbol $BL$ which denotes $\mathbf{P}^N \times \mathbf{P}^N$) and let $(a^0, a^0, \theta^0)$ correspond to the point $(x_i, x_i, L_{x_i})$. So $x_i$ corresponds to the point $a^0$ under the isomorphism $\mathbf{P}^N - H \cong \mathbf{C}^N$. Without loss of generality, we can assume that the first homogeneous coordinate $\theta_i^0$ is not zero and consider the affine coordinates $1/1$. Then, let $(a^0, a^0, \theta^0)$ be corresponding to $(a^0, a^0, \theta^0)$. Let

$$BL_\epsilon = \{(z, w, \eta) \mid z_i - w_i = (z_1 - w_1) \cdot \eta_i, \|z - a^0\| < \epsilon, \|w - a^0\| < \epsilon, \|\eta - \theta^0\| < \epsilon\}$$

where

$$\|z - a^0\| = \sqrt{|z_1 - a^0_1|^2 + \cdots + |z_N - a^0_N|^2}$$

is the distance between $z$ and $a^0$, and so for $\|w - a^0\|$ and $\|\eta - \theta^0\|$. Let $\tilde{D}_\epsilon = \tilde{D} \cap BL_\epsilon$. Let $t$ be a sufficiently small complex number such that $0 < |t| \ll \epsilon$. Then let

$$s_{1,t}(BL_\epsilon) = \text{closure of } \{(z, w, \eta, w = t \cdot \frac{w - z}{\|w - z\|}) \mid z \neq w, (z, w, \eta) \in BL_\epsilon\}$$

where $z + t \cdot (w - z)/\|w - z\|$ means

$$z_1 + t \cdot \frac{w_1 - z_1}{\|w - z\|}, \ldots, z_N + t \cdot \frac{w_N - z_N}{\|w - z\|}.$$
Polar classes and Segre classes

$s_2 h(\tilde{D}_e) \cap s_{1,t}(BL_\epsilon)$ is given by

$$\# \left\{ (z, w, \eta, w) \in s_2 h(\tilde{D}_e) \mid w = z + t \cdot \frac{w - z}{\| w - z \|} \right\}$$

$$= \# \left\{ (z, w, \eta) \in \tilde{D}_e \mid \| w - z \| = |t| \right\}$$

$$= \# \left\{ w \in X^a \cap B_\epsilon(a^0) \mid |t| = \| z - w \|, \exists z \in X^a \cap B_\epsilon(a^0), \mathbf{zw} \ni P \right\},$$

where $\mathbf{zw}$ is the line going through $z$ and $w$.

Here we notice that the transversality of $s_2 h(\tilde{D}_e)$ and $s_{1,t}(BL_\epsilon)$ implies the transversality of $s_{p,t}(X^a)$ and $X^a$ within $B_\epsilon(a^0)$. Hence the following does make sense:

$$\# \left( S_{p,t}(X^a) \cap X^a \cap B_\epsilon(a^0) \right) = \# \left\{ w \in X^a \cap B_\epsilon(a^0) \mid w = z + t \cdot \frac{P - z}{\| P - z \|} \right\}$$

$$= \# \left\{ w \in X^a \cap B_\epsilon(a^0) \mid |t| = \| w - z \|, \exists z \in X^a \cap B_\epsilon(a^0), \mathbf{zw} \ni P \right\}.$$

Thus

$$n_i = \# \left( S_{p,t}(X^a) \cap X^a \cap B_\epsilon(a^0) \right) = n(x_i, P, t, \epsilon).$$

Therefore by Observation 5.8, (5.12), and since taking a generic center $P$ leads us to formula (5.3') (although it is independent of the choice of a generic center $P$), we can conclude both statements (i) and (ii) of Theorem 5.6. This completes the proof of the theorem.

It remains only to prove

**Lemma 5.11.** $s_{1,t}(BL_\epsilon)$ is transverse to $s_2 h(\tilde{D}_e)$ for a sufficiently small $\epsilon$.

**Proof.** (Cf. Figure 5.) Let $\Gamma$ be a sufficiently small branch of $s_1 h(\tilde{D}_e)$ at the point $s_1(a) = s_2(a)$, so $\Gamma - \{ s_1(a) \}$ is smooth. For a sufficiently small $\epsilon$, we may consider that $s_{1,t}(BL_\epsilon)$ and $s_2 h(\tilde{D}_e)$ are in the product $C^{2N} \times C$, where the second factor $C$

![Figure 5](http://www.ams.org/journal-terms-of-use)
indicates the fiber, and \( s_1(BL) \) are in the first factor \( C^{2N} \), and \( s_{1,t}(BL) \) is the translation of \( s_1(BL) \) along the fiber \( C \) by \( t \). Let \( \Delta \) be a small closed disk around the origin in the complex line \( C_w \) where \( w \) denotes the coordinate of \( C \), and let \( f: (\Delta, 0) \to (\Gamma, s_1(a)) \) be the local uniformization (or parameterization) of \( \Gamma \); i.e., \( (f(w) = (f_1(w), \ldots, f_{2N}(w), f_{2N+1}(w)), \) where \( f_j(w) \) is a holomorphic function. Let \( \pi: C^{2N} \times C \to C \) be the projection map, which is clearly holomorphic. Then under the local uniformization of \( \Gamma \), the points where \( \Gamma \) are not transverse to \( s_{1,t}(BL) \), i.e., the points where the tangent lines of \( \Gamma \) are in the translated \( C_{t,2N} \) of \( C_{t,2N} \), correspond to the points \( w \) of \( \Delta \) such that \( (\pi \circ f)'(w) = 0 \). Since \( \pi \circ f: \Delta \to C \) is holomorphic and not a constant map, there are not infinitely many such points \( w \in \Delta \) as \( (\pi \circ f)'(w) = 0 \); otherwise, by a well-known fact in analytic function theory (i.e., identity theorem), \( (\pi \circ f)' \equiv 0 \), which implies that \( \pi \circ f \) is constant, a contradiction. Thus, for each branch (there are only finitely many branches of \( s_2h(D) \) at the point \( s_1(a) = s_2(a) \)), there are only finitely many points where the branch is not transverse to \( s_{1,t}(BL) \). Hence if we take a sufficiently small \( \epsilon \) and \( t, s_{1,t}(BL) \) is always transverse to \( s_2h(D) \). This completes the proof of the lemma.

BIBLIOGRAPHY


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