DENSE IMBEDDING OF TEST FUNCTIONS
IN CERTAIN FUNCTION SPACES

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ABSTRACT. In a recent paper [1], J. U. Kim studies the Cauchy problem for the motion of a Bingham fluid in $R^2$. He points out that the extension of his results to three dimensions depends on proving the denseness of $C^\infty$-functions with compact support in certain spaces. In this note, such a result is proved.

Following Kim's notation [1], we define the following spaces:

$$\tilde{F}_p(R^n) = \{ u \in W^{1,2}(R^n) | \nabla u \in (L^p(R^n))^n \},$$
$$F_p(R^n) = \{ u \in (W^{1,2}(R^n))^n | \nabla u \in (L^p(R^n))^{n \times n}, \text{div} u = 0 \},$$
$$G_p(R^n) = \{ u \in (W^{1,2}(R^n))^n | e(u) = \nabla u + (\nabla u)^T \in (L^p(R^n))^{n \times n}, \text{div} u = 0 \},$$
$$S(R^n) = \{ u \in (C^\infty_0(R^n))^n | \text{div} u = 0 \}.$$

According to Kim's Lemma 1.7 [1], $F_p = G_p$ for $1 < p < \infty$. The results, which will be presented in this paper, are the following.

**Theorem 1.** Let $n$ be arbitrary and $1 \leq p < \infty$. Then $C^\infty_0(R^n)$ is dense in $\tilde{F}_p(R^n)$.

**Theorem 2.** Let $n = 2$ or $n = 3$ and $1 \leq p < \infty$. Then $S(R^n)$ is dense in $F_p(R^n)$ and $G_p(R^n)$.

We remark that the case $p = 2$ of Theorem 2 is well known, even in the context of general domains (see, for example, Heywood [2]). The proofs of both theorems will make use of the following lemma.

**Lemma.** For $x \in R^n$, let

$$\phi_N(x) = \begin{cases} (N^n\Omega_n)^{-1} & \text{if } |x| \leq N, \\ 0 & \text{if } |x| > N, \end{cases}$$

where $\Omega_n$ denotes the volume of the unit ball in $R^n$. Let $1 \leq r < \infty$ and $v \in L^r(R^n)$; if $r = 1$, assume in addition that $\int_{R^n} v = 0$. Then $\phi_N * v \to 0$ in $L^r(R^n)$ as $N \to \infty$.

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Proof of the Lemma. Since \( \|\phi_N\|_{L^1} = 1 \), we have \( \|\phi_N \cdot v\|_{L'} \leq \|v\|_{L'} \), and hence it suffices to show that \( \phi_N \cdot v \to 0 \) for \( v \) in a dense subset of \( L' \). If \( r > 1 \), take \( v \in L^1 \cap L' \). Then \( \|\phi_N \cdot v\|_{L'} \leq \|\phi_N\|_{L'}\|v\|_{L}, \) which tends to zero as \( N \to \infty \). For \( r = 1 \), let \( v \) have compact support, contained in, say, \( \{|x| \leq R\} \), and assume \( \int_{R^n} v = 0 \). Then

\[
\|\phi_N \cdot v\|_{L'} = \int_{R^n} \left| \int_{R^n} \phi_N(x - y)v(y) \, dy \right| \, dx \\
eq \int_{-R \leq |x| \leq N + R} \left| \int_{|y| \leq R} \phi_N(x - y)v(y) \, dy \right| \, dx \\
\leq \int_{-R \leq |x| \leq N + R} \left| \phi_N(x - y) \right| \left| v(y) \right| \, dy \, dx \\
\leq \int_{-2R \leq |z| \leq N} \left| \phi_N(z) \right| \, dz \cdot \int_{|y| \leq R} \left| v(y) \right| \, dy.
\]

This tends to zero as \( N \to \infty \).

Proof of Theorem 1. Clearly it suffices to show that functions of compact support are dense, \( C^\infty \)-regularity can easily be achieved by using a mollifier. If we know that \( u \in L^p(R^n) \) or even that \( u \in L^{p+\varepsilon}(R^n) \) for small enough \( \varepsilon > 0 \), then we can use the standard cut-off procedure to approximate \( u \) by functions of compact support, i.e., if we set \( u_m(x) = u(x)\psi_m(x) \), where, for example,

\[
\psi_m(x) = \begin{cases} 
1 & \text{if } |x| \leq m, \\
2 - |x|/m & \text{if } m \leq |x| \leq 2m, \\
0 & \text{if } |x| \geq 2m,
\end{cases}
\]

then it is easy to show that \( u_m \to u \) in \( \tilde{F}_p \). Therefore, it suffices to show that \( \tilde{F}_p \cap L^{p+\varepsilon} \) is dense in \( \tilde{F}_p \). If \( p > 2 \), then the Sobolev imbedding theorem can be used to show that \( \tilde{F}_p \subset L^p \), and there is nothing left to prove.

For \( p < 2 \), let \( \phi_N \) be as in the lemma above. For \( u \in \tilde{F}_p \), let \( u_N = u - \phi_N \cdot u \). We have \( \nabla u_N = \nabla u - \phi_N \cdot \nabla u \), and, if \( p = 1 \), then \( \int_{R^n} \nabla u = 0 \), since \( u \in L^2 \). Therefore, the lemma implies that \( u_N \to u \) as \( N \to \infty \) in the norm of \( \tilde{F}_p \). It is therefore enough to show that \( u_N \) lies in \( L^{p+\varepsilon} \) for small \( \varepsilon > 0 \). Let \( g \) denote the fundamental solution for Laplace’s equation,

\[
g(x) = \begin{cases} 
|x|^{2-n}/\omega_n(n-2) & \text{if } n \geq 3, \\
\ln|x|/2\pi & \text{if } n = 2,
\end{cases}
\]

where \( \omega_n \) denotes the surface measure of the unit sphere in \( R^n \). In any dimension, \( g \) and its first derivatives are in \( L^{1+\delta}_{\text{loc}} \) for sufficiently small \( \delta > 0 \). We want to consider the behavior of \( g - \phi_N \cdot g \) at infinity. We have

\[
g(x) - \phi_N \cdot g(x) = g(x) - \int_{|y-x|<N} \frac{g(y)}{N^n\Omega_n} \, dy.
\]
By expanding the integrand in a Taylor series about \( x \), we find that this can be bounded by a constant times

\[
N^2 \max_{|y-x| < N} \max_{i,j} \left| \frac{\partial^2 g(y)}{\partial x_i \partial x_j} \right|
\]

Since second derivatives of \( g \) decay like \( |x|^{-n} \) at infinity, it follows that \( g - \phi_N \ast g \) is in \( L^{1+\delta} \) at infinity for any positive \( \delta \), and so are derivatives of \( g \) by the same argument. Hence we conclude that, for small enough \( \delta > 0 \), \( g - \phi_N \ast g \) lies in \( L^{1+\delta} \).

It follows that \( \omega_N = g \ast \nabla u_N = (g - \phi_N \ast g) \ast \nabla u \) lies in \( L^{p+\varepsilon} \) for small positive \( \varepsilon \), and so do its first derivatives. Since \( \text{div} \omega_N = u_N \), this completes the proof.

**Proof of Theorem 2.** For \( p > 1 \), the arguments used by Kim [1] show that Theorem 2 follows from Theorem 1. We may hence concentrate on the case \( p = 1 \).

For \( u \in F_1 \) or \( G_1 \), let \( u_N = u - \phi_N \ast u \) with \( \phi_N \) as before. As in the proof of Theorem 1, it can be shown that \( u_N \to u \) in \( F_1 \) or \( G_1 \), respectively. Moreover, let \( a_N = g \ast \text{curl} u_N = (g - \phi_N \ast g) \ast \text{curl} u \). The convolution \( g \ast \text{curl} u_N \) makes sense because \( G_1 \) and \( F_1 \) are contained in \( F_p \) for \( 1 < p < 2 \), hence the same argument as in the proof of Theorem 1 shows that \( \text{curl} u_N \) as well as \( u_N \) are in \( L^p \) for \( p \in (1, 2] \).

Moreover, \( g \) is integrable at the origin, and its derivative has some power that is integrable at infinity. We can thus decompose \( g \) in the form \( g = g_1 + g_2 \), where \( g_1 \in L^1 \) and \( \nabla g_2 \in L^q \) for some \( q < \infty \). Clearly \( g_1 \ast \text{curl} u_N \) is defined, and \( g_2 \ast \text{curl} u_N \) can be defined by transferring the derivative onto \( g_2 \). We have \( \Delta a_N = \text{curl} u_N \) and \( \text{curl} a_N = g \ast \text{curl} \text{curl} u_N = g \ast (-\Delta u_N) = -u_N \). Since \( G_1 \) and \( F_1 \) are contained in \( F_p \), for every \( p \in (1, 2] \), \( \text{curl} u \) lies in \( L^{1+\varepsilon} \) for \( 0 < \varepsilon < 1 \), and we can conclude as in the proof of Theorem 1 that

\[
a_N = (g - \phi_N \ast g) \ast \text{curl} u \in L^{1+\varepsilon}.
\]

Since \( \Delta a_N \) is also in \( L^{1+\varepsilon} \), it follows that \( a_N \in W^{2, 1+\varepsilon} \).

It thus remains to show that every \( u \in G_1 \) or \( F_1 \) which has the form \( u = \text{curl} a \) with \( a \in W^{2, 1+\varepsilon} \) can be approximated by functions with compact support. This can easily be achieved by multiplying \( a \) with a suitable cut-off function.

**References**