FIXED SETS OF FRAMED G-MANIFOLDS

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ABSTRACT. This note describes restrictions on the framed bordism class of a framed manifold \( Y \) in order that it be the fixed set of some framed \( G \)-manifold \( M \) with \( G \) a finite group. These results follow from a recently proved generalization of the Segal conjecture, and imply, in particular, that if \( M \) is a framed \( G \)-manifold of sufficiently high dimension, and if \( G \) is a \( p \)-group, then the number of "noncancelling" fixed points is either zero or approaches infinity as the dimension of \( M \) goes to infinity. Conversely, we give sufficient conditions on the framed bordism class of a manifold \( Y \) that it be the fixed set of some framed \( G \)-manifold \( M \) of arbitrarily high dimension.

Introduction and statement of results. In this note, we show how the recently proved Segal conjecture on the stable cohomotopy of the classifying space \( BG \) of a finite group \( G \) turns out to place severe restrictions on the fixed-sets of framed \( G \)-manifolds of large dimension.

Conner and Floyd proved the following result in [CF, 40.1]. Let \( G = \mathbb{Z}/p \) (\( p \) an odd prime), and let \( M \) be a smooth compact oriented \( G \)-manifold with fixed set \( Y \) of codimension \( n \) and framed in \( M \). (That is, the normal bundle of \( Y \) in \( M \) is equivariantly framed.) Assume also that the local representation normal to \( Y \) is the same for all components of \( Y \). Then, denoting oriented bordism by \( \Omega_* \), one has \( [Y] \in p^{s(n)}\Omega_* \), where \( s(n) \to \infty \) as \( n \to \infty \). When \( Y \) is discrete, this means that the number of "noncancelling" fixed points is either zero or becomes large as the dimension of \( M \) increases.

Here, we examine this phenomenon in the context of framed \( G \)-manifolds, and give a direct generalization for arbitrary finite groups \( G \). As alluded to above, our proof makes extensive use of the Segal conjecture proved by Carlsson [C1], or, more precisely, its generalization due to Adams, Haeberly, Jackowski, and May [A1]. This suggests that even the "stable" (high-dimensional) properties of fixed sets of \( G \)-manifolds are subtle, and that a generalization of the Conner-Floyd result to oriented \( G \)-manifold for arbitrary \( G \) might require some form of completion result for oriented bordism analogous to the Segal conjecture.

If \( M \) is a (smooth) framed \( G \)-manifold, then there exists an orthogonal \( G \)-module \( V \) such that \( M \) is "modelled locally on \( V \)" in the sense of Pulikowski [P1] and Kosniowski [K1]. This means that if \( x \in M \), then there is a neighborhood \( U \) of \( x \) which is \( G_x \)-diffeomorphic with \( V|G_x \).

Our result is the following.

**Theorem A.** Let \( G \) be a finite group, let \( V \) be an orthogonal \( G \)-module with \( V^G = \{0\} \), and let \( k \geq 0 \). Then there exists an integer \( j \) as well as a sequence \( (s_n) \)
with $s_n \to \infty$ as $n \to \infty$, such that, if $M$ is any framed $G$-manifold modelled locally on the representation $V^n \oplus \mathbb{R}^k$ with $G$-fixed set $Y$, one has $[Y] \in j^*n\Omega^n_{\mathbb{R}}$, where $\Omega^n_{\mathbb{R}}$ denotes nonequivariant framed bordism.

The integer $j$ is determined by the isotropy subgroups of points in $V$ and the algebra of the Burnside ring of $G$, and will be described fully in §1. When $j > 1$, the theorem implies that one cannot have a framed $G$-manifold modelled on $V$ possessing a single fixed point (see §2, Corollary 2). If $G$ is a $p$-group, it will turn out that $j$ is always a power of $p$. When $G$ has odd order and $V$ is so large as to contain arbitrary $G$-orbits, then $j > 1$. On the other hand, if, for example, $G = \mathbb{Z}/p \times \mathbb{Z}/q$ with $p$ and $q$ distinct primes, then there exist $V$'s such that $j = 1$.

Theorem A has the following converse.

**Theorem B.** Let $G$ be a finite group, let $V$ be an orthogonal $G$-module with $V^G = \{0\}$, and let $m, k \geq 0$. Then, with $j$ as in Theorem A and $Y$ an arbitrary framed manifold of dimension $k$, there exists an integer $n$ and a framed $G$-manifold $M$ modelled locally on $V^n \oplus \mathbb{R}^k$ with fixed set framed cobordant with $jnY$.

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1. **A consequence of the Segal conjecture.** Let $G$ be a finite group and let $U = R^\infty$, where $R$ denotes the real regular representation of $G$, endowed with its natural inner product. We shall write $V < U$ to indicate that $V$ is a finite-dimensional $G$-invariant subspace of $U$. The one-point compactification of $V < U$ will be denoted by $S^V$ and, if $X$ is a based $G$-space, the smash product $X \wedge S^V$ will be denoted by $\Sigma^V X$. The stable equivariant cohomotopy of $X$ is given by

$$\omega^G_G(X) = \operatorname{colim}_{U < U}[\Sigma^W \wedge U X, S^V \wedge U]_G,$$

where $\gamma = [V - W] \in RO(G)$ and where $[-,-]_G$ denotes $G$-homotopy classes of based $G$-maps. Dually, the $-\gamma$th stable equivariant homotopy group, $\omega^{-\gamma}_G(X)$, is given by

$$\omega^{-\gamma}_G(X) = \operatorname{colim}_{U < U}[\Sigma^W \wedge U, \omega^V \wedge U X]_G.$$

We shall require the following result.

**Lemma 1.1.** Let $n > 0$, $m \geq 0$, and $V < U$ with $V^G = \{0\}$. Then $\omega^G_{mV+n}$ is finite.

**Proof.** Consider first the case $m = 0$. One has, by a result of Hauschild [H1],

$$\omega^G_n \cong \Sigma(H)\pi^*_n(B(NH/H)_+),$$

for $n \geq 0$, where the sum is taken over a complete set of conjugacy classes $(H)$ of subgroups of $G$. The subscript $+$ denotes addition of a disjoint basepoint. If $n > 0$, then $\pi^*_n(B(NH/H)_+)$ is finite. Now let $m \geq 0$. Then

$$\omega^G_{mV+n} \cong [S^{mV}, \Omega^nQ_GS^0]_G,$$

where $Q_GS^0$ is the equivariant loop space $\operatorname{colim}_{W < U}\Omega^WS^W$, $\Omega^WS^W$ denoting the $G$-space of self-maps of $S^W$ (see, for example, [H1 or CW]). Since $n \geq 0$, all the homotopy groups of all fixed sets of $\Omega^nQ_GS^0$ are finite by the case $n = 0$ applied
to the subgroups $H \subset G$. It now follows by induction over the skeleta of $S^{nV}$ that $\omega^G_{mV+n}$ is finite. □

Let $V < U$ be any $G$-module with $V^G = \{0\}$. Define an associated family $\mathcal{F}(V)$ of subgroups of $V$ by

$$\mathcal{F}(V) = \{H \subset G : V^H \neq \{0\}\}.$$ 

One has a universal $G$-space $E\mathcal{F}(V)$ associated with $\mathcal{F}(V)$; $E\mathcal{F}(V)$ is the unique (up to $G$-homotopy) $G$-CW complex with $E\mathcal{F}(V)^H$ contractible for each $H \in \mathcal{F}(V)$ and empty otherwise. There is then a $G$-cofiber sequence

$$E\mathcal{F}(V)^+ \to S^0 \to E\mathcal{F}(V) \to \cdots$$

associated with the projection of $E\mathcal{F}(V)$ onto a point. Note that, with $S(U)$ denoting the unit sphere in $U < U$, one has

$$E\mathcal{F}(V) \simeq S(\infty V) = \operatorname{colim}_n S(nV),$$

while

$$E\mathcal{F}(V) \simeq S^{\infty V} = \operatorname{colim}_n S^nV,$$

both colimits being taken with respect to the natural inclusions. Passing to stable equivariant cohomotopy gives an exact sequence

$$(1) \quad \cdots \to \omega^G_{\infty}(E\mathcal{F}(V))^+ \overset{\beta}{\to} \omega^G_{\infty}(S^0) \overset{\alpha}{\to} \omega^G_{\infty}(E\mathcal{F}(V)^+^+) \to \cdots$$

in which $\alpha$ is the Segal map in the generalized context of [A1]. In this setting, the Segal conjecture takes the following form. Let $A(G)$ denote the Burnside ring of $G$, and let, for $H \subset G$,

$$d_H : A(G) \to \mathbb{Z}$$

be the homomorphism assigning to the virtual $G$-set $s - t$ the integer $|s^H| - |t^H|$. Denote the ideal $\bigcap_{(H) \in \mathcal{F}(V)} \ker d_H$ by $I(V)$, and $I(V)$-adic completion of the $A(G)$-module $M$ by $\hat{M}$. The conjecture as proved in [A1] then states that $\alpha$ induces an isomorphism

$$\alpha^\wedge : (\omega^G_{\infty}(S^0))^\wedge \to \omega^G_{\infty}(E\mathcal{F}(V)^+)$$

for each $\gamma \in RO(G)$. (In particular, $\omega^G_{\infty}(E\mathcal{F}(V)^+)$ is $I(V)$-adically complete.)

Let $k \in \mathbb{Z}$. The exact sequence (1) is closely related to the exact sequence

$$(2) \quad \cdots \to \omega^G_{nV+k}(S^0) \overset{\beta_n}{\to} \omega^G_{k}(S^0) \overset{\alpha_n}{\to} \omega^G_{nV+k-1}(S(nV)^+) \to \cdots$$

is stable $G$-homotopy induced by the cofiber sequence

$$S(nV)^+ \to D(nV)^+ \to S^{nV} \to \Sigma S(nV)^+ \to \cdots.$$ 

The sequence (2) gives rise to short exact sequences

$$(3) \quad 0 \to \omega^G_k / \operatorname{Im} \beta_n \to \omega^G_{nV+k-1}(S(nV)^+) \to \operatorname{coker} \alpha_n \to 0,$$

where $\omega^G_k = \omega^G_k(S^0)$. One has natural homomorphisms

$$\gamma_* : \omega^G_{(n+1)V+k-1}(S(n+1)V^+) \to \omega^G_{nV+k-1}(SnV^+)$$
(omitting some parentheses), given as follows. Let $\nu: S(n+1)V_+ \to \Sigma^V S_n V_+$ denote the natural quotient, obtained by collapsing about a tubular neighborhood of $S(nV)$ in $S((n+1)V)$, and define $\gamma_*$ as the composite

$$
\omega^G_{(n+1)V+k-1}(S(n+1)V_+) \xrightarrow{\nu} \omega^G_{(n+1)V+k-1}(\Sigma^V S_n V_+) \cong \omega^n_{nV+k-1}(S_n V_+).
$$

It may be checked that, under Spanier Whitehead duality, the maps $\gamma_*$ agree with the inverse system maps

$$
\gamma^*: \omega^{-k}_G(S(n+1)V_+) \to \omega^{-k}_G(S_n V_+)
$$

induced by inclusion. One also has natural homomorphisms

$$
\mu_*: \omega^G_{(n+1)V+k}(S^0) \to \omega^G_{nV+k}(S^0),
$$

given by the composites

$$
\omega^G_{(n+1)V+k}(S^0) \xrightarrow{(\beta_n)} \omega^G_k(S^0) \xrightarrow{(\alpha_n)} \omega^G_{(n+1)V+k-1}(S(n+1)V_+) \to \cdots
$$

where the first map is induced by inclusion $S^0 \to S^V \cong \Sigma^V S^0$. The maps $\gamma_*$ and $\mu_*$ commute the maps in the sequence (2), giving commutative diagrams:

$$
\cdots \to \omega^G_{(n+1)V+k}(S^0) \xrightarrow{(\beta_n)} \omega^G_k(S^0) \xrightarrow{(\alpha_n)} \omega^G_{(n+1)V+k-1}(S(n+1)V_+) \to \cdots \quad \gamma^*
$$

$$
\cdots \to \omega^G_{nV+k}(S^0) \xrightarrow{(\beta_n)} \omega^G_k(S^0) \xrightarrow{(\alpha_n)} \omega^G_{nV+k-1}(S_n V_+) \to \cdots \quad \gamma_*. \quad \gamma^*
$$

Passing the sequences (3) to (inverse) limits gives an exact sequence

$$
0 \to \lim_n \omega^G_k/I\beta_n \xrightarrow{\alpha_n} \lim_n \omega^G_{nV+k-1}(S(nV)_+) \to \lim_n \text{coker} \alpha_n \to 0
$$

since $\text{lim}^1 \omega^G_k/I\beta_n = 0$, the bonding maps being surjections. The map $\alpha = \text{lim}_n \alpha_n$ is reminiscent of the Segal map $\alpha^\wedge$. Write the latter (dually) as

$$
\alpha^\wedge: \lim_n \omega^G_k/I(V)^n \omega^G_k \to \lim_n \omega^G_{nV+k-1}(S(nV)_+).
$$

(The target is $\omega^{-k}_G(E\mathcal{F}(V)_+)$ by vanishing of the $\text{lim}^1$ terms [A1].) Abbreviate $\text{lim}_n \omega^G_k/I\beta_n$ as $(\omega^G_k)^\wedge_\beta$. One then has

**PROPOSITION 1.2.** There exists a natural homomorphism

$$
\psi: (\omega^G_k)^\wedge \to (\omega^G_k)_\beta
$$

making the diagram

$$
(\omega^G_k)^\wedge \xrightarrow{\alpha} \omega^{-k}_G(E\mathcal{F}(V)_+) \xrightarrow{\psi} (\omega^G_k)_\beta
$$

commute. It now follows from injectivity of $\alpha$ (in (4)) that both $\alpha$ and $\psi$ are isomorphisms.

**PROOF.** If $k < 0$, the conclusion is immediate since $\omega^G_k = 0$. Thus assume $k \geq 0$. It suffices to show that, for each $n \geq 0$, there exists an integer $r(n)$ with

$$
I(V)^{r(n)} \subset \text{Im} \beta_n.
$$
(This will then technically define a pro-map from the one inverse system to the other.)

Let $x \in \omega^G_k$. Then $x$ is represented by a $G$-map $S^{W+k} \to S^W$ for some $W < U$. Our object is now to extend a representative of $\rho x$ over $S^{W+k+nV}$ (stably) for arbitrary $\rho \in I(V)^{r(n)}$ with $r(n)$ independent of $x$. Regard the pair $(S^{nV}, S^0)$ as a relative $G$-CW complex with relative $G$-cells of the form $G/H \times D^i$ for $H \in \mathcal{F}(V)$ (which one may assume by the orbit structure of $S^{nV}$).

We define $r(n)$ as the number of relative $G$-cells in $(S^{nV}, S^0)$. Assume, inductively over the skeletons of the pair, that for each $\rho \in I(V)^{s(p)}$, with $s(p)$ the number of relative $G$-cells in the $p$-skeleton $((S^{nV})^p, S^0)$, one has a stably $G$-homotopy commutative diagram:

$$
\begin{array}{c}
(S^{nV})^p \wedge S^{W+k} \\
\cap \swarrow S^{W+k} \\
\downarrow S^{W+k} \\
\quad \uparrow \\
\quad \downarrow S^W \\
\end{array}
$$

Here, $f_p$ represents $\rho x$ and $j$ is inclusion. The obstruction to extending $g_p$ stably over a typical $(p+1)$-cell of the form $G/H \times D^{p+1}$ defines a stable $H$-equivariant map

$$
\theta: S^p \wedge S^{W+k} \to (S^{nV})^p \wedge S^{W+k} \\
\cap j \swarrow S^{W+k} \\
\downarrow S^W,
$$

where $c$ is adjoint to the attaching map for that cell. If $k \in I(V)$, one may represent $k$ by a stable $G$-map $k: S^X \to S^X$ for suitable $X < U$. Consider the diagram:

$$
\begin{array}{c}
(S^{nV})^p \wedge S^{W+k} \wedge S^X \\
\cap \swarrow S^{W+k} \wedge S^X \\
\downarrow S^{W+k} \wedge S^X \\
\quad \uparrow \\
\quad \downarrow S^W \wedge S^X \\
\end{array}
$$

The obstruction to extending $g_p \wedge k$ stably over this cell is now represented by $\theta \wedge k$, regarded as an $H$-equivariant map. Since $k \in I(V)$ and $H \in \mathcal{F}(V)$, this is $H$-homotopy trivial. Thus one may extend $g_p \wedge k$ stably over this cell. Note that $f_p \wedge k$ represents $k \rho x$, so that one may continue this process over the relative $(p+1)$-cells and obtain the inductive step, and hence the result. □

One has the following converse to Proposition 1.2.

**Proposition 1.3.** Let $k \in \mathbb{Z}$. Then there exists a sequence $s(n) \to \infty$ as $n \to \infty$ such that $\text{Im} \beta_n \subset I(V)^{s(n)} \omega^G_k$ for each $n$ sufficiently large.

**Proof.** Define a preliminary sequence $r(n)$ by

$$
r(n) = \min\{n, \max\{j \in \mathbb{N} : \text{Im} \beta_n \subset I(V)^j \omega^G_k\}\}.
$$

(Note that one must allow $\max\{j \in \mathbb{N} : \text{Im} \beta_n \subset I(V)^j \omega^G_k\} = \infty$.) Then, by definition, $\text{Im} \beta_n \subset I(V)^{r(n)} \omega^G_k$. To prove the proposition, it suffices to show that there exists a sequence $q(n)$ of $r(n)$ with $q(n) \to \infty$ as $n \to \infty$. Assume that no such sequence exists. Then there exists an integer $j \in \mathbb{N}$ and a subsequence $t(n)$ of the natural numbers with

$$
\text{Im} \beta_{t(n)} \subset I(V)^j \omega^G_k \quad \text{and} \quad \text{Im} \beta_{t(n)} \not\subset I(V)^{j+1} \omega^G_k.
$$

It follows that there is a sequence of stable $G$-maps

$$
x_{t(n)}: S^{nV+W+k} \to S^W
$$
with the composite
\[ y_{t(n)} : S^{W+k} \to S^{nW+k} \to S^W \]
defining a class \([y_{t(n)}] \in I(V)^j \omega_k^G - I(V)^{j+1} \omega_k^G\) for each \(n\).

If \(k > 0\), then, by Lemma 1.1, \(\omega_k^G\) and \(\omega_k^{G+nV}\) are finite. Since the maps \(\beta_n\) define a map \(\beta\) into the constant system \(\{\omega_k^G\}\), it now follows that there exists an element \(z = ([z_n]) \in \lim_n \omega_k^{G+nV}\), obtained from the \([x_{t(n)}]\) by application of the bonding homomorphisms, with
\[ \beta_n([z_n]) \in I(V)^j \omega_k^G - I(V)^{j+1} \omega_k^G \]
for each \(n \geq 0\). However, \(\beta_n([z_n]) = \beta(z)\) is now independent of \(n\), since it is in a constant system, and \(\beta(z) \in \bigcap_n \text{Im } \beta_n\), by construction. Thus the completion
\[ \omega_k^G \to (\omega_k^G)_{\beta} \]
maps \(\beta(z)\) to zero. Thus, by Proposition 1.2, \((I(V))\text{-adic completion } \omega_k^G \to (\omega_k^G)_{\beta}\) maps \(\beta(z)\) to zero as well. It now follows that \(\beta(z) \in \bigcap_n I(V)^n \omega_k^G\), by definition of \(I(V))\text{-adic completion}. But \(\beta(z) \in I(V)^j \omega_k^G - I(V)^{j+1} \omega_k^G\), a contradiction.

We now consider the case \(k = 0\). Here, by definition of the \(x_n\), one has \(\beta_n(x_n) \in I(V)\omega_0^G = I(V)\), since \(\omega_0^G \cong A(G)\). However, \(I(V)/I(V)^m\) is finite for each \(m \geq 1\), so that there exists a sequence \(([z_n])\) with
\[ z_n : S^{nW} \to S^W \]
such that \(\beta_n[z_n] \in I(V)^j - I(V)^{j+1}\) and such that \(([z_n])\) maps under the natural quotient
\[ \prod_n \omega_{nV}^G \to \prod_n A(G)/I(V)^n \]

\(\omega_k^G \to (\omega_k^G)_{\beta}\)
to an element \(a = ([a_n])\) of \(\lim_n A(G)/I(V)^n = A(G)^\sim\). Thus if \(a_n \in A(G)\) represents \([a_n]\), one has \(a_n - \beta_n[z_n] \in I(V)^n\). Consider \(\psi(a) \in A(G)_{\beta}\). By the construction of \(\psi\), there is a sequence \(q(n)\) with \(q(n) \leq n\) and \(q(n) \to \infty\) such that
\[ a_n - \beta_n[z_n] \in \text{Im } \beta_{q(n)} \).

It now follows that \(\psi(a) = 0\), whence \(a = 0\). But \(a_n = \beta_n[z_n] \in I(V)^j - I(V)^{j+1}\), which is again a contradiction.

When \(k < 0\), \(\omega_k^G = 0\), so the conclusion is automatic in this case. □

**2. Application to framed G-manifolds.** Fix \(V \subset U\), and let \(M\) be a smooth \(G\)-manifold. Then \(M\) is said to have equivariant dimension \(V\) (or to be a \(V\)-manifold) if, for each \(x \in \text{Int } M\), there is a smooth \(G_x\)-equivariant diffeomorphism \(i : V \to M\), taking \(0\) to \(x\). More generally, \(M\) is a \((V - W)\)-manifold for \(V\) and \(W \subset U\) if \(M \times D(W)\) is a \(V\)-manifold. This notion is due originally to Pulikowski [P1] and Kosniowski [K1], but we shall not be requiring such generalizations here.

We shall refer to a \(G\)-manifold of dimension \(V^n \pm R^k\) (where \(R^k\) is given the trivial \(G\)-action) as an \((nV \pm k)\)-manifold, and all \(G\)-manifolds considered will be assumed compact.

The normal bundle of a \(G\)-manifold with equivariant dimension \(V\) has fibers similarly modelled on a fixed representation \(W\) in the sense that the fiber over a
typical point $x$ is $G_x$-isomorphic with $W$. Such $G$-bundles are discussed in [W1 and W2]. $M$ is equivariantly framed if its normal bundle $\mu_M$ with respect to a smooth embedding in some (large) finite-dimensional $G$-module $U$ is a product, $\mu_M \cong \mathcal{E}_M(W)$, where $\mathcal{E}_M(W)$ is the product $G$-bundle $M \times W \to M$, and where $V \oplus W \cong U$, as a $G$-module.

**Remark 2.1.** This last condition, that $V \oplus W \cong U$, is necessary to obtain a well-defined homomorphism from framed $G$-bordism into equivariant stable homotopy. For example, if $G = \mathbb{Z}/p$ ($p$ prime) and $V$ is any nontrivial irreducible $G$-module, then the unit sphere $S(V)$ is equivariantly framed, and may be viewed as either a $(V - 1)$-manifold or a $(v - 1)$-manifold, where $v = \dim V$. However, it is not equivariantly framed, in the above sense, as a $(v - 1)$-manifold.

**Lemma 2.2.** Let $V$ be such that $V^G = \{0\}$, and let $n$ be a nonnegative integer. Then there exists a nonnegative integer $N = N(n, V)$ such that, if $M$ is any framed $(nV + k)$-manifold with $n > N$, then the normal bundle $\gamma_G$ of $M^G$ in $M$ is a product $G$-bundle.

**Proof.** Embed $M$ equivariantly in the (large) $G$-module $U$ and choose a trivialization, $\mu_M \cong \mathcal{E}_M(W)$, of the normal bundle of $M$. Write

$$U = W \oplus V^n \oplus \mathbb{R}^n \cong U_0 \oplus V^r \oplus V^n \oplus \mathbb{R}^k,$$

where $U_0$ has no summands isomorphic with a summand of $V$. Then $\gamma_G \oplus \mathcal{E}(V^r)$ has fiber dimension $(n + r)V$, and is canonically a product $G$-bundle. The $G$-bundle $\gamma_G$ is classified by the space $BO_G(nV)$, where $O_G(jV)$ is the group of equivariant orthogonal isomorphisms of $jV = V^j$. The composite

$$M^G \to BO_G(nV) \to \text{colim} BO_G(jV)$$

of the natural inclusion with a classifying map is therefore null-homotopic. Since the second arrow is an $n$-equivalence for sufficiently large $m$ (depending only on $n$ and $V$), the result now follows. □

It follows from the lemma that the fixed-sets of framed $G$-manifolds admit stable framings, given sufficiently large “codimension” $n$. The above argument may easily be elaborated to show that, for each $H \subset G$, $M^H$ is equivariantly framed as an $NH/H$-manifold.

Denote by $\Omega_{fr}$ nonequivariant framed bordism (stable homotopy). If $H \subset G$, then let $J(H) \subset \mathbb{Z}$ be the ideal

$$J(H) = \text{Im} d_H: I(V) \to \mathbb{Z}.$$

We reformulate Theorem A, including a description of the integer $j$.

**Theorem A.** Let $G$ be a finite group, let $V$ be any orthogonal $G$-module with $V^G = \{0\}$, and let $k \geq 0$. Let $H \subset G$ be such that $V^H = \{0\}$. Then there exists a sequence $(s_n)$ with $s_n \to \infty$ as $n \to \infty$ such that, if $M$ is any framed $(nV + k)$-manifold with $H$-fixed set $Y^k$, one has $[Y] \in J(H)^s \Omega^k_H$.

**Corollary 1.** Let $G$ be a $p$-group, let $V$ be any orthogonal $G$-module with $V^G = \{0\}$, and let $k \geq 0$. Let $H \subset G$ be such that $V^H = \{0\}$. Then there exists a sequence $(s_n)$ with $s_n \to \infty$ as $n \to \infty$ such that, if $M$ is any framed $(nV + k)$-manifold with $H$-fixed set $Y^k$, one has $[Y] \in p^{s_n} \Omega^k_H$.

**Proof.** This is now an immediate consequence of the fact that, for a $p$-group, $J(H) \subset p\mathbb{Z}$. □
Corollary 2. If $G$ is any $p$-group, there does not exist any framed $V$-manifold possessing a single fixed point.

Proof. If $M$ were a framed $V$-manifold with a single fixed point, then the sequence $M_n = (X \times X \times \cdots \times X)$ (n times) is a sequence of framed $nV$-manifolds each possessing a single fixed point, contradicting Corollary 1. □

Remark 2.3. Corollary 2 fails if $G$ is not a $p$-group. For example, let $G = \mathbb{Z}/p \times \mathbb{Z}/q$, with $p$ and $q$ distinct primes. Choose integers $m$ and $n$ with $mp + nq = 1$, and let $V = \rho$, any one-dimensional semifree irreducible complex $\mathbb{Z}/pq$-module. The element $a = [1 - m\mathbb{Z}/p - n\mathbb{Z}/q] \in A(\mathbb{Z}/pq)$ lies in $I(V)$, since $\mathcal{F}(V) = \{1\}$ (where 1 is the trivial subgroup). By the proof of Proposition 1.1, there exists an integer $r(n)$ with $I(V)^r(n) \subset \text{Im} \beta_n$ for any $n \geq 0$. Choose any such $n$, and let $f_n: S^{nV+W} \to S^W$ be such that $\beta_n[f_n] = a^n$. One may $G$-homotope $f$ to a $G$-map transverse to $0 \in S^W$, so that $f_n^{-1}(0)$ is a framed $nV$-manifold, $M$. The fixed-set of $M$ corresponds to the class of $\beta_n: (S^W)^G \to (S^W)^G \in \pi^*_G \cong \mathbb{Z}$. By definition of $f_n$, however, one has

$$\deg f_n^G = d_G(a^n) = 1,$$

so that $M$ possesses only a single “essential” (noncancelling) fixed point in the sense of [K2]. One can thus attach copies of $S(V) \times I$ to $M$ to obtain a framed $G$-manifold of dimension $nV$ possessing a single fixed point.

3. Proof of Theorems A and B. We first prove Theorem A. If $M_n$ is a framed $G$-manifold of dimension $nV + k$, then the Pontryagin-Thom construction defines a $G$-map

$$f_n: S^{nV+k+W} \to S^W$$

for some $W$. Let $s(n)$ be the sequence obtained in Proposition 1.2. Then the composite

$$S^{k+W} \to S^{nV+k+W} \to S^W$$

of $f_n$ with inclusion defines a class $x \in I(V)^{s(n)} \omega_k^G$. Let $H$ be such that $V^H = \{0\}$. Then restriction of a $G$-map to the $H$-fixed subset defines a homomorphism $\omega_k^G \to \pi^*_H$ such that, if $a \in A(G)$, then $\varphi(ay) = d_H(a)\varphi(y)$, where $\varphi: A(G) \to A(H)$ is the forgetful homomorphism. This may be seen directly from the definition of the $A(G)$-action on $\omega_k^G$. Thus

$$\varphi(x) = \varphi(ay) = d_H(a)\varphi(y)$$

for some $a \in I(V)^n$, where $d_H(I(V)^n) \subset J(H)^n$. Since the $H$-fixed set of $M$ corresponds to the class $\varphi(x)$, the result now follows. □

Remarks 3.1. If $H \in \mathcal{F}(V)$, then all information on the $H$-fixed set is lost upon application of $\beta_n$, so no analogous result can be drawn.

Turning to the proof of Theorem B, and with $j$ a generator of the ideal $J(G)$, let $[Y] \in \pi^*_k$ be the stable homotopy class determined by the framed manifold $Y$. Then, under the natural map $\pi^*_k \to \omega^G_k$, $[Y]$ determines a stable homotopy class of $G$-maps $\varsigma: S^{U+k} \to S^U$ with $U < \mathcal{U}$. Following the proof of Proposition 1.2, one extends $\rho^n \varsigma$ (stably) to a $G$-map

$$\varsigma': S^{U+mV+k} \to S^U$$

for suitable $n$ and arbitrary $\rho \in I(V)$. Now $G$-homotope $\varsigma'$ to a $G$-map $\varsigma$ transverse to $0 \in S^U$, and let $M$ be the framed $G$-manifold $\varsigma^{-1}(0)$. Then $M$ has dimension...
mV + k, and its G-fixed set Z is the preimage of $(S^U)^G$ under the restriction $\zeta|((S^U + mV + k)^G = \zeta|(S^U + k)^G$, since $V^G = \{0\}$. Since $\zeta$ is stably $G$-homotopic to an extension of $p^n \zeta$, restricting to the $G$-fixed set gives a framed cobordism of $Z$ with $d_G(p)^n Y$. The theorem now follows by choosing $p \in d_G^{-1}(j)$. □

BIBLIOGRAPHY


