TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 298, Number 2, December 1986

ON THE ZEROS OF SUCCESSIVE DERIVATIVES OF EVEN LAGUERRE-POLYA FUNCTIONS

LI-CHIEN SHEN

ABSTRACT. Using "method of steepest descent", we prove that the final set (in the sense of Polya) of a class of even Laguerre-Polya functions is the entire real axis.

Introduction. We say that f is a Laguerre-Polya function if it has the form

$$f(z) = Kz^m e^{-az^2+bz} \prod \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

where K, b and the a_n are real, $a \geq 0$, m is a nonnegative integer and $\sum a_n^{-2} < \infty$.

One of the most important properties of a Laguerre-Polya function f is that all the derivatives of f have only real zeros. In fact, it was conjectured by Polya in 1914 that the converse is also true, that is, if an entire function f is real (i.e. $f(x)$ is real whenever x is real) and all the derivatives of f have only real zeros, then f is a Laguerre-Polya function. This conjecture was confirmed by S. Hellerstein and J. Williamson in two remarkable papers [**2** and **3**].

In an address before the American Mathematical Society in 1942, Polya asserted that if the order of a real entire function f is greater than 1, then differentiation tends to concentrate the zeros; the zeros of $f^{(n)}$ tend to move in from ∞ as n increases; their distribution becomes denser. If the order of f is less than 1, then differentiation tends to scatter the zeros; the zeros of $f^{(n)}$ tend to move out to ∞ as n increases; their distribution becomes thinner. However, we must keep in mind that the above phenomenon described by Polya is very general and qualitative, and the study of the distribution of zeros of the successive derivatives of an entire function remains very difficult. For more details concerning the zeros of the derivatives of entire functions, we refer the readers to the original text of Polya's speech [**5**].

Following Polya we say that the point z_0 belongs to the final set of f if

(i) f is analytic at z_0,

(ii) for every $\varepsilon > 0$, the disk $|z - z_0| < \varepsilon$ contains zeros of $f^{(n)}$ for infinitely many values n.

In view of the particularly simple distribution of the zeros of the derivatives of a Laguerre-Polya function (that is, they all lie on the real axis), we propose to study the final set of a Laguerre-Polya function. Using Polya's assertion as a guide, it seems reasonable to expect that if a Laguerre-Polya function has order > 1, then its final set will be the whole real axis. We are unable to prove this assertion.

Received by the editors November 16, 1984 and, in revised form, November 27, 1985.
1980 *Mathematics Subject Classification*. Primary 30A66; Secondary 30A70.
Key words and phrases. The final set, Laguerre-Polya function, method of steepest descent, Phragmen-Lindelöf indicator, proximate order.

However, we will establish

THEOREM 1. *Let f be a Laguerre-Polya function of order > 1. If f is even and its Phragmen-Lindelöf indicator function $h(\theta)$ has the following property*

$$h(\pi/2) > h(\pi/2 \pm \theta) \geq h(\pi/2 \pm \phi) \tag{1}$$

whenever $0 < \theta < \phi < \pi/2$, then the final set of f is the whole real axis.

We remark that if f is an even Laguerre-Polya function, then f has the form

$$f(z) = Kz^{2m}e^{-cz^2} \prod (1 - z^2/a_n^2) \tag{2}$$

where $c \geq 0$, K and the a_n are real, m is a nonnegative integer and $\sum 1/a_n^2 < \infty$. Clearly, we see that the inequality (1) is automatically fulfilled if c is > 0. Also, "the order of $f > 1$" is trivially satisfied if $c > 0$, whereas it is equivalent to $\sum |a_n|^{-1-\delta}$ for some $\delta > 0$ if $c = 0$.

We remind the readers that the Phragmen-Lindelöf indicator function of an entire function f is defined as

$$h(\theta) = \limsup_{r\to\infty} \frac{\ln|f(re^{i\theta})|}{r^{\rho(r)}},$$

where $\rho(r)$ is a proximate order of f [**4**, p. 31]. We assume that the readers are familiar with the basic properties of proximate order of an entire function.

We will be using "method of steepest descent" on Cauchy's formula for the derivatives

$$f^{(n)} = \frac{n!}{2\pi i} \int_{|s|=r} \frac{f(s)}{(s-z)^{n+1}}\, ds$$

to obtain an asymptotic formula for $f^{(n)}(z)$ for $z \in D$ as n and r tend to ∞ along two properly chosen sequences $\{n_j\}$ and $\{r_j\}$, where D is a compact region in the complex plane.

Our method requires

(i) the adequate selection of a monotonically increasing sequence $\{n_j\}$ tending to ∞,

(ii) the approximation of $\ln f(z)$ near the point $z = ir$ by using the following

LEMMA A [**1**, p. 78]. *Let f be analytic and $f(z) \neq 0$ in the disk*

$$D = \{z\colon |z - w| < 2\eta w,\ 0 < \eta \leq \tfrac{1}{2}\}.$$

Define

$$a(z) = zf'(z)/f(z), \qquad b(z) = za'(z). \tag{3}$$

If there exists a positive constant C such that $|b(z)| \leq C|b(w)|$ for all $z \in D$, then

$$\ln f(we^{i\theta}) = \ln f(w) + i\theta a(w) - \theta^2 b(w)/2 + E(w,\theta),$$

where

$$|E(w,\theta)| \leq C|b(w)\theta^3|/2\eta$$

for $|\theta| \leq \eta$.

1. Preliminaries. Let f be an even Laguerre-Polya function. From (2) and (3),

$$a(z) = 2m - 2cz^2 + 2z^2 \sum \frac{1}{z^2 - a_n^2}, \tag{1.1}$$

hence

$$0 \le a(ir) \uparrow \infty, \qquad a(ir)/r^2 \to 2c. \tag{1.2}$$

$$b(z) = -4cz^2 - 4z^2 \sum \frac{a_n^2}{(z^2 - a_n^2)^2}, \tag{1.3}$$

hence

$$b(ir) \ge 0, \qquad b(ir)/r^2 \to 4c. \tag{1.4}$$

From (2),

$$|f(re^{i\theta})| = |K| e^{-cr^2 \cos 2\theta} \prod \left| 1 - \frac{2r^2 \cos 2\theta}{a_n^2} + \frac{r^4}{a_n^4} \right|^{1/2}.$$

It follows that, if $0 < \theta < \phi < \pi/2$,

$$|f(re^{i(\pi/2 \pm \phi)})| < |f(re^{i(\pi/2 \pm \theta)})| < |f(re^{i\pi/2})|;$$

and since f is real, $|f(z)| = |f(\bar{z})|$, we conclude that

$$f(ir) = M(r, f) \quad \text{and} \quad h(\pi/2 \pm \phi) \le h(\pi/2 \pm \theta) \quad \text{if } \theta \le \phi, \tag{1.5}$$

where $M(r, f) = \mathrm{Max}_{|z|=r} |f(z)|$.

From (1.2),

$$\begin{aligned} \ln\left(\frac{f(2ir)}{f(i)}\right) &= \int_1^{2r} \frac{if'(it)}{f(it)}\, dt = \int_1^{2r} \frac{a(it)}{t}\, dt \ge \int_r^{2r} \frac{a(it)}{t}\, dt \\ &\ge \frac{a(ir)}{2} \qquad (r > 1) \end{aligned} \tag{1.6}$$

and

$$\ln\left(\frac{f(ir)}{f(ir/2)}\right) = \int_{r/2}^{r} \frac{a(it)}{t}\, dt \le a(ir) \ln 2. \tag{1.7}$$

We now turn to the properties of $b(z)$. In order to be able to apply Lemma A, we need the following property of $b(z)$.

LEMMA 1.1. *Let f be an even Laguerre-Polya function. If $|z - ir| \le r/2$, then $|b(z)| \le 72b(ir)$.*

PROOF. Let $S_0 = \{z\colon \pi/3 \le |\arg z| \le 2\pi/3\}$. If $z \in S_0$, then

$$\begin{aligned} |z^2 - a_n^2|^2 &= (|z|^2 + a_n^2)^2 - 2|a_n z|^2 (1 + \cos 2\theta) \\ &\ge \tfrac{1}{2}(|z|^2 + a_n^2)^2 \qquad (\theta = \arg z). \end{aligned}$$

Hence, together with (1.3), one sees that if $z \in S_0$, then

$$|b(z)| \le 8|z|^2 \left(c + \sum a_n^2 (|z|^2 + a_n^2)^{-2}\right). \tag{1.8}$$

For any $r > 0$, let z be a point such that $|z - ir| \le r/2$. Then $z \in S_0$ and $(r^2 + a_n^2)/(|z|^2 + a_n^2) \le 4$. Therefore, from (1.8), one has

$$\begin{aligned} |b(z)| &\le \frac{8|z|^2}{r^2} \left\{ cr^2 + \sum \frac{a_n^2 r^2 (r^2 + a_n^2)^2}{(r^2 + a_n^2)^2 (|z|^2 + a_n^2)^2} \right\} \\ &\le 18 \left(cr^2 + 16 \sum \frac{a_n^2 r^2}{(r^2 + a_n^2)^2} \right) \le 72b(ir) \end{aligned}$$

for $|z - ir| < r/2$. This proves Lemma 1.1.

Using Lemmas A and 1.1, we can now approximate $f(z)$ near the point $z = ir$ for all $r > 0$.

LEMMA 1.2. *Let f be an even Laguerre-Polya function. Then, for any $r > 0$,*

$$\ln f(ire^{i\theta}) = \ln f(ir) + i\theta a(ir) - \theta^2 b(ir)/2 + E(r,\theta),$$

where $|E(r,\theta)| < 144b(ir)|\theta|^3$ for $|\theta| \le \frac{1}{4}$.

We shall see in the following section that in order that Lemma 1.2 can be used to derive an asymptotic formula of $f^{(n_j)}$ for some subsequence $\{n_j\}$ tending to ∞, it is essential that

$$\limsup_{r\to\infty} b(ir) = \infty. \tag{1.9}$$

The following lemma gives a more accurate description of the growth rate of $b(ir)$ than that of (1.9).

LEMMA 1.3. *Let f be an even Laguerre-Polya function of order ρ. If its indicator function h has the property* (1), *then there exists a monotonically increasing sequence $\{r_j\}$ tending to ∞ such that*

$$0 < \liminf_{j\to\infty} \frac{b(ir_j)}{r_j^{\rho(r_j)}} \le \limsup_{r\to\infty} \frac{b(ir)}{r^{\rho(r)}} < \infty, \tag{i}$$

$$\limsup_{r\to\infty} \frac{a(ir)}{r^{\rho(r)}} \le 2^{\rho+1} h\left(\frac{\pi}{2}\right), \tag{ii}$$

$$\lim_{j\to\infty} \frac{a(ir_j)}{r_j} = \infty \quad \text{if } \rho > 1. \tag{iii}$$

We note that if $c > 0$, then $\rho(r) \equiv 2$ and, by (1.4), we see that this lemma is trivial.

PROOF. From Lemma 1.2, one sees that for $|z| \le \frac{1}{4}$,

$$\frac{\theta^2 b(ir)}{2}(1 + \tilde{E}(r,\theta)) = \ln f(ir) - \ln f(re^{i(\theta+\pi/2)}) + ia(ir)\theta, \tag{1.10}$$

where

$$|\tilde{E}(r,\theta)| \le k|\theta| \quad \text{for some finite constant } k > 0. \tag{1.11}$$

From (1.11), we can select a $\theta_0 > 0$ such that $\mathrm{Re}(1 + \tilde{E}(r,\theta)) \le 2$ if $0 \le \theta \le \theta_0$. Taking the real part of (1.10), we have

$$\theta^2 b(ir) \ge \ln f(ir) - \ln|f(ire^{i\theta})| \tag{1.12}$$

for $0 \le \theta \le \theta_0$.

We now select a sequence $r_j \to \infty$ such that

$$\frac{\ln f(ir_j)}{r_j^{\rho(r_j)}} \to h\left(\frac{\pi}{2}\right).$$

From (1.12), we clearly have

$$\theta^2 \liminf_{j\to\infty} \frac{b(ir_j)}{r_j^{\rho(r_j)}} \ge h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{2} + \theta\right) > 0 \tag{1.13}$$

if $0 < \theta \le \theta_0$. On the other hand, from (1.10), we always have

$$\limsup_{r\to\infty} \frac{b(ir)}{r^{\rho(r)}} < \infty. \tag{1.14}$$

Part (i) of this lemma now follows from (1.13) and (1.14).

From (1.6), we have

$$\frac{\ln f(2ir)}{r^{\rho(r)}} - \frac{\ln f(i)}{r^{\rho(r)}} \geq \frac{1}{2}\frac{a(ir)}{r^{\rho(r)}}. \tag{1.15}$$

Since $r^{\rho(r)-\rho}$ is slowly increasing [**4**, p. 32], we have

$$\lim_{r\to\infty} \frac{(2r)^{\rho(2r)}}{r^{\rho(r)}} = 2^{\rho}. \tag{1.16}$$

Combining (1.15) and (1.16), we obtain

$$\limsup_{r\to\infty} \frac{a(ir)}{r^{\rho(r)}} \leq 2^{\rho+1} h\left(\frac{\pi}{2}\right).$$

This establishes (ii).

From (1.7), one has

$$\frac{a(ir)}{r^{\rho(r)}} \geq \frac{1}{\ln 2}\left[\frac{\ln f(ir)}{r^{\rho(r)}} - \frac{\ln f(ir/2)}{r^{\rho(r)}}\right]. \tag{1.17}$$

Since

$$\limsup_{r\to\infty} \ln f\left(\frac{ir}{2}\right) \Big/ r^{\rho(r)} \leq \left(\frac{1}{2}\right)^{\rho} h\left(\frac{\pi}{2}\right),$$

(1.17) implies that

$$\liminf_{j\to\infty} \frac{a(ir_j)}{r_j^{\rho(r_j)}} > 0. \tag{1.18}$$

Since $\rho(r) \to \rho$ as $r \to \infty$ [**4**, p. 32], (iii) follows from (1.18).

REMARK 1. If the order of f in Lemma 1.3 is < 1, then (1.15) clearly shows that $\lim_{r\to\infty} a(ir)/r = 0$.

Let $\{r_j\}$ be the sequence in Lemma 1.3. For each j, we define n_j to be the largest integer $\leq a(ir_j)$. Since $a(ir)$ is monotonically increasing and tending to ∞ as $r \to \infty$, $n_j \to \infty$ as $j \to \infty$.

Let $\delta_j = (b(ir_j))^{-2/5}$. Then, from (i) of Lemma 1.3, we see that $\delta_j \to 0$ as $j \to \infty$.

In the next two lemmas, we will estimate the two quantities which appear in the integrand of Cauchy's integral formula for the derivatives.

LEMMA 1.4. *Let D be an arbitrary compact region in the complex plane. If we let $n = n_j$ and $r = r_j$, then as $j \to \infty$*

(i)

$$(1 + zie^{-i\theta}/r)^{-n-1} = (1 + o(1))e^{-cz^2-(n+1)(i+\theta)z/r}$$

uniformly for $|\theta| \leq \delta_j$ and uniformly for z in D.

(ii)

$$\left(1 + \frac{zie^{-i\theta}}{r}\right)^{-n-1} = (1 + o(1)) \exp\left\{-cz^2(1 + S(2\theta)) - \frac{(n+1)(1 + S(\theta))zi}{r}\right\}$$

uniformly for $\delta_j \le |\theta| \le \pi/2$ *and* $z \in D$, *where* $S(\theta) = e^{-i\theta} - 1$. *We also note that* $|S(\theta)/\theta| = |2\theta^{-1}\sin(\theta/2)| \le 1$ *for all* θ *real.*

PROOF. By definition, $n_j \le a(ir_j) \le n_j+1$, therefore from (1.2) $(n_j+1)/r_j^2 \to 2c$ as $j \to \infty$. This implies that, if $r = r_j$ and $n = n_j$, then

(1.19)
$$\left(1 + \frac{zie^{-i\theta}}{r}\right)^{-n-1} = \exp\left\{\frac{-(n+1)zie^{-i\theta}}{r} + \frac{(n+1)(zi)^2e^{-2i\theta}}{2r^2} + \cdots\right\}$$
$$= (1+o(1))\exp\left\{-cz^2e^{-2i\theta} - \frac{(n+1)zie^{-i\theta}}{r}\right\} \qquad (j \to \infty).$$

If we further assume that $|\theta| \le \delta_j$, then from (i) and (ii) of Lemma 1.3, we have

$$\frac{(n_j+1)\theta^2}{r_j} \le \frac{(n_j+1)\delta_j^2}{r_j} = \frac{(n_j+1)(b(ir_j))^{-4/5}}{r_j} \le kr_j^{-1} + \frac{1}{5}\rho(r_j)$$

for some constant $k > 0$. Since $\lim_{r\to\infty}\rho(r) \le 2$, we conclude that $(n_j+1)\theta^2/r_j = o(1)$. Therefore, for $|\theta| \le \delta_j$, we have

$$e^{-2i\theta} = 1 + o(1), \tag{1.20}$$

$$\frac{-(n_j+1)e^{-i\theta}}{r_j} = \frac{-(n_j+1)(1-i\theta)}{r_j} + o(1) \tag{1.21}$$

as $j \to \infty$.

From (1.19), (1.20) and (1.21), we deduce (i).

Let $e^{-i\theta} = 1 + S(\theta)$. Then, by replacing $e^{-2i\theta}$ and $e^{-i\theta}$ in (1.19) by $1 + S(2\theta)$ and $1 + S(\theta)$, respectively, we have (ii).

LEMMA 1.5. *Let* f *satisfy the assumptions of Lemma* 1.3. *Then, there exist constants* θ_0 *and* $k > 0$ *such that, for* $r = r_j$, *we have, as* $j \to \infty$,

(i)
$$f(ire^{i\theta}) = (1+o(1))f(ir)\exp\left\{ia(ir)\theta - \frac{b(ir)\theta^2}{2}\right\}$$

uniformly for $|\theta| \le \delta_j$,

(ii)
$$|f(re^{i(\theta+\pi/2)})| \le |f(ir)|\exp\left(-\frac{b(ir)\theta^2}{3}\right)$$

uniformly for $\delta_j \le |\theta| \le \theta_0$,

(iii)
$$|f(re^{i(\theta+\pi/2)})/f(ir)| \le \exp(-kr^{\rho(r)})$$

uniformly for $\theta_0 \le |\theta| \le \pi/2$.

PROOF. From Lemma 1.2, we see that if $|\theta| \le \delta_j = (b(ir_j))^{-2/5}$, then

$$|E(r_j,\theta)| \le 144b(ir_j)(b(ir_j))^{-6/5} = O((b(ir_j))^{-1/5}) = o(1)$$

as $j \to \infty$. The conclusion (i) is established.

Again, from Lemma 1.2, we see that, for $|\theta| \le \frac{1}{4}$,

$$\ln|f(re^{i(\theta+\pi/2)})| = \ln|f(ir)| - \theta^2 b(ir)(\tfrac{1}{2} + E_1(r,\theta)),$$

where $|E_1(r,\theta)| \leq K|\theta|$ for some constant K. We can select a θ_0 in the interval $(0, \frac{1}{4})$ such that $\frac{1}{2} + E_1(r,\theta) > \frac{1}{3}$ whenever $0 \leq |\theta| \leq \theta_0$. Hence, for $0 \leq |\theta| \leq \theta_0$, we have

$$\ln|f(re^{i(\theta+\pi/2)})| < \ln|f(ir)| - \theta^2 b(ir)/3$$

and this implies (ii).

Given any $\varepsilon > 0$, it is well known that [**4**, p. 71]

$$\ln|f(re^{i\theta})| \leq (h(\theta) + \varepsilon/2)r^{\rho(r)} \tag{1.22}$$

for all $r > r(\varepsilon)$ uniformly for all $0 \leq \theta \leq \pi$.

Since the sequence $\{r_j\}$ is chosen so that

$$\lim_{j\to\infty} \frac{\ln|f(r_j e^{i\pi/2})|}{r_j^{\rho(r_j)}} = h\left(\frac{\pi}{2}\right),$$

hence, from (1.22), one sees that

$$\ln|f(r_j e^{i\theta})| - \ln|f(ir_j)| \leq (h(\theta) - h(\pi/2) + \varepsilon)r_j^{\rho(r_j)} \tag{1.23}$$

for all $0 \leq \theta \leq \pi$ if j is large enough, say $j > j_0$.

Since h has the property (1), we can select an $\varepsilon > 0$ so that

$$h(\pi/2 + \theta_0) - h(\pi/2) + \varepsilon = -k < 0.$$

But, from (1.4), it follows that

$$h(\pi/2 + \theta) - h(\pi/2) + \varepsilon \leq -k \tag{1.24}$$

for all $\theta_0 \leq |\theta| \leq \pi/2$. From (1.23) and (1.24) we prove (iii).

2. Proof of Theorem 1. Using Lemma 1.4 and Lemma 1.5, we will obtain an asymptotic formula of $f^{(n_j)}$. (The explicit form of $f^{(n_j)}$ is described in (2.14).)

Let D be a compact region which is to be fixed throughout this section. From Cauchy's formula for the derivatives, we have, for all $R > R_0$ and $z \in D$,

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi i}\int_{|s|=R} \frac{f(s)}{(s-z)^{n+1}}\,ds \\ &= \frac{n!}{2\pi i}\int_{|s|=R} \frac{f(s)}{s^{n+1}}\left(\frac{s-z}{s}\right)^{-n-1} ds \\ &= \frac{n!}{2\pi R^n}\int_0^{2\pi} f(Re^{it})e^{-int}\left(1 - \frac{z}{Re^{it}}\right)^{-n-1} dt. \end{aligned} \tag{2.1}$$

We now let

$$F_j(t,z) = f(r_j e^{it})e^{-in_j t}\left(1 - \frac{z}{r_j e^{it}}\right)^{-n_j-1},$$

where n_j and r_j, $j = 1, 2, \ldots$, are defined as in the previous section.

We now let

$$I_1 = \int_0^{\pi} F_j(t,z)\,dt \quad \text{and} \quad I_2 = \int_{\pi}^{2\pi} F_j(t,z)\,dt.$$

Then

$$I_1 + I_2 = \int_0^{2\pi} F_j(t,z)\,dt = \frac{2\pi r^n f^{(n)}(z)}{n!} \qquad (n = n_j, r = r_j).$$

(One should keep in mind that I_1 and I_2 are both functions of z and j.)

To compute I_1 we decompose the interval $(0,\pi)$ into five subintervals:

$$\left(0,\frac{\pi}{2}-\theta_0\right),\quad \left(\frac{\pi}{2}-\theta_0,\frac{\pi}{2}-\delta_j\right),\quad \left(\frac{\pi}{2}-\delta_j,\frac{\pi}{2}+\delta_j\right),$$
$$\left(\frac{\pi}{2}+\delta_j,\frac{\pi}{2}+\theta_0\right),\quad \left(\frac{\pi}{2}+\theta_0,\pi\right).$$

Let J_1, J_2, J_3, J_4 and J_5 denote, respectively, the integrals of F_j over those five subintervals. We now estimate J_i, $i=1,2,3,4,5$, separately.

It should be mentioned here that J_3 is the most important term and all the other J_i, $i\neq 3$, are insignificant when compared with J_3.

From (ii) of Lemma 1.4 and (iii) of Lemma 1.5, we have, for all j large,

$$|J_1| < k_1 f(ir_j)\exp\{-kr_j^{\rho(r_j)} + (n_j+1)|z|/r_j\},$$

where k_1 is some constant depending on D.

Since f is even, the estimate for J_5 is the same as that of J_1.

We now estimate J_2. We first make a change of variable: $t=\pi/2-\theta$. Then from (ii) of Lemmas 1.4 and 1.5, we have

(2.3)

$$\begin{aligned}|J_2| &\le \int_{\delta_j}^{\theta_0} |f(r_j e^{i(-\theta+\pi/2)})|\left|\left(1+\frac{ize^{i\theta}}{r_j}\right)^{-n_j-1}\right| d\theta\\ &\le k_2\left|\exp\left(\frac{-iz(n_j+1)}{r_j}\right)\right| f(ir_j)\int_{\delta_j}^{\theta_0}\exp\left[\frac{-\theta^2 b(ir_j)}{3}+\frac{(n_j+1)|z|\theta}{r_j}\right] d\theta\\ &= k_2 f(ir_j)\left|\exp\left(\frac{-iz(n_j+1)}{r_j}\right)\right|\int_{\delta_j}^{\theta_0}\exp\left[\frac{-\theta^2 b(ir_j)}{3}\left(1-\frac{3(n_j+1)|z|}{\theta r_j b(ir_j)}\right)\right] d\theta,\end{aligned}$$

where k_2 is a constant depending on D.

We observe that, from (i) and (ii) of Lemma 1.3 and $\lim_{r\to\infty}\rho(r)=\rho\le 2$,

$$\frac{n_j+1}{b(ir_j)}\le\frac{a(ir_j)+1}{b(ir_j)}\le K_1<\infty\quad\text{for all } j \text{ large}$$

and

$$r_j\delta_j = r_j b(ir_j)^{-2/5}\ge K_2 r_j^{1-(2/5)\rho(r_j)}\to\infty\quad\text{as } j\to\infty,$$

where K_1 and K_2 are some finite positive constants independent of j. Hence, for $\delta_j\le\theta\le\theta_0$,

$$\frac{(n_j+1)|z|}{r_j\theta b(ir_j)}\le\frac{(n_j+1)|z|}{r_j\delta_j b(ir_j)}\to 0\quad\text{as } j\to\infty. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}|J_2| &\le k_2 f(ir_j)\left|\exp\left(\frac{-(n_j+1)iz}{r_j}\right)\right|\int_{\delta_j}^{\theta_0}\exp\left(\frac{-\theta^2 b(ir_j)}{4}\right) d\theta\\ &= k_2 f(ir_j)(b(ir_j))^{-1/2}\left|\exp\left(\frac{-i(n_j+1)z}{r_j}\right)\right|\int_{b(ir_j)^{1/10}}^{\theta_0\sqrt{b(ir_j)}} e^{-t^2/4}\,dt\\ &= \left|\exp\left(\frac{-i(n_j+1)z}{r_j}\right)\right|(b(ir_j))^{-1/2}o(f(ir_j)).\end{aligned}\tag{2.5}$$

We have similar estimate for J_4.

We now compute J_3. We first make a change of variable: $t = \pi/2 + \theta$. Using (i) of Lemmas 1.4 and 1.5, we have

$$J_3 = (-i)^{n_j} \exp(-cz^2 - (n_j+1)zi/r_j) \cdot f(ir_j) \cdot J_6, \tag{2.6}$$

where

$$J_6 = \int_{-\delta_j}^{\delta_j} (1+o(1)) \exp\left\{ i(a(ir_j) - n_j)^\theta - \frac{(n_j+1)z\theta}{r_j} - \frac{\theta^2 b(ir_j)}{2} \right\} d\theta.$$

After making a further change of variable $\theta = u/\sqrt{b(ir_j)}$, the above integral J_6 becomes

(2.7)

$$(b(ir_j))^{-1/2} \int_{-\omega_j}^{\omega_j} (1+o(1)) \exp\left(-\frac{u^2}{2} - \frac{(n_j+1)zu}{r_j\sqrt{b(ir_j)}} \right) \exp\left\{ -iu \frac{(n_j - a(ir_j))}{\sqrt{b(ir_j)}} \right\} du,$$

where $\omega_j = (b(ir_j))^{1/10}$.

Since $0 \le a(ir_j) - n_j \le 1$,

$$\frac{n_j - a(ir_j)}{\sqrt{b(ir_j)}} u \to 0 \quad \text{as } j \to \infty \tag{2.8}$$

for $-\omega_j \le u \le \omega_j$.

From (1.2) and (1.4)

$$\frac{a(ir_j)}{r_j^2} \to 2c \quad \text{and} \quad \frac{b(ir_j)}{r_j^2} \to 4c \quad \text{as } j \to \infty.$$

Therefore, as $j \to \infty$

$$(n_j+1)/r_j\sqrt{b(r_j)} \to \sqrt{c}. \tag{2.9}$$

From (2.7), (2.8) and (2.9), one sees that the integral

$$J_6 \sim \frac{e^{cz^2/2}}{\sqrt{b(ir_j)}} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{\frac{\pi}{2b(ir_j)}} e^{cz^2/2}. \tag{2.10}$$

Thus, from (2.10) and (2.6),

$$J_3 \sim (-i)^{n_j} \sqrt{\frac{\pi}{2b(ir_j)}} f(ir_j) e^{-cz^2/2} \exp\left(\frac{-(n_j+1)zi}{r_j} \right) \quad \text{as } j \to \infty. \tag{2.11}$$

From (2.11), (2.5) and (2.2), we obtain

$$I_1 = (1+o(1)) \left(\frac{\pi}{2}\right)^{1/2} (-i)^{n_j} \frac{f(ir_j)}{\sqrt{b(ir_j)}} e^{-cz^2/2} \exp\left(\frac{-(n_j+1)zi}{r_j} \right). \tag{2.12}$$

Using the same method, we also obtain

$$I_2 = (1+o(1))(i)^{n_j} \left(\frac{\pi}{2}\right)^{1/2} \frac{f(-ir_j)}{\sqrt{b(-ir_j)}} e^{-cz^2/2} \exp\left(\frac{(n_j+1)zi}{r_j} \right). \tag{2.13}$$

Since f is even, $f(-ir_j) = f(ir_j)$ and $b(-ir_j) = b(ir_j)$. We can conclude from (2.12) and (2.13) that

(2.14)
$$f^{(n)}(z) = \frac{n!f(ir)}{r^n\sqrt{2\pi b(ir)}}e^{-cz^2/2}\left\{\cos\left(\frac{n\pi}{2} + \frac{(n+1)z}{r}\right) + o(1)\right\} \quad (n = n_j, r = r_j).$$

We now prove Theorem 1. We are mainly interested in counting the number of zeros of $f^{(n)}$ in an arbitrarily selected region in the complex plane. Since we have noted earlier that all the zeros of $f^{(n)}$ are real, therefore we choose D to be an interval on the real axis, say $D = [a, b]$.

Let N_n be the number of zeros of $f^{(n)}$ in the interval $[a, b]$. From (iii) of Lemma 1.3, we see that if the order of f is > 1, then $\lim_{j\to\infty}(n_j + 1)/r_j = \infty$. This fact combined with (2.14) gives

$$N_{n_j} \sim \frac{b-a}{\pi}\frac{n_j+1}{r_j} \to \infty.$$

This completes the proof.

REMARK 2. From (2.14), we also see that if the order of f is < 1, then $\lim(n_j + 1)/r_j = 0$ and the zeros of $f^{(n_j)}$ tend to move out to ∞ and the set of all the limit points of the zeros of $f^{(n_j)}$, $j = 1, 2, \ldots$, is $\{0\}$ if infinitely many n_j are odd, and it is empty if only finitely many of n_j are odd.

3. Acknowledgment. I would like to thank the referee for improving the original manuscript and making this paper easier to read.

REFERENCES

1. W. K. Hayman, *A generalization of Stirling's formula*, J. Reine Angew. Math. **196** (1956), 67–95.
2. S. Hellerstein and J. Williamson, *Derivatives of entire functions and a question of Polya*, Trans. Amer. Math. Soc. **227** (1977), 227–249.
3. ——, *Derivatives of entire functions and a question of Polya.* II, Trans. Amer. Math. Soc. **234** (1977), 497–503.
4. B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monos., vol. 5, Amer. Math. Soc., Providence, R. I., 1963.
5. G. Polya, *On the zeros of the derivatives of a function and its analytic character*, Bull. Amer. Math. Soc. **49** (1943), 178–191.

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13210

Current address: Department of Mathematics, University of Florida, Gainesville, Florida 32611