COMPLETELY REDUCIBLE OPERATORS
THAT COMMUTE WITH COMPACT OPERATORS

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ABSTRACT. It is shown that if $T$ is a completely reducible operator on a
Banach space and $TK = KT$, where $K$ is an injective compact operator with
a dense range, then $T$ is a scalar type spectral operator. Other related results
are also obtained.

Let $\mathcal{A}$ be an algebra of bounded linear operators on a Banach space $X$. lat
$\mathcal{A}$ is the lattice of (closed) invariant subspaces of $\mathcal{A}$. We say that $\mathcal{A}$ is completely
reducible if for every $M \in \text{lat} \mathcal{A}$ there is $N \in \text{lat} \mathcal{A}$ with $M + N = X$ (that is,
$M \cap N = 0$ and the algebraic sum $M + N$ coincides with $X$). An operator $T$
is completely reducible if the algebra generated by $T$ is. It is unknown whether a
weakly closed unital completely reducible algebra must be reflexive; that is, must
contain every operator which leaves invariant its invariant subspaces. Some partial
solutions of this problem can be found in [1, 6, 7].

In this paper we show that every completely reducible operator commuting with
an injective compact operator with a dense range is a scalar type spectral operator.
In particular, the weakly closed unital algebra generated by such an operator must
be reflexive. This result seems to be unknown even for operators on a Hilbert space.
Also, we show that every compact completely reducible operator must be a scalar
type spectral operator. This answers a question raised by E. Azoff and A. Lubin
(see the last page of [1]) and, independently, by V. Lomonosov. Finally, our result
generalizes the results of Loginov and Šul'man [2] and Rosenthal [5] on reductive
Hilbert space operators that commute with compact operators.

The following theorem is the central result of the author's paper [4], where it
was stated in a slightly different form:

\textbf{THEOREM 1.} Let $\mathcal{A}$ be a commutative operator algebra on a Banach space $X$. If
the commutant of $\mathcal{A}$ is completely reducible and the ranges of compact operators
in $\mathcal{A}$ span $X$, then every operator in $\mathcal{A}$ is a scalar type spectral operator. If, in
addition, $\mathcal{A}$ is a weakly closed unital completely reducible algebra, then $\mathcal{A}$ is gener-
ated, as a uniformly closed algebra, by a complete totally atomic Boolean algebra
of projections. Moreover, $\mathcal{A}$ is reflexive and admits spectral synthesis (i.e., every
invariant subspace of $\mathcal{A}$ is spanned by its one-dimensional invariant subspaces).

Thus, in order to prove the result described above, it suffices to show that if
$\mathcal{A}$ is a commutative completely reducible algebra which has enough hyperinvariant
subspaces, then the commutant of $\mathcal{A}$ is also completely reducible. This will be done
in Theorem 8 below. The above result then follows easily, a sufficient supply of

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hyperinvariant subspaces being provided by Lomonsov's theorem. It can be shown, by a slight variation of the proof of Theorem 8, that the word "hyperinvariant" in its statement can be replaced with "invariant."

Let us introduce some definitions and notation. For Banach spaces $X$ and $Y$, $\mathcal{L}(X,Y)$ denotes the collection of all bounded linear operators from $X$ to $Y$; $\mathcal{L}(X,X)$ is denoted by $\mathcal{L}(X)$. $X^*$ means the conjugate space of the Banach space $X$. For $M \subseteq X$, $M^\perp$ is an annihilator of $M$ in $X^*$. An operator $E$ in $\mathcal{L}(X)$ is a projection if $E^2 = E$. If $E$ and $F$ are projections, we write $E \leq F$ provided $EF = FE = E$. Clearly, $E \leq F$ if and only if $E(X) \subseteq F(X)$ and $\text{Ker } E \supseteq \text{Ker } F$. If $E$ is a projection, we write $E^\perp$ for $I - E$. If $A$ is a subalgebra of $\mathcal{L}(X)$, then $A'$ denotes the commutant of $A$; that is, the set of all operators in $\mathcal{L}(X)$ that commute with every operator in $A$. Hyperinvariant subspaces of $A$ are those invariant for $A'$. We will write $\mathcal{P}(A)$ for the family of all projections in $A'$, and $\mathcal{P}_0(A)$ for the set of those projections in $\mathcal{P}(A)$ whose range is hyperinvariant for $A$. Finally, for $E$ in $\mathcal{P}(A)$, we define $\text{int}_E A$ as the set of all $T \in \mathcal{L}(E^\perp(X), E(X))$ such that

$$EAET = TEE^\perp$$

or, equivalently,

$$(A|E(X))(T) = T(A|E^\perp(X))$$

for each $A \in A$.

Clearly, an operator algebra $A$ is completely reducible if and only if for every $M$ in $\text{lat } A$ there is a projection in $\mathcal{P}(A)$ with range $M$. Note also that for $A$ completely reducible and $M$ in $\text{lat } A$, the restriction of $A$ to $M$, $A|M$, is also completely reducible.

We shall need some very elementary lemmas. The first is well known.

**Lemma 2.** Let $X$ be a Banach space and let $X_1$ and $X_2$ be subspaces of $X$ with $X_1 + X_2 = X$. Then $X$ is isomorphic to the exterior direct sum $X_1 \oplus X_2$ defined as a vector space of ordered pairs $(x_1, x_2)$, $x_i \in X_i$, endowed with the norm $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$.

**Lemma 3.** Let $A$ be a subalgebra of $\mathcal{L}(X)$ and $E \in \mathcal{P}(A)$. Then $E^\perp(X)$ is in $\text{lat } A'$ if and only if $\text{int}_E A = 0$.

**Proof.** Suppose $E^\perp(X) \in \text{lat } A'$. For each $T \in \text{int}_E A$, $ETE^\perp$ is in $A'$, so that $ETE^\perp = 0$ and $T = 0$. Conversely, for each $B \in A'$, $EBE^\perp E^\perp(X)$ is in $\text{int}_E A$, and $\text{int}_E A = 0$ implies $EBE^\perp = 0$, so that $E^\perp(X)$ is invariant under $B$.

**Lemma 4.** Let $A$ be a subalgebra of $\mathcal{L}(X)$ and $F \in \mathcal{P}_0(A)$. Let $X_1 = F(X)$ and $X_2 = \text{Ker } F$.

(i) For each $M \in \text{lat } A'$, $F(M) \in \text{lat } A'$.

(ii) A subspace $Y$ which contains $X_1$ belongs to $\text{lat } A'$ if and only if $Y = X_1 + Y_1$, where $Y_1 \subseteq X_2$ and $Y_1 \in \text{lat } (A|X_2)'$.

**Proof.** (i) For each $B \in A'$, $BF(M) \subseteq F(X)$, since $F(X)$ is hyperinvariant, and $BF(M) \subseteq M$, since $M$ is hyperinvariant and $F \in A'$. Hence, $BF(M) \subseteq F(X) \cap M = F(M)$.

(ii) Let $Y \supseteq X_1$ and $Y \in \text{lat } A'$. In particular, $Y$ is invariant under $F$, so that $Y = X_1 + Y_1$ for some $Y_1 \subseteq X_2$. For each $C \in (A|X_2)'$, $F^\perp CF^\perp \in A'$, and $C(Y_1) = F^\perp CF^\perp(X_1 + Y_1) \subseteq F^\perp(Y) = Y_1$; that is, $Y_1 \in \text{lat } (A|X_2)'$. Conversely,
if \( Y_1 \in \text{lat}(\mathcal{A}\mathcal{X}_2)' \) and \( B \in \mathcal{A}' \), then \( F^\perp B F^\perp |X_2 \) is in \((\mathcal{A}\mathcal{X}_2)' \) and \( F^\perp B F = 0 \). It follows that \( B(Y) = (FB + F^\perp B F^\perp)(Y) \subseteq X_1 \downarrow Y_1 = Y \).

The following two lemmas will enable us to reduce the proof of the main result to the case when the completely reducible commutative algebra has no nonzero finite-dimensional invariant subspaces.

**Lemma 5.** Let \( \mathcal{A} \subseteq \mathcal{L}(X) \) be a completely reducible algebra such that the one-dimensional subspaces in \( \text{lat} \mathcal{A} \) span \( X \). Then \( \mathcal{A}' \) is completely reducible.

**Proof.** Clearly, \( \mathcal{A} \) is commutative. We claim that \( \mathcal{A} \) admits spectral synthesis. Indeed, let \( M \in \text{lat} \mathcal{A} \). Then there exists \( F \in \mathcal{P}(\mathcal{A}) \) such that \( E(X) = M \). Since \( E \) transforms every one-dimensional invariant subspace of \( \mathcal{A} \) into an invariant subspace of \( \mathcal{A} \) of dimension no greater than one, \( E(X) \) is spanned by one-dimensional elements of \( \text{lat} \mathcal{A} \).

Now suppose \( X_1 \in \text{lat} \mathcal{A}' \). Since \( \mathcal{A}' \supseteq \mathcal{A} \), \( X_1 \) is also in \( \mathcal{A} \), and one can find \( X_2 \in \text{lat} \mathcal{A} \) with \( X_1 \downarrow X_2 = X \). We shall show that \( X_2 \) is also in \( \text{lat} \mathcal{A}' \). Suppose not. Denote by \( E \) the projection onto \( X_1 \) along \( X_2 \). Then, by Lemma 3, \( \text{int}_E \mathcal{A} \neq 0 \), and, by our claim above, there exist such \( T \in \text{int}_E \mathcal{A} \) and one-dimensional \( N \in \text{lat}(\mathcal{A}\mathcal{X}_2) \) such that \( M = T(N) \neq 0 \). It is very easy to see that \( M \in \text{lat} \mathcal{A} \) and the algebra \( \mathcal{A}'(M \downarrow N) \) consists only of multiples of the identity. Denote by \( S \) an operator which maps \( M \) into \( N \) and is identically zero on some invariant complement to \( M \downarrow N \). Then \( S \in \mathcal{A}' \), but \( X_1 \) is not invariant for \( S \), a contradiction.

**Lemma 6.** Let \( \mathcal{A} \subseteq \mathcal{L}(X) \) be a completely reducible algebra. Suppose \( X_1 \) is spanned by all one-dimensional subspaces in \( \text{lat} \mathcal{A} \) and \( X_2 \) is in \( \text{lat} \mathcal{A} \) with \( X_1 \downarrow X_2 = X \). Then both \( X_1 \) and \( X_2 \) are in \( \text{lat} \mathcal{A}' \).

**Proof.** Obviously, \( X_1 \) lies in \( \text{lat} \mathcal{A}' \). Suppose \( X_2 \) does not. Then, denoting by \( E \) a projection onto \( X_1 \) along \( X_2 \), we conclude that \( \text{int}_E \mathcal{A} \neq 0 \). Choose nonzero \( T \in \text{int}_E \mathcal{A} \). Since \( \text{cl} T(X_2) \in \text{lat}(\mathcal{A}\mathcal{X}_1) \) and, as has been noted in the proof of the previous lemma, \( \mathcal{A}\mathcal{X}_1 \) admits spectral synthesis, we can find a one-dimensional \( P \in \mathcal{P}(\mathcal{A}\mathcal{X}_1) \) such that \( PT \neq 0 \). However, \( PT \in \text{int}_E \mathcal{A} \), so that \( \text{Ker} PT \subset \text{lat}(\mathcal{A}\mathcal{X}_2) \). On the other hand, \( \text{codim} \text{Ker} PT = 1 \) and, since \( \mathcal{A}\mathcal{X}_2 \) is completely reducible, \( \mathcal{A}\mathcal{X}_2 \) has a one-dimensional invariant subspace, which contradicts the definition of \( X_1 \) and therefore completes the proof.

**Lemma 7.** Suppose \( \mathcal{A} \subseteq \mathcal{L}(X) \) is a commutative completely reducible algebra which has the following property: for every nonzero \( M \in \text{lat} \mathcal{A}' \), there is \( N \in \text{lat} \mathcal{A} \) such that \( N \subset M, N \neq 0, N \neq M \). Let \( M_1, M_2, \ldots, M_n, \ldots \) be an infinite sequence of nonzero subspaces in \( \text{lat} \mathcal{A}' \). Then there exists such an \( F \in \mathcal{P}_0(\mathcal{A}) \) that \( F^\perp (M_1) \neq 0 \) and \( F(M_n) \neq 0 \) for infinitely many \( n \).

**Proof.** Choose hyperinvariant \( N \subset M_1, N \neq 0, N \neq M_1 \). Since \( \mathcal{A}' \supseteq \mathcal{A} \), \( N \in \text{lat} \mathcal{A} \), and there exists \( P \in \mathcal{P}_0(\mathcal{A}) \) with range \( N \). Now consider two cases.

**Case 1.** \( P(M_n) \neq 0 \) for only finitely many \( n \).

Then there is an infinite set of positive integers \( J \) such that \( M_n \subset P^\perp (X) \) for all \( n \) in \( J \). Let \( F \) denote a projection of \( \mathcal{P}_0(\mathcal{A}) \) onto a subspace \( \bigvee_{n \in J} M_n \) such that \( F \leq P^\perp \). Then \( F(M_n) = M_n \neq 0 \) for every \( n \in J \) and \( F^\perp (M_1) \supseteq N \neq 0 \).

**Case 2.** \( P(M_n) \neq 0 \) for infinitely many \( n \).

Then take \( F = P \). Clearly, \( F(M_n) \neq 0 \) for infinitely many \( n \). On the other hand, \( F^\perp (M_1) \neq 0 \), since \( N \neq M_1 \).
Now we are ready for the proof of our main result.

**Theorem 8.** Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a commutative completely reducible algebra. Suppose that for every hyperinvariant subspace $M$ of $\mathcal{A}$ of dimension and codimension greater than 1, there exist nontrivial hyperinvariant subspaces of $\mathcal{A}$, $M_1$ and $M_2$, other than $M$, such that $M_1 \subseteq M \subseteq M_2$. Then $\mathcal{A}'$ is completely reducible.

**Proof.** Choose $X_1$ in lat $\mathcal{A}'$. Since $\mathcal{A}' \supseteq \mathcal{A}$, $X_1$ is also in lat $\mathcal{A}$ and, since $\mathcal{A}$ is completely reducible, there is in $X_2 \in \text{lat} \mathcal{A}$ such that $X_1 + X_2 = X$. We claim that $X_2$ is in lat $\mathcal{A}'$ and therefore that $X_2$ is the unique complement to $X_1$ in lat $\mathcal{A}$.

The claim will be established by contradiction; suppose $X_2$ is not in lat $\mathcal{A}'$. Denote by $E$ the projection onto $X_1$ along $X_2$. The proof will be divided into three parts. In the first part, we shall construct two infinite sequences of pairwise orthogonal projections, $\{E_n\}_{n=1}^{\infty}$ in $(\mathcal{A}|X_2)'$ and $\{F_n\}_{n=1}^{\infty}$ in $(\mathcal{A}|X_1)' = \mathcal{A}'|X_1$, and a sequence $\{T_n\}_{n=1}^{\infty}$ in int$_E \mathcal{A}$ such that $E_nT_nF_n \neq 0$ for all $n$.

Note that for $T \in \text{int}_E \mathcal{A}$, $B \in \mathcal{A}'|X_1$, and $C \in (\mathcal{A}|X_2)'$, BTC $\in \text{int}_E \mathcal{A}$. For an arbitrary projection $G$ in $\mathcal{L}(X_2)$ let $M(G)$ denote the subspace of $X_1$ spanned by all $TG(X_2)$ with $T \in \text{int}_E \mathcal{A}$. Clearly, $M(G)$ is always in lat($\mathcal{A}'|X_1$).

Now denote by $Y$ the intersection of the kernels of all operators in int$_E \mathcal{A}$. By our assumption that $X_2$ is not in lat $\mathcal{A}'$ and Lemma 3, it follows that $Y \neq X_2$. On the other hand, $Y$ lies in $\text{lat}(\mathcal{A}|X_2)'$ and, by Lemma 4(ii), $X_1 + Y$ lies in lat $\mathcal{A}'$. Let $Q$ be a projection in $\mathcal{P}(\mathcal{A}|X_2)$ onto a subspace which is complementary to $Y$ in $X_2$. By hypothesis, $X_1 + Y$ is contained in some larger nontrivial hyperinvariant subspace of $\mathcal{A}$. By Lemma 4(ii), this larger subspace has the form $X_1 + Y + E_1(X_2)$ for some nonzero $E_1 \in \mathcal{P}(\mathcal{A}|X_2)$, $E_1 \leq Q$, $E_1 \neq Q$. Repeating the same argument, one can find nonzero $E_2 \in \mathcal{P}(\mathcal{A}|X_2)$ with $E_2 \leq Q - E_1$, $E_2 \neq Q - E_1$. Proceeding by induction, we get an infinite sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise orthogonal nonzero projections in $\mathcal{P}(\mathcal{A}|X_2)$ with $E_n \leq Q$ for all $n$. It follows from the definition of Q that $M(E_n) \neq 0$ for all $n$.

Now Lemma 7 provides $G_1 \in \mathcal{P}_0(\mathcal{A}|X_1)$ such that $G_1^\perp M(E_1) \neq 0$ and $G_1 M(E_n) \neq 0$ for every $n$ in the infinite set $J$ of positive integers.

Renumber the elements of $J_1$ by $2, 3, \ldots$. By Lemma 4(i), $G_1 M(E_n) \in \text{lat}(\mathcal{A}|X_1)'$ for $n \geq 2$. Since $$((\mathcal{A}|G_1(X_1))' = \mathcal{A}|G_1(X_1),$$
we may again apply Lemma 7 to the algebra $\mathcal{A}|G_1(X_1)$ and a sequence $G_1 M(E_2)$, $G_1 M(E_3), \ldots$ of its hyperinvariant subspaces. As a result, we obtain $G_2 \in \mathcal{P}_0(\mathcal{A}|X_1)$ such that $G_2 \leq G_1$, $(G_1 - G_2)G_1 M(E_2) = (G_1 - G_2)M(E_2) \neq 0$, and $G_2 G_1 M(E_n) = G_2 M(E_n) \neq 0$ for every $n$ from an infinite subset $J_2$ of $J_1$.

Proceeding by induction (renumbering the elements of $J_n$ by $n + 1, n + 2, \ldots$), we get a sequence $I = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$, where $G_n \in \mathcal{P}_0(\mathcal{A}|X_1)$ and $$(G_{n-1} - G_n)M(E_n) \neq 0, \quad n = 1, 2, \ldots.$$ Let $F_n = G_{n-1} - G_n$, $n = 1, 2, \ldots$. The $F_n$'s are pairwise orthogonal projections in $\mathcal{P}(\mathcal{A}|X_1)$, and $F_n M(E_n) \neq 0$ for $n = 1, 2, \ldots$. Finally, from the definition of $M(E_n)$, it follows that there is a sequence $\{T_n\}_{n=1}^{\infty}$ in int$_E \mathcal{A}$ such that $F_n T_n E_n \neq 0$ for all $n$.

In the second part of the proof we will construct a closed unbounded linear transformation $T$ defined on the linear manifold $D \subseteq X_1$, with range in $X_2$, such

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that its graph \( \{ Tx + x, \ x \in \mathcal{D} \} \) is invariant under \( \mathcal{A} \). For this, define an operator 
\[ T_0 \in \mathcal{L}(X_2, X_1) \] 
as follows:
\[ T_0 = \sum_{n=1}^{\infty} 2^{-n} \| F_n T_n E_n \|^{-1} F_n T_n E_n, \]
where the series converges in the sense of the norm in \( \mathcal{L}(X_2, X_1) \). It is easy to see
that \( T_0 \in \text{int} \mathcal{A} \). Now let \( L = \bigcap_{n=1}^{\infty} \text{Ker } F_n \). Then \( L \in \text{lat}(\mathcal{A}|X_1) \); let \( N \) be in
\( \text{lat}(\mathcal{A}|X_1) \) with \( L \perp N = X_1 \). Denote by \( P \) the projection onto \( N \) along \( L \). Let us observe that for every \( n \) the operator \( P F_n | F_n(X_1) \) is injective; it follows that
\( P F_n T_n E_n \neq 0 \) for \( n \geq 1 \). Now define \( S \in (\mathcal{A}|X_2)' \) as follows:
\[ S = \sum_{n=1}^{\infty} 2^{-2n} \| P F_n T_n E_n \| \| F_n T_n E_n \|^{-1} \| E_n \|^{-1} E_n \]
(again, the series is convergent in the sense of the norm).

We claim that the subspace
\[ M = \text{cl}\{(PT_0 x, S x), \ x \in X_2 \} \subseteq X_1 \oplus X_2 \]
is a graph of some linear transformation \( T: \mathcal{D} \to X_1, \mathcal{D} \subseteq X_2 \).

Indeed, the conjugate space to \( X_1 \oplus X_2 \) is a linear space of vectors \((x_1^*, x_2^*)\),
\( x_i^* \in X_i^* \), endowed with the norm
\[ \| (x_1^*, x_2^*) \| = \text{sup}(\| x_1^* \|, \| x_2^* \|). \]
It is easy to see that \((x_1^*, x_2^*) \in M^\perp \) if and only if
\[ T_0^* P^* x_1^* + S^* x_2^* = 0. \]
Note that, by the definition of \( L \),
\[ \text{weak}^* \text{cl} \left( \bigvee_{n=1}^{\infty} F_n^* (X_1^*) \right) = \text{weak}^* \text{cl} \left( \bigvee_{n=1}^{\infty} (\text{Ker } F_n)^\perp \right) \]
\[ = \left( \bigcap_{n=1}^{\infty} \text{Ker } F_n \right)^\perp = L^\perp. \]
Note also that \( L^\perp = P^* (X_1^*), \ N^\perp = \text{Ker } P^*, \) and \( L^\perp \perp N^\perp = X_1^* \). Let
\[ \mathcal{L} = \bigcup_{n=1}^{\infty} \left( \sum_{i=1}^{n} F_i^* (X_1^*) \right). \]
Clearly, \( \mathcal{L} \) is weak* dense in \( L^\perp \). Now, for \( m \geq 1 \),
\[ T_0^* P^* F_m = T_0^* F_m = \left( \sum_{n=1}^{\infty} 2^{-n} \| F_n T_n E_n \|^{-1} E_n^* T_n^* F_n^* \right) F_m^* \]
\[ = 2^{-m} \| F_m T_m E_m \|^{-1} E_m^* T_m^* F_m^*. \]
Similarly, we conclude that \( E_m^* (X_2^*) \) is contained in the range of \( S^* \) for \( m \geq 1 \). It follows that for each \( x_i^* \in \mathcal{L} + N^\perp \) there exists \( x_2^* \in X_2^* \) such that
\( T_0^* P^* x_1^* + S^* x_2^* = 0 \) (if \( x_1^* \in N^\perp \), then \( P^* x_1^* = 0 \) and we can take \( x_2^* = 0 \)), or \((x_1^*, x_2^*) \in M^\perp \). Now suppose \((x, 0) \in M \) for some \( x \in X_1 \). Then, for each \( x_i^* \in \mathcal{L} + N^\perp, \ x_1^* (x) = 0 \), and, since \( \mathcal{L} + N^\perp \) is weak* dense in \( X_1^*, \ x = 0 \). This proves our claim.
Now we shall show that \( T \) is unbounded. Indeed, for every \( m \geq 1 \), \( E_m(X_2) \subseteq S(X_2) \) and therefore \( E_m(X_2) \subseteq D \). Furthermore, \( TSE_m = PT_0E_m \), or
\[
T(2^{-2m} \| PF_m T_m E_m \| \| F_m T_m E_m \|^{-1} \| E_m \|^{-1}) E_m
= 2^{-m} \| F_m T_m E_m \|^{-1} PF_m T_m E_m.
\]
Since \( PF_m T_m E_m \neq 0 \),
\[
T E_m = 2^m \| E_m \| \| PF_m T_m E_m \|^{-1} PF_m T_m E_m.
\]
Hence, \( \| T E_m \| = 2^m \| E_m \| \), which proves that \( T \) is unbounded.

Now let \( M_0 \) be the closure of \( \{ PT_0x + Sx, \ x \in X_2 \} \) in \( X \). Lemma 2 allows us to identify \( M_0 \) with \( M \). That is, we may suppose that \( M_0 \cap X_1 = 0 \) and \( (M_0 + X_1) \cap X_2 = D \). For each \( A \) in \( \mathcal{A} \), \( A = EAE + E^\perp AE^\perp \); it follows, since \( PT_0 \in \text{int} \mathcal{A} \) and \( S \in (\mathcal{A}|X_2)' \), that for \( x \in X_2 \),
\[
A(PT_0x + Sx) = (EAE + E^\perp AE^\perp)(PT_0x + Sx)
= EAEPT_0x + E^\perp AE^\perp x = PT_0E^\perp AE^\perp x + SE^\perp AE^\perp x,
\]
which shows that \( M_0 \in \text{lat} \mathcal{A} \). This ends the second part of the proof.

In the last part of the proof we obtain a contradiction. To do this, it would suffice to refer to a simple result of Fong [1], but we prefer to give a direct proof.

Since \( M_0 \in \text{lat} \mathcal{A} \) and \( \mathcal{A} \) is completely reducible, one can find \( M_1 \in \text{lat} \mathcal{A} \) such that \( M_0 \cap M_1 = X \). Let \( E_0 \) denote the projection onto \( M_0 \) along \( M_1 \). Then \( E \in \mathcal{A}' \) and hence \( X_1 \) is invariant under \( E_0 \). This implies that
\[
X_1 = X_1 \cap M_0 \cap X_1 \cap M_1.
\]
However, as we have seen, \( X_1 \cap M_0 = 0 \); that means \( X_1 \subseteq M_1 \). From the fact that \( M_0 \cap M_1 = X \) and Lemma 2, it follows that the manifold \( M_0 + X_1 \) is closed. Then \( D = (M_0 + X_1) \cap X_2 \) is also closed. But \( D \) is the domain of definition for a closed unbounded transformation \( T \), and, by the Closed Graph Theorem, cannot be closed. This contradiction completes the proof of the theorem.

**THEOREM 9.** Let \( \mathcal{A} \subseteq \mathcal{L}(X) \) be a commutative unital weakly closed completely reducible algebra. Suppose that the intersection of the kernels of all compact operators in \( \mathcal{A}' \) is zero and the subspace spanned by ranges of all compact operators in \( \mathcal{A}' \) is \( X \). Then \( \mathcal{A} \) is generated, as a uniformly closed algebra, by a complete bounded totally atomic Boolean algebra of projections; in particular, \( \mathcal{A} \) is an algebra of scalar type spectral operators. Furthermore, \( \mathcal{A} \) is reflexive and admits spectral synthesis.

**PROOF.** Let \( X_1 \) denote the subspace spanned by all one-dimensional subspaces in \( \text{lat} \mathcal{A} \), and let \( X_2 \) be a complement to \( X_1 \) in \( \text{lat} \mathcal{A} \). By Lemma 6, \( X_1 \) and \( X_2 \) are in \( \text{lat} \mathcal{A}' \). We shall show that for \( \mathcal{A}|X_2 \) the conditions of the previous theorem are satisfied. Denote by \( \mathcal{C} \) the family of all compact operators in \( (\mathcal{A}|X_2)' \). Clearly, intersections of the kernels of all operators in \( \mathcal{C} \) is zero, and the subspace spanned by all their ranges is \( X_2 \). Let \( M \) be a nonzero subspace in \( \text{lat} \mathcal{A}|X_2 \) and \( E \) be in \( P_0(\mathcal{A}|X_2) \) with \( E(X_2) = M \). Then there is \( K_1 \in \mathcal{C} \) such that \( K_1|M \neq 0 \) and (note that \( M \) is infinite-dimensional), by Lomonosov's theorem [3], there is a nonzero \( M_1 \subseteq M \) such that \( M_1 \in \text{lat} \mathcal{A}|X_2 \) and \( M_1 \neq M \). On the other hand, there exists \( K_2 \in \mathcal{C} \) such that \( E^\perp K_2 E^\perp \neq 0 \), for otherwise \( E^\perp KE = E^\perp KE^\perp = 0 \) for each \( K \) in \( \mathcal{C} \), hence \( K = EK \) and \( K(X_2) \subseteq E(X_2) \), which contradicts our
hypothesis. Again, by Lomonosov's theorem, the algebra $A|E^+(X_2)$ has a nontrivial hyperinvariant subspace, and now Lemma 4(iii) implies that $A|X_2$ has a nontrivial hyperinvariant subspace $M_2$ strictly containing $M$. So the conditions of Theorem 8 are satisfied for $A|X_2$.

Choose a subspace in $\text{lat } A'$ Clearly, it can be written as $M \oplus N$, where $M \subseteq X_1$ and $N \subseteq X_2$. By Lemma 5, there exists $M_1$ in $\text{lat}(A|X_1)'$ such that $M \oplus M_1 = X_1$, and, by Theorem 8, we can find $N_1 \in \text{lat}(A|X_2)'$ such that $N \oplus N_1 = X_2$. But then $M_1 \oplus N_1$ lies in $\text{lat } A'$ and $(M \oplus N) \oplus (M_1 \oplus N_1) = X$. It follows that $A'$ is completely reducible. Now, to conclude the proof of the theorem, it suffices to apply Theorem 1.

The following corollary follows immediately from Theorem 9.

**Corollary 10.** Let $T \in \mathcal{L}(X)$ be a completely reducible operator. If $TK = KT$, where $K$ is an injective compact operator such that $\text{cl } K(X) = X$, then $T$ is a scalar type spectral operator, and spectral synthesis holds for $T$.

**Corollary 11.** Let $A \subseteq \mathcal{L}(X)$ be a commutative unital weakly closed completely reducible algebra. If the intersection of the kernels of all the compact operators in $A$ is zero, or if the ranges of all the compact operators in $A$ span $X$, then the conclusions of Theorem 9 hold for $A$.

**Proof.** We shall show that the assumption about the kernels is equivalent to that about the ranges. Then the result would be an immediate consequence of Theorem 9. Let $M$ be intersection of the kernels of all compact operators in $A$. Clearly, $M \in \text{lat } A$. Let $N \in \text{lat } A$ be such that $M \oplus N = X$. Let $N_0$ be the subspace spanned by all $K(N)$, where $K$ runs over the set of all compact operators in $A$. Obviously, $N_0 \in \text{lat } A$ and $N_0 \subseteq N$. Let $N_1 \in \text{lat } A$, $N_0 \oplus N_1 = N$. Then, by the definition of $N_0$, all the compact operators in $A$ vanish on $N_1$; that is, $N_1 \subseteq M \cap N = 0$ and $N_0 = N$. On the other hand, the range of every compact operator in $A$ is contained in $N$. It follows that the subspace spanned by the ranges of the compact operators in $A$ is exactly $N$. But $M = 0$ implies $N = X$, and vice versa.

To end this paper, we give a characterization of completely reducible compact operators.

**Corollary 12.** Every compact, completely reducible operator $K \in \mathcal{L}(X)$ is a scalar type spectral operator.

**Proof.** It suffices to note that $\text{Ker } K \oplus \text{cl } K(X) = X$ [1].

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**References**

