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CONSTRUCTION OF HIGH DIMENSIONAL KNOT GROUPS FROM CLASSICAL KNOT GROUPS

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ABSTRACT. In this paper we study constructions of high dimensional knot groups from classical knot groups. We study certain homomorphic images of classical knot groups. Specifically, let K be a classical knot group and w any element in K. We are interested in the quotient groups G obtained by centralizing w, i.e. $G = K/[K, w]$, and ask whether G is itself a knot group.

For certain K and w we show that G can be realized as the group of a knotted 3-sphere in 5-space, but G is not realizable by a 2-sphere in 4-space. By varying w, we also obtain quotients that are groups of knotted 2-spheres in 4-space, but they cannot be realized as the groups of classical knots.

We have examples of quotients $K/[K, w]$ that have nontrivial second homology. Hence these groups cannot be realized as knot groups of spheres in any dimension. However, we show that these groups are groups of knotted tori in S^4.

Introduction. In this paper we study certain constructions of high dimensional knot groups from classical knot groups. As described by R. Litherland [**L**] the groups of twist spun knots and 2-knots obtained by the method of rolling are obtained from classical knot groups by centralizing peripheral elements. In an attempt to generalize the process of centralizing peripheral elements to produce 2-knot groups, we will study the general process of centralizing any element in a classical knot group. Thus if K is a classical knot group and w is any element in K, we study the quotient group $G = K/[K, w]$ and ask whether G is itself a knot group. We will show that the only obstruction to G being a high dimensional knot group is if $H_2(G) \neq 0$. We have examples of a group K and elements w such that $H_2(K/[K, w]) \neq 0$.

For certain K and w we show that G can be realized as the group of a knotted 3-sphere in 5-space, but G is not realizable by a 2-sphere in 4-space. By varying w, we also obtain quotients $K/[K, w]$ that are groups of knotted 2-spheres in 4-space. However these quotients cannot be realized as the groups of classical knots.

1. Preliminaries. In this paper we work in the smooth category. S^n denotes the n-sphere. An n-dimensional knot is a smooth embedding of S^n into S^{n+2} and we denote it by (S^{n+2}, S^n). We refer to (S^{n+2}, S^n), $n \geq 3$, as a high dimensional knot, (S^4, S^2) as a 2-knot and (S^3, S^1) as a classical knot.

If K is a group and a, b are elements in K then the commutator $[a, b] = a^{-1}b^{-1}ab$. If w is any element in K then $[K, w]$ is the smallest normal subgroup of K containing $\{[k, w]; k \in K\}$. Similarly if A and B are subgroups of K then $[A, B]$ denotes the

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smallest normal subgroup of K containing $\{[a,b];\ a \in A,\ b \in B\}$. The commutator subgroup $[K,K]$ is denoted by K'. If k is an element of a group K, then the equivalence class of k in K/K' is denoted by $[k]$.

All homology groups are taken with integer coefficients. We shall use the following three equivalent definitions for the second homology of a group G.

DEFINITION 1.1. If X is a connected CW-complex with $\pi_1(X) = G$ and $\pi_n(X) = 0$, $n \geq 2$, then for each p, $H_p(G)$ is defined to be $H_p(X)$.

DEFINITION 1.2. If Y is a connected CW-complex with $\pi_1(Y) \cong G$, and $\Sigma(Y)$ denotes the subgroup of $H_2(Y)$ generated by all singular 2-cycles representable by maps of a 2-sphere into Y, then $H_2(G) \cong H_2(Y)/\Sigma(Y)$.

DEFINITION 1.3. If F is a free group, $\theta\colon F \to G$ an epimorphism and $R = \ker\theta$, then $H_2(G) \cong R \cap [F,F]/[F,R]$.

Note that Definition 1.1 is independent of the choice of X. The equivalence of 1.2 and 1.3 is shown in [**H**].

We define the deficiency of a group presentation with n generators and m relators to be the integer $n - m$.

2. The second homology of $K/[K,w]$. Let K be a classical knot group and let w be any element in K. We construct a group G by centralizing the element w, i.e., G is of the form $G = K/[K,w]$. In [**K**], M. Kervaire gives necessary and sufficient conditions for a group to be the group of a (S^{n+2}, S^n) knot if $n \geq 3$. Since K is a knot group and hence satisfies the following:

(i) K is finitely presented.

(ii) $H_1(K) \cong \mathbf{Z}$.

(iii) There exists $t \in K$, such that $K/\langle\langle t\rangle\rangle = 1$.

(iv) $H_2(K) = 0$.

It is straightforward to check that G satisfies conditions (i)–(iii). Hence the only obstruction to G being a high dimensional knot group is if $H_2(G) \neq 0$.

THEOREM 2.1. *Let K be a knot group with homology generator t, let w be any word in K and let $G = K/[K,w]$. Then $H_2(G)$ is isomorphic to the cyclic subgroup generated by $[t,w]$ in the group $C = K/[K,[K,w]]$.*

PROOF. Let φ be the quotient map of K onto G. Let $N = \ker\varphi$; then $N = [K,w]$ in K. Since N is normal in K and $G = K/N$, by J. Stallings [$\mathbf{S_2}$] there is an exact sequence $H_2(K) \to H_2(G) \to N/[K,N] \to H_1(K) \to H_1(G) \to 0$. Since K is a knot group, $H_2(K) = 0$. Moreover, $H_1(K) \to H_1(G)$ is an isomorphism since both groups are isomorphic to $\mathbf{Z}$ and the map is onto. Hence $H_2(G) \cong N/[K,N]$. The subgroup $[K,N]$ is the kernel of the quotient map ψ from K onto C. Hence $H_2(G)$ is isomorphic to the image of N under this map. If K has a presentation of the form $\langle t, g_1, \ldots, g_n; R_1 \cdots R_m\rangle$ where t is a homology generator and $g_i \in K'$, $1 \leq i \leq n$ (i.e., a preabelian presentation for K), then $\psi(N)$ is the subgroup of C generated by $[t,w]$ and $[g_i,w]$, $1 \leq i \leq n$. (Note that these elements generate a normal subgroup of C.) To complete the proof of the theorem we need the following lemma.

LEMMA 2.1.1. *For all $g \in K'$; $[g,w] = 1$ in C.*

PROOF OF LEMMA 2.1.1. First observe that $[k,w]^{-1} = [w,k]$. Moreover, for every element k in K, $[[k,w],k^{-1}] = 1$ in C. Thus in C we have the following:

$$1 = [[k,w],k^{-1}] = [w,k]k[k,w]k^{-1} = [w,k]kk^{-1}w^{-1}kwk^{-1}.$$

Hence $[w,k] = [k^{-1},w]$ in C. Let g be any commutator, i.e., $g = [a,b]$. Then using the commutator identities on p. 290 in [**MKS**] and the fact that $[x,w]$ is central for all x in K,

$$[g,w] = [a^{-1}b^{-1}ab,w] = [a^{-1},w][a,w][b^{-1},w][b,w] = 1.$$

If $g = h[a,b]h^{-1}$, then

$$[g,w] = [h[a,b]h^{-1},w] = [h,w][[a,b],w][h^{-1},w] = [h,w][h^{-1},w] = 1.$$

As every element in K' is the product of conjugates of commutators, we have $[g,w] = 1$ for all $g \in K'$. □

We now complete the proof of the theorem as follows. By Lemma 2.2.1 for each i, $[g_i,w] = 1$ in C. Thus the subgroup $\psi(N)$ of C is the cyclic group generated by $[t,w]$. □

By using Stallings sequence as in the proof of Theorem 2.1, we obtain the following useful proposition.

PROPOSITION 2.1.2. *If $H_2(K) = 0$ and $H_1(K) = Z$ and $w \in K'$, then $H_2(K/\langle\langle w\rangle\rangle)$ is isomorphic to the cyclic subgroup generated by w in $K/[K,w]$.*

DEFINITION. A group G has exponent m if m is the least positive integer such that $g^m = 1$ for all $g \in G$.

THEOREM 2.2. *Let K be a knot group with homology generator t. Let w be any word in K with $[w] = t^{\pm m}$ in K/K', $m > 0$, and let $G = K/[K,w]$. Then $H_2(G)$ is a cyclic group of exponent less than or equal to m.*

PROOF. By Theorem 2.1 we know that $H_2(G)$ is isomorphic to the cyclic subgroup generated by $[t,w]$ in $C = K/[K,[K,w]]$. Any word in K that abelianizes to $t^{\pm m}$ can be written in the form $w = gt^{\pm m}$ where $g \in K'$. In the group C, $[t^m,w] = [t,w]^m$ and since $g \in K'$, $[g,w] = 1$. Thus

$$1 = [w,w] = [gt^{\pm m},w] = [g,w][[g,w],t^{\pm m}][t^{\pm m},w] = [t^{\pm m},w].$$

Moreover, $[t^{\pm m},w] = 1$ implies that $[t,w]^m = 1$. □

REMARK. Theorem 2.3 below shows that it is quite possible for $H_2(G)$ to be much smaller than Z_m, in particular zero. In the case where $w \in K'$, the above analysis just says that $H_2(G)$ is a (possible infinite) cyclic group. In fact, the only example we have been able to find for which $H_2(G) \neq 0$ is a case where $w \in K'$ but $H_2(G) = \mathbf{Z}_2$. It remains to be seen whether there is an example where $w \in K'$ and $H_2(G) = \mathbf{Z}$.

COROLLARY 2.2.1. *Let K and G be as in Theorem 2.2. If $[w] = t^{\pm 1}$ in K/K' then G is a high dimensional knot group.*

PROOF. According to the discussion preceding Theorem 2.1 to show that $G = K/[K,w]$ is a high dimensional knot group, it suffices to show that $H_2(G) = 0$. This is the case $m = 1$ in Theorem 2.2. □

In the case where K is a (p,q)-torus knot group we obtain the following result.

THEOREM 2.3. *Let K be a (p,q)-torus knot group with meridian t, $[w] = t^m$ in K/K' and $G = K/[K,w]$. If $m = \pm 1 \pmod{pq}$ then $H_2(G) = 0$. More generally, $H_2(G)$ is cyclic of exponent less than or equal to* $\gcd(m, pq)$.

PROOF. By Theorem 2.1 we know that $H_2(G)$ is isomorphic to the cyclic subgroup generated by $[t,w]$ in the group $C = K/[K,[K,w]]$. Moreover, by Theorem 2.2, $[t,w]^m = 1$. Let λ be the longitude of K. Since K is a torus knot group it has nontrivial center generated by $t^{pq}\lambda$. Thus $[t^{pq}\lambda, w] = 1$ in K and consequently in C. In the group C we also have $[\lambda, w] = 1$ since $\lambda \in K'$. Furthermore the commutator $[t^{pq}, w]$ commutes with every element of K, in particular with λ. Using the commutator identities on p. 290 in [**M,K,S,**], we thus have

$$1 = [t^{pq}\lambda, w] = [t^{pq}, w][[t^{pq}, w], \lambda][\lambda, w] = [t^{pq}, w].$$

Note that in C, $[t^{pq}, w] = [t,w]^{pq}$. Thus $[t,w]^m = [t,w]^{pq} = 1$ and it follows that $H_2(G)$ is cyclic of exponent less than or equal to $\gcd(pq, m)$. □

In an attempt to learn some more about $H_2(K/[K,w])$ when w is a word in the commutator subgroup K', we look at the relationship between $H_2(K/[K,w])$ and $H_2(K/\langle\langle w\rangle\rangle)$.

THEOREM 2.4. *If K is a classical knot group (or more generally, K is finitely presented, $H_1(K) = \mathbf{Z}$ and $H_2(K) = 0$) and $w \in K'$, then there is a map of $H_2(K/\langle\langle w\rangle\rangle)$ onto $H_2(K/[K,w])$.*

PROOF. Let φ be the natural map of $K/[K,w]$ onto $K/\langle\langle w\rangle\rangle$ and let $A = \ker\varphi$. We then have an exact sequence $0 \to A \to K/[K,w] \to K/\langle\langle w\rangle\rangle \to 0$. The subgroup A is the normal subgroup generated by w in $G = K/[K,w]$. Since w is central in G, we see that A is the cyclic subgroup generated by w. As A is in the center of G, the above sequence is a central extension of A by $K/\langle\langle w\rangle\rangle$. By Theorem 1.1 in [**G**] there exists a homomorphism ρ such that the following sequence is exact.

$$H_1(K/\langle\langle w\rangle\rangle) \otimes A \xrightarrow{\rho} H_2(G) \to H_2(K/\langle\langle w\rangle\rangle) \to H_1(A)$$
$$\to H_1(G) \xrightarrow{\theta} H_1(K/\langle\langle w\rangle\rangle) \to 0.$$

Since $H_1(G) \cong H_1(K/\langle\langle w\rangle\rangle) \cong \mathbf{Z}$ and θ is onto, θ is an isomorphism. Furthermore $H_1(A) = A$ since A is cyclic and hence abelian. As $H_1(K/\langle\langle w\rangle\rangle) \cong \mathbf{Z}$ and $\mathbf{Z}\otimes A \cong A$ we get the following exact sequence:

$$A \to H_2(G) \to H_2(K/\langle\langle w\rangle\rangle) \xrightarrow{\psi} A \to 0.$$

The fact that A is isomorphic to the cyclic subgroup generated by w in $K/[K,w]$ allows us to use Proposition 2.1.2 to conclude that $A \cong H_2(K/\langle\langle w\rangle\rangle)$. Moreover ψ is an isomorphism since A is finitely generated and ψ is onto. Thus the sequence reduces to $A \to H_2(G) \to 0$. In other words $H_2(K/\langle\langle w\rangle\rangle) \to H_2(K/[K,w])$ is surjective. □

COROLLARY 2.4.1. *Let K and w be as in Theorem 2.4. If w is trivial in $K/[K,w]$ then $H_2(K/[K,w]) = 0$.*

PROOF. If $w = 1$ in $K/[K,w]$ then by Proposition 2.1.2 $H_2(K/\langle\langle w\rangle\rangle) = 0$. The result now follows from Theorem 2.4. □

If $w \in K'$ then by Theorem 2.4 it is necessary that $H_2(K/\langle\langle w\rangle\rangle) \neq 0$ in order for $H_2(K/[K,w])$ to be nontrivial. However this condition is not sufficient. To illustrate this, consider the case where K is the group of the trefoil knot. Then K has a presentation of the form $\langle t,a,b; tat^{-1}=b,\ tbt^{-1}=a^{-1}b\rangle$. M. Farber showed that if $H=K/K''$ then $H_2(H)\neq 0$ [**F**, p. 770]. In this particular presentation for K the longitude $\lambda = ab^{-1}a^{-1}b$ and $K''=\langle\langle\lambda\rangle\rangle$. Thus $H_2(K/\langle\langle\lambda\rangle\rangle)\neq 0$. However $H_2(K/[K,\lambda])=0$ since $[t,\lambda]=1$ in K and consequently in $K/[K,[K,\lambda]]$. ($K/[K,\lambda]$ is the group of a deform spun trefoil; see [**L**].)

3. An example where $H_2(K/[K,w])\neq 0$. We will now consider the group K of the trefoil knot. It is convenient to use the following presentation for K; $K=\langle t,a,b;\ tat^{-1}=b,\ tbt^{-1}=a^{-1}b\rangle$. We shall centralize an element $w\in K'$. Let $w=b^n$, $n\geq 2$, and let $G=K/[K,w]$. To obtain a presentation for G we add the relations $[t,b]^n=[a,b^n]=[b,b^n]=1$ to the presentation for K. The resulting presentation can be simplified as follows. First, $1=[t,b^n]=t^{-1}b^{-n}tb^n=a^{-n}b^n$. Hence we can add the relation $a^n=b^n(1)$. The three relations $[t,b^n]=[a,b^n]=[b,b^n]=1$ are consequences of (1) in the presence of the two original relations in K. Thus G has a presentation of the form $\langle t,a,b; tat^{-1}=b,\ tbt^{-1}=a^{-1}b,\ a^n=b^n\rangle$. The second homology $H_2(G)$ is isomorphic to the subgroup generated by $[t,w]=a^{-n}b^n$ in the group $C=K/[K,[K,w]]$. A presentation for C is obtained from a presentation for K by adding nine relators stating that $[t,w]$, $[a,w]$ and $[b,w]$ are central. Note that since $w=b^n$, $[b,w]=[b,b^n]=1$ in K. Moreover, by Lemma 2.1.1 $[a,w]=1$ in C since $a\in K'$. Hence the three relators that state $[a,w]$ is central can be replaced by $[a,w]=1$. Thus C has a presentation $C=\langle t,a,b;\ tat^{-1}=b,\ tbt^{-1}=a^{-1}b,\ [a,w]=[t,[t,w]]=[a,[t,w]]=[b,[t,w]]=1\rangle$. By using the fact that $[a,w]=[a,b^n]=1$ we can eliminate the relators $[a,[t,w]]$ and $[b,[t,w]]$ from the above presentation for C to get the presentation

$$\langle t,a,b;\ tat^{-1}=b,\ tbt^{-1}=a^{-1}b,\ [a,b^n]=[t,a^{-n},b^n]\rangle.$$

Finally we use the first two relations to eliminate t from $[t,a^{-n}b^n]$. Then by using $[a,b^n]=1$, we can rewrite $[t,a^{-n}b^n]=1$ as $b^{2n}a^{-n}(b^{-1}a)^n=1$ to obtain a final presentation for C.

$$C=\langle t,a,b; tat^{-1}=b, tbt^{-1}=a^{-1}b, b^{2n}a^{-n}(b^{-1}a)^n=[a,b^n]=1\rangle.$$

We wish to compute the order of the cyclic subgroup of C generated by $[t,w]$. Since $[t,w]$ is an element of C' we can compute the order of the cyclic subgroup of C' generated by $[t,w]$. The group C' is generated by a and b. By considering the Reidemeister-Schreier process, with coset representatives $\{t^n\}$, we see that a sufficient set of relators for C' is obtained from the two relators $b^{2n}a^{-n}(b^{-1}a)^n$ and $[a,b^n]$ in C by conjugating by powers of t. After several simplifications we get a presentation of C' of the form $C'=\langle a,b; a^n=(a^{-1}b)^n=b^{-n}, b^{4n}=1\rangle$. $H_2(G)$ is isomorphic to the cyclic subgroup generated by b^{2n} in C'. Thus $H_2(G)$ is either trivial or $\mathbf{Z}_2$.

THEOREM 3.1. *Let K be the trefoil knot group with a presentation $\langle t,a,b; tat^{-1}=b, tbt^{-1}=a^{-1}b\rangle$, $w=b^n$, $n\geq 3$. Then $H_2(K/[K,w])=\mathbf{Z}_2$.*

PROOF. Let $R=\langle a,b; a^n=(a^{-1}b)^n=b^{-n}\rangle$. We will show that R is isomorphic to the fundamental group of an orientable Seifert fibered 3-manifold. We then use

that fact to show that the order of b in R is infinite. Once we have established these facts, the rest of the argument goes as follows. The kernel of the natural map of R onto C' is the cyclic subgroup of R generated by b^{4n}. Thus if $b^{2n} = 1$ in C', then $b^{2n} = b^{4nk}$ for some $k \in \mathbf{Z}$, but this contradicts the fact that b has infinite order in R.

Let M be a Seifert fiber manifold of genus zero with three exceptional fibers; we denote M by $M(0;(n,1),(n,1),(n,1))$, $n \geq 3$. Then

$$\pi_1(M) = \langle g_1, g_2, g_3, h;\ [g_i, h] = g_i^n h = g_1 g_2 g_3 = 1\rangle \qquad (1 \leq i \leq 3)$$

[**J,N**].

In $\pi_1(M)$ we can use the relation $g_1g_2g_3 = 1$ to eliminate g_3. Moreover since $g_1^n = g_2^n = g_3^n = h^{-1}$ we can eliminate h. Hence we obtain the following presentation for $\pi_1(M)$; $\pi_1(M) = \langle g_1, g_2; g_1^n = g_2^n = (g_2^{-1}g_1^{-1})^n\rangle$ which is isomorphic to R under the isomorphism that maps $g_1 \to a^{-1}b$ and $g_2 \to b^{-1}$.

Claim. The order of g_2^n in $\pi_1(M)$ is infinite.

PROOF OF CLAIM. M is a Seifert fiber space over an orbifold X where the orbifold fundamental group of X is $\langle g_1, g_2; g_1^n = g_2^n = (g_2^{-1}g_1^{-1})^n = 1\rangle$. Since $n \geq 3$ and hence $1/n + 1/n + 1/n \leq 1$ the above groups are hyperbolic or Euclidean triangle groups. Hence X has an orbifold cover $\tilde{X}$ which is homeomorphic to R^2. According to P. Scott [$\mathbf{S_1}$] there is a natural covering space $\tilde{M}$ of M with an orbit space $\tilde{X}$. Since $\tilde{X} \cong R^2$ it follows that $\tilde{M}$ must be $\mathbf{R}^3$. Thus M is not covered by S^3 and we use Lemma 3.2 in [$\mathbf{S_1}$] to conclude that the cyclic subgroup of $\pi_1(M)$ generated by a regular fiber is infinite or equivalently g_2^n has infinite order. □

Since g_2^n has infinite order and the isomorphism between $\pi_1(M)$ and R maps g_2 to b^{-1} we conclude that b has infinite order in R.

One remaining question is to see what happens to $H_2(K/[K,w])$ when $w = b^2$.

THEOREM 3.2. *If K is the trefoil knot group with a presentation $\langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b\rangle$, $w = b^2$ and $G = K[K,w]$ then $H_2(G) = 0$.*

PROOF. $H_2(G)$ is isomorphic to the subgroup generated by $a^{-2}b^2$ in $C' = \langle a, b; a^2 = (a^{-1}b)^2 = b^{-2}, b^8 = 1\rangle$. Since $(a^{-1}b)^2$ is central we get $a(a^{-1}b)^2a^{-1} = (a^{-1}b)^2$ which reduces to $(ba^{-1})^2 = (a^{-1}b)^2$. We then have

$$a^{-2}b^2 = (ab^{-1})^2(b^{-1}a)^2 = ab^{-1}ab^{-2}ab^{-1}a = b^{-4}a^4 = b^{-8} = 1. \quad \square$$

The above shows that $G_n = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b, a^n = b^n\rangle$ is a high dimensional knot group if $n = 2$. Even more, we shall see that G_2 is isomorphic to the group of the 3-twist spun trefoil. By E. Zeeman [$\mathbf{Z}$] the group of the n-twist spun knot is obtained from the group of the original knot by centralizing t^n where t is a meridian. In particular if K is the trefoil knot group with the presentation $K = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b\rangle$ then the group of the 3-twist spun trefoil is $H = K/[K, t^3]$. A presentation of H is obtained from a presentation of K by adding the relations $[a, t^3] = [b, t^3] = 1$. In the presence of the relation $tat^{-1} = b$, the second relation is a consequence of $[a, t^3] = 1$. Next we use the original relations in K to eliminate t from $[a, t^3] = 1$ to obtain $ab^{-1}ab$ which in turn is equivalent to $a^2 = b^2$. Hence $H = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b, a^2 = b^2\rangle$, and we conclude that $G_2 = H$. Note that G_2 is not the group of a classical knot since G_2' which is generated by a and b is not a free group.

In Theorem 3.1 we showed $H_2(G_n) \cong Z_2$ if $n \geq 3$. Hence G_n cannot be realized as the group of a knotted sphere in any dimension.

REMARK. Similar results were obtained by Murasugi [**M**] and Farber [**F**] in 1977. They both exhibit examples of homomorphic images of classical knot groups that have nontrivial second homology.

PROPOSITION 3.3. *The group* $G = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b, a^n = b^n \rangle$ *is the group of a knotted torus in* S^4.

PROOF. It suffices to show that G has a Wirtinger presentation of deficiency 0 [**Y**]. We rewrite G by setting $t = x$, $a = yx^{-1}$, $b = xyx^{-2}$. The first two relations in G then reduces to $xyx = yxy$ which in turn is equivalent to $yxy^{-1} = x^{-1}yx$. Furthermore,

$$a^n b^{-n} = (yx^{-1})^n (xyx^{-2})^n = (yx^{-1})^{n-1} y (xy^{-1})^{n-1} xy^{-1}x^{-1}.$$

Setting $a^n b^{-n} = 1$ yields the relation $y = (xy^{-1})^{n-1} xyx^{-1} (yx^{-1})^{n-1}$. Thus G has a presentation of the form

$$G = \langle x, y; x = y^{-1}x^{-1}yxy, y = (xy^{-1})^{n-1} xyx^{-1} (yx^{-1})^{n-1} \rangle. \quad \square$$

4. The quotient groups $K/[K, w]$**.** In an attempt to answer the question of what dimension knot if any can realize the group $G = K/[K, w]$, we will first address the question of when G will be infinite cyclic, i.e., G is the group of the unknot. To do this it might in some cases be easier to investigate the group $K/\langle\langle w \rangle\rangle$. The following theorem states a useful relationship between the two groups.

THEOREM 4.1. *If K is a classical knot group with homology generator t and $[w] = t^{\pm m}$ in K/K', $m \geq 0$, then $K/[K, w] \cong \mathbf{Z}$ if and only if $K/\langle\langle w \rangle\rangle \cong \mathbf{Z}_m$. (Note $\mathbf{Z}_0 = \mathbf{Z}$.)*

PROOF. Let $G = K/[K, w]$ and $H = K/\langle\langle w \rangle\rangle$, then there is a natural map of G onto H. Thus if $G \cong \mathbf{Z}$, H must be a cyclic group. In particular H is abelian. Hence H' is trivial. A presentation for H can be obtained from a presentation for K by adding the relation $w = 1$. Thus we have $H = \langle K; w = 1 \rangle$ and $H/H' = \langle K/K'; [w] = 1 \rangle = \langle t; t^{\pm m} = 1 \rangle = \mathbf{Z}_m$. Since H' is trivial, we get that $H \cong \mathbf{Z}_m$. On the other hand suppose that $K/\langle\langle w \rangle\rangle \cong \mathbf{Z}_m$. Then $K' \subset \langle\langle w \rangle\rangle$. Let $\langle t, g_1, \ldots, g_n; R_1 \cdots R_m \rangle$, t is a homology generator, $g_i \in K'$, be a presentation for K. Since $K' \subset \langle\langle w \rangle\rangle$, each commutator $[x, y]$ is of the form $[x, y] = a_1 w^{\varepsilon_1} a_1^{-1} \cdots a_k w^{\varepsilon_k} a_k^{-1}$. In the group $G = K/[K, w]$ we get $[x, y] = w^{\varepsilon_1 + \cdots + \varepsilon_k}$. The remaining part of the proof will be divided into two cases.

Case 1. Let $[w] = t^{\pm m}$ in K/K', $m > 0$. In this case $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_K = 0$. Hence all commutators are killed when we take the quotient $K/[K, w]$. It follows that $K/[K, w] \cong K/K' \cong \mathbf{Z}$.

Case 2. Let $w \in K'$. We have shown that when we set $[K, w] = 1$ every element in K' is a power of w. In particular the g_i's are powers of w. Since K abelianizes to $\mathbf{Z}$ each R_j has exponent sum 0 in t. Thus when we centralize w, each R_j becomes a power of w, i.e., $R_j = w^{l_j}$ in G, $1 \leq j \leq m$. Furthermore for $1 \leq i \leq n$, $[g_i, w] = 1$. Let $d = \gcd(l_1, \ldots, l_m)$ then $G = \langle t, w; [t, w] = w^d = 1 \rangle \cong \mathbf{Z} \oplus \mathbf{Z}_d$. As $G/G' \cong \mathbf{Z}$, d must equal 1 and $G \cong \mathbf{Z}$. $\square$

When $[w]$ is a homology generator in K/K' we get a further characterization of the group $K/[K, w]$.

THEOREM 4.2. *If K is a classical knot group with homology generator t, w any word in K with $[w] = t^{\pm 1}$ in K/K', then $K/[K,w] \cong K/\langle\langle w\rangle\rangle \times \mathbf{Z}$.*

PROOF. Let ϕ be the natural map of $K/[K,w]$ onto $K/\langle\langle w\rangle\rangle$. Let $A = \ker\phi$; then A is the normal subgroup generated by w in $G = K/[K,w]$. As w is central in G, A is just the cyclic subgroup generated by w in G. The exact sequence $0 \to A \to K/[K,w] \to K/\langle\langle w\rangle\rangle \to 1$ is a central extension of A by $K/\langle\langle w\rangle\rangle$. By Theorem 5.1.2 in [**W**] there is a 1-1 correspondence between the equivalence classes of extensions of A by $K/\langle\langle w\rangle\rangle$ and $H^2(K/\langle\langle w\rangle\rangle; A)$. The Universal Coefficient Theorem [**R**] states that $H^2(K/\langle\langle w\rangle\rangle, A) = \mathrm{Hom}(H_2(K/\langle\langle w\rangle\rangle), A) \oplus \mathrm{Ext}(H_1(K/\langle\langle w\rangle\rangle), A)$. Let $H = K/\langle\langle w\rangle\rangle$; then H is a perfect group. Hence $H_1(K/\langle\langle w\rangle\rangle) = H/H' = 0$ and it follows that $\mathrm{Ext}(H_1(K/\langle\langle w\rangle\rangle), A) = 0$. In order to continue the proof we need the following lemma.

LEMMA 4.2.1. *Let K and w be as in Theorem 4.2. Then $H_2(K/\langle\langle w\rangle\rangle) = 0$.*

PROOF OF LEMMA 4.2.1 As K is a classical knot group it has a deficiency 1 presentation. Hence $K/\langle\langle w\rangle\rangle$ has a deficiency 0 presentation obtained from the presentation for K by adding the relation $w = 1$. Construct a 2-simplex C with 1 vertex, n 1-simplices (one for each generator), n 2-simplices (one for each relator) such that $\pi_1(C) \cong K/\langle\langle w\rangle\rangle$. The Euler characteristic of C is $X(C) = 1 - n + n = 1$. Furthermore $H_0(C) = \mathbf{Z}$ and $H_1(C) = 0$. Hence the first two Betti numbers are: $\beta_0 = 1$, $\beta_1 = 0$. As $1 = X(C) = \beta_0 - \beta_1 + \beta_2$ we have that $\beta_2 = 0$. Thus $H_2(C) = 0$ since C is 2-dimensional. The result now follows since $H_2(K/\langle\langle w\rangle\rangle)$ is a quotient of $H_2(C)$. □

The fact that $H_2(K/\langle\langle w\rangle\rangle) = 0$ implies that $\mathrm{Hom}(H_2(K\langle\langle w\rangle\rangle), A) = 0$, and we conclude that $H^2(K/\langle\langle w\rangle\rangle; A) = 0$. Hence there is only one equivalence class of extensions of A by $K/\langle\langle w\rangle\rangle$. Thus $0 \to A \to K/[K,w] \to K/\langle\langle w\rangle\rangle \to 1$ must be a split exact sequence and we get $G = K/[K,w] \cong K/\langle\langle w\rangle\rangle \times A$. Since $K/\langle\langle w\rangle\rangle$ is a perfect group and A is cyclic, $G/G' \cong A$. On the other hand, $G/G' \cong \mathbf{Z}$. Thus $A \cong \mathbf{Z}$. □

Let K be a classical knot group with meridian t and longitude λ. If $w = t\lambda^n$ then $G = K/[K,w]$ is the group a n-roll, 1-twist spun knot as defined by R. Litherland [**L**]. The following corollary which is due to Litherland [**L**, Corollary 5.3] is an easy consequence of Theorem 4.2.

COROLLARY 4.2.2. *If K is the group of a knot with property* P, *then the n-roll, 1-twist spun knot is nontrivial.*

PROOF. The group of the n-roll, 1-twist spun knot is $K/[K, t\lambda^n] = K/\langle\langle t\lambda^n\rangle\rangle \times \mathbf{Z}$ which is isomorphic to $\mathbf{Z}$ only if $K/\langle\langle t\lambda^n\rangle\rangle = 1$. However, since the knot has property P, no nontrivial surgery on this knot yields a simply connected 3-manifold or equivalently $K/\langle\langle t\lambda^n\rangle\rangle \neq 1$. □

EXAMPLES. Let K be the group of the trefoil knot. Then K has a presentation of the form $K = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b\rangle$. We will give some examples of the different groups that can be obtained by centralizing certain w's in K'.

THEOREM 4.3. *Let $K = \langle t, a, b; tat^{-1} = b, tbt^{-1} = a^{-1}b\rangle$ and let $w = a^n b$, $n \neq -2, -1, 0, 1$. If $G = K/[K,w]$, then G is a high dimensional knot group that cannot be realized as a 2-knot group.*

PROOF. A presentation for G is obtained from a presentation for K by adding the relations $[t,w] = [a,w] = [b,w] = 1$. Note that $[t,w] = 1$ if and only if $[t^{-1},w] = 1$. Thus we have

$$1 = [t^{-1},w] = tb^{-1}a^{-n}t^{-1}a^n b = b^{-1}ab^{-n}a^n b$$

which reduces to

$$a^{n+1} = b^n. \tag{1}$$

The relation $[a, a^n b] = 1$ holds if and only if

$$[a,b] = 1. \tag{2}$$

The relation $[b, a^n b] = 1$ is a consequence of (2). As $a^{n+1} = b^n$, we have $ta^{n+1}t^{-1} = tb^n t^{-1}$. Using the original relations in K we get $ta^{n+1}t^{-1} = b^{n+1}$ and $tb^n t^{-1} = (a^{-1}b)^n$. Moreover $(a^{-1}b)^n = a^{-n}b^n$ since a and b commute. Thus we get $b^{n+1} = a^{-n}b^n$ which reduces to

$$b = a^{-n}. \tag{3}$$

Now we use (3) to substitute for b in $tat^{-1} = b$ and $tbt^{-1} = a^{-1}b$ to get $tat^{-1} = a^{-n}$ and $ta^{-n}t^{-1} = a^{-n-1}$ which imply that

$$a^{n+1} = a^{-n^2}. \tag{4}$$

The relations (1) and (2) are consequences of (3) and (4). Finally $tat^{-1} = a^{-n}$ and $a^{n+1} = a^{-n^2}$ imply $ta^{-n}t^{-1} = a^{-n-1}$. Thus G has a presentation $G = \langle t, a; tat^{-1} = a^{-n}, a^{n^2+n+1} = 1\rangle$. In order to show that G is a high dimensional knot group we must show that $H_2(G)$ is trivial. We use the exact sequence

$$\cdots \to H_{k+1}(G) \to H_k(G') \xrightarrow{t_*-1} H_k(G') \to H_k(G) \to \cdots$$

where t_* is the action of the group $G/G' \cong \mathbf{Z}$ on $H_*(G')$ [**F**]. In our case $H_1(G') = G' \cong \mathbf{Z}_{n^2+n+1}$ and $t_*\colon H_1(G') \to H_1(G')$ is given by multiplication by $-n$.

LEMMA 4.3.1. $H_2(G') = 0$

PROOF OF LEMMA 4.3.1. Let F be the free group $\mathbf{Z}$, and let R be the kernel of the natural map of F onto G'. Then $H_2(G') \cong [F,F] \cap R/[F,R]$. Thus $H_2(G') = 0$ as the commutator subgroup $[F,F] = 1$. □

Since $H_2(G') = 0$ the above exact sequence reduces to $0 \to H_2(G) \to H_1(G') \xrightarrow{t_*-1} H_1(G') \to \cdots$ and it follows that $H_2(G) \cong \ker(t_* - 1)$. By standard calculations it is not hard to see that this kernel must be trivial. □

So far we have established that G can be realized as the group of a high dimensional knot. Let $T = \mathrm{Tors}(G'/G'')$; then $T = \mathbf{Z}_{n^2+n+1}$. Suppose that G is the group of a knotted 2-sphere in 4-space; then there exists a nondegenerate symmetric form $L\colon T \times T \to Q/\mathbf{Z}$ such that $L(x,y) = L(-nx,-ny)$ for all $x,y \in T$ [**F**, Theorem 5.2]. Let a be a generator for T; then $L(a,a) = s/(n^2+n+1) \bmod 1$ where $s \in \mathbf{Z}$ and $(s, n^2+n+1) = 1$. Moreover $L(-na,-na) = n^2 s/(n^2+n+1) \bmod 1$. Since $L(a,a) = L(-na,-na)$ this implies that $(n^2-1)s/(n^2+n+1) \in \mathbf{Z}$. As $(s, n^2+n+1) = 1$ we conclude that n^2+n+1 divides n^2-1 which is possible only if $n = -2, -1, 0$ or 1. Thus for any other values of n, G cannot be a 2-knot group. □

Finally we will see what happens to the group G when $n = -2, -1, 0, 1$. Recall that G has the following presentation: $G = \langle t, a; tat^{-1} = a^{-n}, a^{n^2+n+1} = 1\rangle$. We will denote the four groups by G_{-2}, G_{-1}, G_0 and G_1. It is easy to verify that $G_0 = G_1 = \mathbf{Z}$ and $G_{-2} = G_1 = \langle t, a; tat^{-1} = a^{-1}, a^3 = 1\rangle$ which is the group of the 2-twist spun trefoil. Note since this group has an element of finite order it is not a classical knot group.

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