COUNTABLY GENERATED DOUGLAS ALGEBRAS

KEIJI IZUCHI

ABSTRACT. Under a certain assumption of $f$ and $g$ in $L^\infty$ which is considered by Sarason, a strong separation theorem is proved. This is available to study a Douglas algebra $[H^\infty, f]$ generated by $H^\infty$ and $f$. It is proved that (1) ball($B/H^\infty + C$) does not have exposed points for every Douglas algebra $B$, (2) Sarason's three functions problem is solved affirmatively, (3) some characterization of $f$ for which $[H^\infty, f]$ is singly generated, and (4) the $M$-ideal conjecture for Douglas algebras is not true.

Let $H^\infty$ be the space of bounded analytic functions on the unit disk. A uniformly closed subalgebra between $H^\infty$ and $L^\infty$ is called a Douglas algebra. By Chang-Marshall's theorem [3, 19], a Douglas algebra is generated by $H^\infty$ and complex conjugates of some inner functions. A Douglas algebra is called singly (countably, respectively) generated if it is generated by $H^\infty$ and a complex conjugate of only one (countably many) inner function(s). In this paper, we investigate a Douglas algebra $[H^\infty, f]$ which is generated by $H^\infty$ and $f$ in $L^\infty$. By Chang-Marshall's theorem, it is easy to see that $[H^\infty, f]$ is countably generated. To study $[H^\infty, f]$, we have to study the behavior of $f$ on $M(L^\infty)$. Let $N(f)$ equal the closure of

$$\bigcup \{\text{supp } \mu_x; x \in M(H^\infty + C) \text{ and } f|\text{supp } \mu_x \not\in H^\infty|\text{supp } \mu_x\};$$

roughly speaking $N(f)$ is a subset of $M(L^\infty)$ on which $f$ is not analytic. Properties of $N(f)$ play important roles in studying Douglas algebras. $N(f)$, especially $N(I)$ where $I$ is an inner function, is studied in [13].

The key theorem (Theorem 2.1 given in §2) is that if either $f|\text{supp } \mu_x$ or $g|\text{supp } \mu_x$ belongs to $H^\infty|\text{supp } \mu_x$ for every $x \in M(H^\infty + C)$ then $N(f) \cap N(g) = \emptyset$. When $f$ and $g$ are inner functions, this fact is already proved in [13]. The above assumption is considered by Sarason [22], and he showed that either $f|Q$ or $g|Q$ belongs to $H^\infty|Q$ for every QC-level set $Q$ under the above assumption. Our theorem with Corollary 2.1 gives more striking separation than Sarason's. Using our separation theorem, we study singly or countably generated Douglas algebras. In [14], the author showed that a Douglas algebra $B$ is singly generated if and only if ball($B/H^\infty + C$) has extreme points. In §3, we shall give also a geometrical characterization of countably generated Douglas algebras. And we shall show that ball($B/H^\infty + C$) does not have exposed points for every Douglas algebra $B$. In [22, p. 471], Sarason proposed a problem that the above mentioned Sarason theorem is still true for three functions. In §4, we shall give an affirmative answer. In §5, we study a special sequence of QC-level sets which will be called strongly discrete. Using a property of such a sequence, given in Theorem 5.1, we shall prove...
a theorem which is more precise than Gorkin’s given in [8, Theorem 2.1]. In §6, we shall give equivalent conditions on \( f \) for which \([H^\infty, f]\) is singly generated. This answers Marshall’s problem given in [19]. In §7, we shall give a negative answer of the \( M \)-ideal conjecture [18].

1. Preliminaries. Let \( A \) be a uniformly closed subalgebra of \( C(K) \), the space of continuous functions on a compact Hausdorff space \( K \). We denote by \( M(A) \) the maximal ideal space of \( A \) equipped with the weak*-topology and by \( \partial A \) the Shilov boundary for \( A \). For \( f \in C(\partial A) \), ||f|| means the supremum norm of \( f \) and \( \bar{f} \) means the complex conjugate of \( f \). A closed subset \( E \) of \( \partial A \) is called a peak set for \( A \) if there is a function \( f \) in \( A \), which is called a peaking function for \( E \), such that \( ||f|| = 1 \) and \( E = \{ x \in \partial A; |f(x)| = 1 \} = \{ x \in \partial A; f(x) = 1 \} \). If \( E \) is an intersection of peak sets, it is called a weak peak set for \( A \). A measure \( \mu \) on \( \partial A \) is called an annihilating measure for \( A \), if \( \int_{\partial A} f d\mu = 0 \) for every \( f \in A \).

For a point \( x \), \( A \) is called a QC-level set.

Let \( D \) be the open unit disk. Let \( L^\infty \) be the space of bounded measurable functions on \( \partial D \) with respect to the normalized Lebesgue measure \( d\theta/2\pi \). We identify a function \( f \) in \( H^\infty \) with its boundary function. Then \( H^\infty \) is an essentially uniformly closed subalgebra of \( L^\infty \). \( H^\infty + C \) is the smallest Douglas algebra which contains \( H^\infty \) properly [21], where \( C \) is the space of continuous functions on \( \partial D \). We put \( X = M(L^\infty) \), then \( X \) may be identified with \( \partial H^\infty \). We note that \( M(H^\infty + C) = M(H^\infty) \setminus D \), and \( D \) is weak*-dense in \( M(H^\infty) \) by the corona theorem (see [6]). For a subset \( E \) of \( M(H^\infty) \), we denote by \( c\ell E \) the weak*-closure of \( E \) in \( M(H^\infty) \). For a point \( x \) in \( M(H^\infty) \), we denote by \( \mu_x \) the unique representing measure for \( x \), and by \( \text{supp } \mu_x \) the closed support set for \( \mu_x \). \( \text{supp } \mu_x \) is a weak peak set for \( H^\infty \) [10, p. 207], and it is easy to see that \( H^\infty |\text{supp } \mu_x \) does not contain any nonconstant real functions. By Sarason [20],

\[ H^\infty + C = \{ f \in L^\infty; f |\text{supp } \mu_x \in H^\infty |\text{supp } \mu_x \text{ for every } x \in M(H^\infty + C) \} \]

We use the notation \( m \) for the representing measure for the point 0 in \( D \), that is,

\[ \int_X f dm = \int_{\partial D} f d\theta/2\pi \text{ for every } f \in H^\infty \]

For \( f \in L^\infty \) and a Douglas algebra \( B \), we put \( ||f + B|| = \inf \{|||f + h||; h \in B\} \), the quotient norm of \( L^\infty /B \). For a subset \( E \) of \( L^\infty \), we denote by \( [E] \) the uniformly closed subalgebra generated by \( E \).

Put \( QC = (H^\infty + C) \cap (H^\infty + C) \) and \( QA = H^\infty \cap QC \). By [20],

\[ QC = \{ f \in L^\infty; f |\text{supp } \mu_x \text{ is constant for every } x \in M(H^\infty + C) \} \]

Then \( QC \) is a \( C^* \)-subalgebra of \( L^\infty \). Hence there is a continuous onto map \( \pi: X \to M(QC); f(\pi(x)) = f(x) \) for every \( f \in QC \). A closed subset \( \pi^{-1}(y), y \in M(QC) \), is called a \( QC \)-level set. A \( QC \)-level set is a weak peak set for \( QA \). For \( x \in M(H^\infty + C) \), there is a unique \( QC \)-level set \( Q_x \) such that \( Q_x \supset \text{supp } \mu_x \). We denote by \( m_0 \) the probability measure on \( M(QC) \) such that \( \int_{M(QC)} f dm_0 = \int_{\partial D} f d\theta/2\pi \) for every \( f \in QA \). Then we have \( m(\pi^{-1}(E)) = m_0(E) \) for measurable subsets \( E \) of \( M(QC) \).

For \( f \in L^\infty \), we put

\[ N(f) = \text{the closure of } \bigcup \{ \text{supp } \mu_x; f |\text{supp } \mu_x \not\in H^\infty |\text{supp } \mu_x \} \]

and

\[ Q(f) = \bigcup \{ \pi^{-1}(y); f |\pi^{-1}(y) \not\in H^\infty |\pi^{-1}(y), y \in M(QC) \} \]

Generally \( Q(f) \) is not a closed subset of \( X \).
A Blaschke product with zeros \( \{z_n\}_{n=1}^{\infty} \) in \( D \) is a function of the form

\[
b(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{\bar{z}_n - z_n} \frac{z_n - \bar{z}_n}{1 - z_n z_n}
\]

for \( z \in D \), where \( \sum_{n=1}^{\infty} 1 - |z_n| < \infty \). If \( \{z_n\}_{n=1}^{\infty} \) satisfies moreover

\[
\inf \prod_{n \neq m} \frac{|z_n - z_m|}{1 - z_m z_n} > 0 \quad \left( \lim_{n \to \infty} \prod_{n \neq m} \frac{|z_n - z_m|}{1 - z_m z_n} = 1 \right)
\]

then \( \{z_n\}_{n=1}^{\infty} \) is called interpolating (sparse), and \( b(z) \) is called an interpolating (sparse) Blaschke product. These Blaschke products are inner functions, where a function \( I \in H^\infty \) with \( |I| = 1 \) on \( X \) is called inner. If \( I \) is an inner function, put \( Z(I) = \{x \in M(H^\infty + C): I(x) = 0 \} \). Then \( N(\bar{I}) = Q(\bar{I}) = \bigcup \{Q_x; x \in Z(I)\} \) [13, Theorem 1]. If \( b \) is an interpolating Blaschke product with zeros \( \{z_n\}_{n=1}^{\infty} \), then \( \text{cl}\{z_n\}_{n=1}^{\infty} \) is homeomorphic to the Stone-Čech compactification of \( \{z_n\}_{n=1}^{\infty} \) and \( Z(b) = \text{cl}\{z_n\}_{n=1}^{\infty} \setminus \{z_n\}_{n=1}^{\infty} \) [10, p. 205].

Let \( Y \) be a Banach space. We denote by ball \( \text{ball} Y \) the closed unit ball of \( Y \). A point \( y \) in ball \( \text{ball} Y \) is called extreme if \( \|y + x\| \leq 1, \ x \in Y, \) implies \( x = 0 \). A point \( y \) in ball \( \text{ball} Y \) is called exposed if there is a bounded linear functional \( \psi \) of \( \text{ball} Y \) such that \( \|\psi\| = 1, \ \psi(y) = 1 \) and \( \psi(x) \neq 1 \) for every \( x \in \text{ball} Y \) with \( x \neq y \). We note that an exposed point is extreme. A closed subspace \( Z \) of \( Y \) is called an \( M \)-ideal of \( Y \) if there is a projection \( P \) from \( Y^* \), the dual space of \( Y \), onto the annihilating subspace of \( Z \) in \( Y^* \), \( \{\psi \in Y^*; f = 0 \text{ on } Z\} \), such that \( \|x\| = \|Px\| + \|x - Px\| \) for every \( x \) in \( Y^* \).

2. The main theorem. In this section, we shall show the following theorem and give its applications.

**Theorem 2.1.** Let \( f \) and \( g \) be functions in \( L^\infty \). If for every \( x \in M(H^\infty + C) \) either \( f|_{\text{supp } \mu_x} \in H^\infty|_{\text{supp } \mu_x} \) or \( g|_{\text{supp } \mu_x} \in H^\infty|_{\text{supp } \mu_x} \), then \( N(f) \cap N(g) = \emptyset \).

To show Theorem 2.1, we need some lemmas.

**Lemma 2.1** [24]. For an inner function \( I \), there is an interpolating Blaschke product \( b \) such that \( [H^\infty, b] = [H^\infty, I] \).

**Lemma 2.2.** Let \( B \) be a Douglas algebra. Then the following assertions are equivalent.

(i) There is a function \( f \) in \( L^\infty \) with \( B = [H^\infty, f] \).

(ii) There is a sequence of interpolating Blaschke products \( \{I_n\}_{n=1}^{\infty} \) with \( B = [H^\infty, \{I_n\}_{n=1}^{\infty}] \).

**Proof.** Let \( f \in L^\infty \) with \( B = [H^\infty, f] \). By Chang-Marshall’s theorem, there is a sequence of inner functions \( \{I_n\}_{n=1}^{\infty} \) such that \( I_n \in [H^\infty, f] \) and \( \|I_n f + H^\infty\| \to 0 \) \( (n \to \infty) \). Then \( [H^\infty, f] \subset [H^\infty, \{I_n\}_{n=1}^{\infty}] \subset [H^\infty, f] \), so \( [H^\infty, f] = [H^\infty, \{I_n\}_{n=1}^{\infty}] \). By Lemma 2.1, we may take \( I_n \) as an interpolating Blaschke product. Conversely suppose that \( B = [H^\infty, \{I_n\}_{n=1}^{\infty}] \) for a sequence of inner functions \( \{I_n\} \). We put \( f = \sum_{n=1}^{\infty} |I_n| + 1/3^n \). If \( I_n|_{\text{supp } \mu_x} \in M(H^\infty) \), is not constant, then
\( I_n(\supp \mu_x) = \partial D \). Hence \( f|\supp \mu_x \) is constant if and only if \( I_n|\supp \mu_x \) is constant for every \( n \). Since real functions in \( H^\infty|\supp \mu_x \) are constant functions for each \( x \in M(H^\infty) \), \( M([H^\infty, f]) = M([H^\infty, \{ I_n \}_{n=1}^\infty]) \). By Chang-Marshall’s theorem, \( f \) is the desired function.

The following lemma is a special case of Theorem 2.1 proved in [13, Corollary 3].

**Lemma 2.3.** Let \( I \) and \( J \) be inner functions. If for every point \( x \) in \( M(H^\infty+C) \) either \( \tilde{I}|\supp \mu_x \in H^\infty|\supp \mu_x \) or \( \tilde{J}|\supp \mu_x \in H^\infty|\supp \mu_x \), then \( N(\tilde{I}) \cap N(\tilde{J}) = \emptyset \).

**Lemma 2.4.** Let \( I \) be an interpolating Blaschke product. Let \( E \) be a closed subset of \( D \) such that \( \partial E \notin E \subset \{ x \in M(H^\infty+C); |I(x)| = 1 \} \). Then for each \( \varepsilon \) with \( 0 < \varepsilon < 1 \), there is an interpolating Blaschke product \( b \) satisfying that \( |b| \geq \varepsilon \) on \( E \).

**Proof.** Let \( \{ z_n \}_{n=1}^\infty \) be the zero sequence of \( I \). We denote by \( I_k \) the interpolating Blaschke product with zeros \( \{ z_{n,k} \}_{n=k}^\infty \). By our assumption, there exists a constant \( r \) such that \( 0 < r < 1 \) and \( |I| \geq \varepsilon \) on \( \{ z \in E; |z| > r \} \). Since \( |I_k| \to 1 \) \((k \to \infty)\) uniformly on each compact subset of \( D \), \( |I_k| \geq \varepsilon \) on \( \{ z \in E; |z| \leq r \} \) for sufficiently large \( k \). Put \( b = I_k \), then \( b \) satisfies our assertion.

The following is a key lemma to prove Theorem 2.1.

**Lemma 2.5.** Let \( \{ I_n \}_{n=1}^\infty \) be a sequence of interpolating Blaschke products such that \( \prod_{n=1}^\infty I_n \) is a Blaschke product. Let \( g \) be a function in \( L^\infty \). Suppose that for every \( x \in M(H^\infty+C) \) either \( g|\supp \mu_x \in H^\infty|\supp \mu_x \) or \( \tilde{I}_n|\supp \mu_x \in H^\infty|\supp \mu_x \) for all \( n \). Then there exists a Blaschke product \( I \) such that

(i) \( (\prod_{n=1}^\infty I_n)|\tilde{I} \in H^\infty \); consequently \( N(\tilde{I}) \subset N(\prod_{n=1}^\infty I_n) \);

(ii) either \( \tilde{I}|\supp \mu_x \in H^\infty|\supp \mu_x \) or \( g|\supp \mu_x \in H^\infty|\supp \mu_x \) for every \( x \in M(H^\infty+C) \); and

(iii) \( N(I_n) \subset N(\tilde{I}) \) for all \( n \).

**Proof.** By Lemma 2.2, there is a sequence of interpolating Blaschke products \( \{ J_m \}_{m=1}^\infty \) such that

\[
[H^\infty, g] = [H^\infty, \{ J_m \}_{m=1}^\infty].
\]

By our assumption, for every \( x \in M(H^\infty+C) \), either \( \tilde{I}_n|\supp \mu_x \in H^\infty|\supp \mu_x \) for all \( n \) or \( \tilde{J}_m|\supp \mu_x \in H^\infty|\supp \mu_x \) for all \( m \). By Lemma 2.3,

\[
N(\tilde{I}_n) \cap N(\tilde{J}_m) = \emptyset \quad \text{for every } n \text{ and } m.
\]

Let \( \{ z_{n,k} \}_{k=1}^\infty \) be the zero sequence of \( I_n \). Put \( I_0 = \prod_{n=1}^\infty I_n \). Since \( I_0 \) is a Blaschke product, we have

\[
\sum_{n=1}^\infty \sum_{k=1}^\infty (1 - |z_{n,k}|) < \infty.
\]

For each \( m \), we put

\[
U_{m,i} = \{ z \in D; |J_m(z)| \leq 1 - 1/i, \quad |z| \geq 1 - 1/i \}
\]

for \( i = 1, 2, \ldots \). Then \( U_{m,i} \) is a closed subset of \( D \). By (2) and (4), \( I_n (= I) \) and \( U_{m,i} (= E) \) satisfy the assumptions of Lemma 2.4. Because, if there is \( x \)
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in \( c \{ U_{m,i} \setminus U_{m,i} \} \) with \(|I_n(x)| \neq 1\), then \(|J_m(x)| \leq 1 - 1/i\), so we get \( \supp \mu_x \subset N(\overline{T_n}) \cap N(\overline{T_m}) \).

First we shall work on \( J_1 \), and we shall find a sequence of interpolating Blaschke products \( \{ b_{1,n} \}_{n=1}^{\infty} \) satisfying the following two conditions by induction.

5) \( I_n \bar{b}_{1,n} \) is a finite Blaschke product, and

6) \( \inf \{ |b_{1,i}b_{1,i+1} \cdots b_{1,n}(z)| ; z \in U_{1,i} \} > 1 - 1/i \) for \( 1 \leq i \leq n \).

Applying Lemma 2.4 for \( I_1 \) and \( U_{1,1} \), there is an interpolating Blaschke product \( b_{1,1} \) such that \( I_1 \bar{b}_{1,1} \) is a finite Blaschke product and \( \inf \{ |b_{1,1}(z)| ; z \in U_{1,1} \} > 0 \). Suppose that \( \{ b_{1,1}, b_{1,2}, \ldots, b_{1,N} \} \) satisfies (5) and (6) for \( 1 \leq i \leq n \leq N \). For \( 1 \leq i \leq N \), we put

\[ c(N, i) = \inf \{ |b_{1,i}b_{1,i+1} \cdots b_{1,N}(z)| ; z \in U_{1,i} \}. \]

By (6), \( c(N, i) > 1 - 1/i \). Also we put

\[ E = \bigcup \{ U_{1,i} ; 1 \leq i \leq N + 1 \}, \]

then \( I = I_{N+1} \) and \( E \) satisfy the assumptions of Lemma 2.4. Let \( \varepsilon \) be a constant satisfying

\[ 1 > \varepsilon > \max \left\{ 1 - \frac{1}{N+1}, \frac{1 - 1/i}{c(N, i)} ; 1 \leq i \leq N \right\}. \]

By Lemma 2.4, there is an interpolating Blaschke product \( b_{1,N+1} \) such that \( I_{N+1} \bar{b}_{1,N+1} \) is a finite Blaschke product and

\[ |b_{1,N+1}| \geq \varepsilon \quad \text{on } E. \]

Thus we get the following inequalities.

For \( 1 \leq i < N + 1 \);

\[ \inf \{ |b_{1,i}b_{1,i+1} \cdots b_{1,N+1}(z)| ; z \in U_{1,i} \} \]

\[ \geq \inf \{ |b_{1,i}b_{1,i+1} \cdots b_{1,N}(z)| ; z \in U_{1,i} \} \inf \{ |b_{1,N+1}(z)| ; z \in U_{1,i} \} \]

\[ > c(N, i) \varepsilon \quad \text{by (7), (8) and (10)} \]

\[ > 1 - 1/i \quad \text{by (9)}. \]

For \( i = N + 1 \);

\[ \inf \{ |b_{1,N+1}(z)| ; z \in U_{1,N+1} \} \geq \varepsilon > 1 - 1/N + 1 \quad \text{by (8), (9) and (10)}. \]

Consequently \( \{ b_{1,1}, b_{1,2}, \ldots, b_{1,N+1} \} \) satisfies (5) and (6). This completes the construction of \( \{ b_{1,n} \}_{n=1}^{\infty} \).

In the above proof, we use only the fact \( N(\overline{J_1}) \cap N(\overline{T_n}) = \emptyset \) for \( n = 1, 2, \ldots \). By (2) and (5), we have \( N(\overline{J_2}) \cap N(\overline{b_{1,n}}) = \emptyset \). So we can repeat the above argument for \( J_2 \) and \( \{ b_{1,n} \}_{n=2}^{\infty} \), we remark that \( n \) starts from 2. Then there is a sequence of interpolating Blaschke products \( \{ b_{2,n} \}_{n=2}^{\infty} \) such that \( b_{1,n} \bar{b}_{2,n} \) is a finite Blaschke product for \( n \geq 2 \) and

\[ \inf \{ |b_{2,i}b_{2,i+1} \cdots b_{2,n}(z)| ; z \in U_{2,i} \} > 1 - 1/i \quad \text{for } 2 \leq i \leq n. \]

Repeating the above argument several times, for each \( m \) there is a sequence of interpolating Blaschke products \( \{ b_{m,n} \}_{n=m}^{\infty} \) such that

\[ b_{m,n} \bar{b}_{m+1,n} \] is a finite Blaschke product for \( m + 1 \leq n \),
and
\begin{equation}
\inf\{b_{m,i}b_{m,i+1}\ldots b_{m,n}(z) ; z \in U_{m,i}\} > 1 - 1/i \quad \text{for } m \leq i \leq n.
\end{equation}

We put \( I = \prod_{n=1}^{\infty} b_{n,n} \). By (3) and (11), \( I \) is a Blaschke product and \( I_0I \in H^\infty \), so we get (i). We shall prove that \( I \) satisfies (ii) and (iii).

To prove (ii), let \( x \in M(H^\infty + C) \) with \( |\text{supp } \mu_x| \notin H^\infty |\text{supp } \mu_x| \). We shall prove \( |\text{supp } \mu_x| \in H^\infty |\text{supp } \mu_x| \), that is, \( |J_m(x)| < 1 \). Take a positive integer \( i_0 \) with \( m \leq i_0 \) and
\begin{equation}
|J_m(x)| < 1 - 1/i_0.
\end{equation}

Let \( i \geq i_0 \). By (4), (13) and the corona theorem, \( x \in \text{cl } U_{m,i_0} \setminus U_{m,i_0} \). Since \( |b_{m,n}| \leq |b_{n,n}| \) on \( D \) for \( m \leq n \) by (11), we have
\[
\inf\left\{ \prod_{n=i}^{\infty} b_{n,n}(z) ; z \in U_{m,i} \right\} \geq \inf\left\{ \prod_{n=i}^{\infty} b_{n,n}(z) ; z \in U_{m,i} \right\} \\
\geq 1 - 1/i \quad \text{by (12)}.
\]

Then
\[
\left| \prod_{n=i}^{\infty} b_{n,n} \right| \geq 1 - \frac{1}{i} \quad \text{on } \text{cl } U_{m,i} \setminus U_{m,i}.
\]

By (5) and (11), \( |I_n| = |b_{n,n}| \) on \( M(H^\infty + C) \) for \( n = 1, 2, \ldots \). By (2) and (4), \( |I_n| = 1 \) on \( \text{cl } U_{m,i} \setminus U_{m,i} \) for \( n = 1, 2, \ldots \). Thus
\[
|I| = \left| \prod_{n=1}^{\infty} b_{n,n} \right| \geq 1 - \frac{1}{i} \quad \text{on } \text{cl } U_{m,i} \setminus U_{m,i}.
\]

Since \( \text{cl } U_{m,i} \setminus U_{m,i} \subset \text{cl } U_{m,j} \setminus U_{m,j} \) for \( i \leq j \) by (4), we get
\[
|I| \geq 1 - 1/i \quad \text{on } \text{cl } U_{m,i_0} \setminus U_{m,i_0} \quad \text{for every } i \geq i_0.
\]

Thus \( |I| = 1 \) on \( \text{cl } U_{m,i_0} \setminus U_{m,i_0} \). Since \( x \in \text{cl } U_{m,i_0} \setminus U_{m,i_0} \), \( |I(x)| = 1 \). Hence \( I \) is constant on \( \text{supp } \mu_x \), and \( |\text{supp } \mu_x| \in H^\infty |\text{supp } \mu_x| \). This completes the proof of (ii).

Since \( |I_n| = |b_{n,n}| \) on \( M(H^\infty + C) \) for each \( n \),
\[
|I| = \left| \prod_{n=1}^{\infty} b_{n,n} \right| \leq |b_{n,n}| = |I_n| \quad \text{on } M(H^\infty + C).
\]

Thus we get \( N(I) \supset N(I_n) \). This completes the proof.

**Proof of Theorem 2.1.** Let \( f \) and \( g \) be functions in \( L^\infty \) such that for every \( x \in M(H^\infty + C) \) either \( f|\text{supp } \mu_x| \in H^\infty |\text{supp } \mu_x| \) or \( g|\text{supp } \mu_x| \in H^\infty |\text{supp } \mu_x| \). We shall show the existence of a Blaschke product \( I \) such that
\[
\text{(a)} \quad \text{either } |I| \text{supp } \mu_x \in H^\infty |\text{supp } \mu_x| \text{ or } |I| \text{supp } \mu_x \in H^\infty |\text{supp } \mu_x| \text{ for every } x \in M(H^\infty + C),
\]
\[
\text{(b)} \quad N(I) \supset N(f).
\]

If the above fact is proved, applying it again, we get a Blaschke product \( J \) such that
\[
\text{(a')} \quad |J| \text{supp } \mu_x \in H^\infty |\text{supp } \mu_x| \text{ or } |J| \text{supp } \mu_x \in H^\infty |\text{supp } \mu_x| \text{ for every } x \in M(H^\infty + C),
\]
\[
\text{(b')} \quad N(J) \supset N(g).
\]
Then by Lemma 2.3, $N(\bar{T}) \cap N(J) = \emptyset$, so we get our assertion.

Using Lemma 2.5, we shall show the existence of a Blaschke product $I$ satisfying (a) and (b). By Lemma 2.2, there is a sequence of interpolating Blaschke products $\{I_n\}_{n=1}^{\infty}$ such that

$$[H^\infty, f] = [H^\infty, \{I_n\}_{n=1}^{\infty}].$$

We note that if $f|\text{supp} \mu_x \in H^\infty|\text{supp} \mu_x$ for some $x \in M(H^\infty + C)$, then we get $\bar{T}_n|\text{supp} \mu_x \in H^\infty|\text{supp} \mu_x$ for all $n$. Let $\{z_{n,k}\}_{k=1}^{\infty}$ be the zero sequence of $I_n$. Replacing $I_n$ by $I'_n$ such that $I_n I'_n$ is a finite Blaschke product, we may assume that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (1 - |z_{n,k}|) < \infty.$$

Then $\prod_{n=1}^{\infty} I_n$ is a Blaschke product. By our assumption, $\{I_n\}_{n=1}^{\infty}$ and $g$ satisfy the assumptions of Lemma 2.5. Hence there is a Blaschke product $I$ satisfying (a) and $N(\bar{T}) \supset N(\bar{T}_n)$ for all $n$. Since $N(f)$ coincides with the closure of $\bigcup\{N(\bar{T}_n); n = 1, 2, \ldots\}$, we get (b). This completes the proof.

To prove the corollaries, we give two lemmas.

**LEMMA 2.6** (Sarason's unpublished result, see [8, Theorem 2.8]). Let $f \in L^\infty$ with $f^2 = f$, and let $Q$ be a QC-level set. If $f|Q \in H^\infty|Q$, then $f|Q$ is a constant.

**LEMMA 2.7.** Let $b$ be a sparse Blaschke product with zeros $\{w_n\}_{n=1}^{\infty}$ and $I$ be an inner function. Then $N(b) \cap N(\bar{T}) = \emptyset$ if and only if $|I(w_n)| \rightarrow 1$ ($n \rightarrow \infty$).

**PROOF.** Suppose $N(\bar{b}) \cap N(\bar{T}) = \emptyset$. Then $|I| = 1$ on $Z(b)$. Since $Z(b) = \text{cl}\{w_n\}_{n=1}^{\infty} \setminus \{w_n\}_{n=1}^{\infty}$, $|I(w_n)| \rightarrow 1$ ($n \rightarrow \infty$). Next suppose that $|I(w_n)| \rightarrow 1$ ($n \rightarrow \infty$). Then $|I| = 1$ on $Z(b)$. Let $x \in M(H^\infty + C)$ with $|b(x)| < 1$. Then there is a point $x_0$ in $Z(b)$ with $\text{supp} \mu_{x_0} = \text{supp} \mu_x$ by the proof of Lemma 1 in [9]. Since $|I(x_0)| = 1$, we have $|I(x)| = 1$. Thus

$$\{x \in M(H^\infty + C); |I(x)| < 1\} \cap \{y \in M(H^\infty + C); |b(y)| < 1\} = \emptyset.$$

By Lemma 2.3, we have $N(\bar{b}) \cap N(\bar{T}) = \emptyset$.

The following corollary shows that $N(f)$ consists of QC-level sets, which is a generalization of Theorem 1 in [13].

**COROLLARY 2.1.** For $f \in L^\infty$, $N(f) = \pi^{-1}(\pi(N(f)))$ and $N(f)$ is a weak peak set for QA.

**PROOF.** The inclusion $N(f) \subset \pi^{-1}(\pi(N(f)))$ is trivial. Suppose that $N(f) \subsetneq \pi^{-1}(\pi(N(f)))$. Then there is a QC-level set $Q$ with $N(f) \cap Q \neq \emptyset$ and $Q \notin N(f)$. Take an open and closed subset $U$ of $X$ with $U \cap N(f) = \emptyset$ and $U \cap Q \neq \emptyset$. Then $f$ and $\chi_U$, the characteristic function of $U$, satisfy the assumption of Theorem 2.1. Thus $N(f) \cap N(\chi_U) = \emptyset$. By Lemma 2.6, $\chi_U|Q \notin H^\infty|Q$. Since $Q$ is a weak peak set for $H^\infty$, there is $x \in M(H^\infty + C)$ such that $\text{supp} \mu_x \subset Q$ and $\chi_U|\text{supp} \mu_x \notin H^\infty|\text{supp} \mu_x$. Thus $N(\chi_U) \cap Q \neq \emptyset$. But this contradicts $N(f) \cap Q \neq \emptyset$ and $N(f) \cap N(\chi_U) = \emptyset$. Thus $N(f) = \pi^{-1}(\pi(N(f)))$. By Wolff's theorem [23, Theorem 1 and Lemma 2.3] as the proof of Theorem 1 in [13], $N(f)$ is a weak peak set for QA.

The following follows Corollary 2.1.
COROLLARY 2.2. For \( f \in L^\infty \), \( Q(f) \subset N(f) \) and \( \overline{cl\, Q(f)} = N(f) \).

For \( f \in L^\infty \), we put \( Q_0(f) = \bigcup \{ \pi^{-1}(y) ; y \in M(QC) \} \) and \( f|\pi^{-1}(y) \) is not constant.

COROLLARY 2.3. For \( f \in L^\infty \), \( Q(f) \cup Q(\tilde{f}) \subset Q_0(f) \subset N(f) \cup N(\tilde{f}) \).

PROOF. By the definitions, \( Q(f) \cup Q(\tilde{f}) \subset Q_0(f) \). Suppose that \( Q_0(f) \not\subset N(f) \cup N(\tilde{f}) \). By Corollary 2.1, there is a \( QC \)-level set \( Q \) with \( Q \cap (N(f) \cup N(\tilde{f})) = \emptyset \) and \( Q \subset Q_0(f) \). Take a function \( q \) in \( QC \) such that \( q \) is constant on \( Q_0(f) \). By \( 20 \), \( f \in QC \), so \( f \) is constant on \( Q_0(f) \). This fact contradicts \( Q \subset Q_0(f) \).

REMARK. In §6, we will prove that \( Q(f) = N(f) \) if and only if \( [H^\infty, f] \) is singly generated. If \( f \in H^\infty \), \( Q(f) \subset Q_0(f) \subset N(f) \) by Corollary 2.3. Moreover if there is a \( QC \)-level set \( Q \) such that \( f | Q \) is real nonconstant, then \( Q(\tilde{f}) \subset Q_0(\tilde{f}) \), and \( [H^\infty, \tilde{f}] \) is not singly generated.

COROLLARY 2.4. Let \( f \in L^\infty \). If \( I \) is an interpolating Blaschke product with \( N(I) \subset N(f) \), then \( \tilde{I} \in [H^\infty, f] \).

PROOF. Suppose \( \tilde{I} \notin [H^\infty, f] \). Then there is a point \( x_0 \) in \( M([H^\infty, f]) \) with \( I(x_0) = 0 \). Let \( \{w_k\}_{k=1}^\infty \) be the zero sequence of \( I \). Then \( x_0 \in \overline{cl\{w_k\}_{k=1}^\infty} \). By Lemma 2.2, \( [H^\infty, f] = [H^\infty, \{I_n\}_{n=1}^\infty] \) for some sequence of interpolating Blaschke products \( \{I_n\}_{n=1}^\infty \). Since \( |I_n(x_0)| = 1 \), there is a subsequence \( \{w_{k_n}\}_{n=1}^\infty \) of \( \{w_k\}_{k=1}^\infty \) such that \( |I_n(w_{k_n})| \to 1 \) \((k \to \infty)\) for every \( n \). Taking again its subsequence, we may assume that \( \{w_{k_n}\}_{n=1}^\infty \) is a sparse sequence. Let \( b \) be the sparse Blaschke product with zeros \( \{w_{k_n}\}_{n=1}^\infty \). By Lemma 2.7, \( N(b) \cap N(I_n) = \emptyset \) for every \( n \). Hence \( b \) and \( f \) satisfy the assumption of Theorem 2.1. Then \( N(b) \cap N(f) = \emptyset \). This contradicts \( N(I) \subset N(f) \), because \( N(b) \subset N(I) \).

COROLLARY 2.5 (CF. [13, COROLLARY 5]). Let \( f \) and \( g \) be functions in \( L^\infty \). Then \( N(f) \subset N(g) \) if and only if \( [H^\infty, f] \subset [H^\infty, g] \).

PROOF. Suppose \( N(f) \subset N(g) \). Let \( I \) be an interpolating Blaschke product with \( \tilde{I} \in [H^\infty, f] \). Then \( N(\tilde{I}) \subset N(f) \subset N(g) \). By Corollary 2.4, we have \( \tilde{I} \in [H^\infty, g] \). By Chang-Marshall’s theorem, \( [H^\infty, f] \subset [H^\infty, g] \). The converse assertion is trivial.

For a Douglas algebra \( B \), let \( N(B) \) equal the closure of

\[
\bigcup \{ \text{supp} \mu_x ; x \in M(H^\infty + C) \setminus M(B) \}.
\]

We note that \( N([H^\infty, f]) = N(f) \).

COROLLARY 2.6 (CF. [13, COROLLARIES 4 AND 6]). Let \( B \) be a Douglas algebra.

(i) If \( f \in L^\infty \) satisfies \( N(B) \subset N(f) \), then \( B \subset [H^\infty, f] \).

(ii) Let \( f \in B \). Then \( N(f) = N(B) \) if and only if \( B = [H^\infty, f] \). Consequently \( B \) is countably generated if and only if there is \( f \in B \) with \( N(f) = N(B) \).

PROOF. (i) Let \( I \) be an inner function with \( \tilde{I} \in B \). Then \( N(\tilde{I}) \subset N(B) \subset N(f) \). By Corollary 2.4, \( \tilde{I} \in [H^\infty, f] \). Thus \( B \subset [H^\infty, f] \).

(ii) By (i),

\[
N(f) = N(B) \iff B \subset [H^\infty, f] \subset B \iff B = [H^\infty, f].
\]
3. Geometrical properties of quotient spaces of Douglas algebras. In [14], the author showed that a Douglas algebra \( B \) is singly generated if and only if ball\((B/H^\infty + C)\) has extreme points. In this section, we shall prove two theorems as applications of \( \S 2 \). The first one, Theorem 3.1, is a geometrical characterization of countably generated Douglas algebras. In Theorem 3.2, we shall show that there are no exposed points in ball\((B/H^\infty + C)\). This is already proved in [15, Theorem 4] for \( B = [H^\infty, \bar{b}] \), where \( \bar{b} \) is a sparse Blaschke product. To state Theorem 3.1, we define an extreme family.

Let \( Y \) be a Banach space. If a subset \( E \) of ball \( Y \) satisfies the following conditions, we shall call it an extreme family:

(a) \( \|y\| = 1 \) for every \( y \in E \), and

(b) if a point \( y_0 \) in \( Y \) satisfies \( \|y + y_0\| \leq 1 \) for every \( y \in E \), then \( y_0 = 0 \).

By our definition, an extreme family consisting of only one element is an extreme point of ball \( Y \).

**Theorem 3.1.** Let \( B \) be a Douglas algebra with \( B \supseteq H^\infty + C \). Then \( B \) is countably generated if and only if \( B/H^\infty + C \) has an extreme family consisting of countably many elements.

**Lemma 3.1** [13, Theorem 1]. For an inner function \( I \), we have \( N(I) = Q(I) = \bigcup \{Q_x; x \in Z(I)\} \).

**Proof of Theorem 3.1.** First, suppose that \( B = [H^\infty, \{\overline{I_n}\}_{n=1}^\infty] \) for a sequence of interpolating Blaschke products \( \{\overline{I_n}\}_{n=1}^\infty \). It is easy to see

\[ \|\overline{I_n} + H^\infty + C\| = 1. \]

We shall show that \( \{\overline{I_n} + H^\infty + C\}_{n=1}^\infty \) is an extreme family of ball\((B/H^\infty + C)\).

Let \( g \in B \) with

\[ (1) \quad \|\overline{I_n} \pm g + H^\infty + C\| \leq 1 \quad \text{for every } n. \]

By Corollary 2.1 (or see [13, Theorem 1]), \( N(\overline{I_n}) \) is a weak peak set for \( QA \). Then \( B_n = \{f \in L^\infty; f|N(\overline{I_n}) \subseteq H^\infty|N(\overline{I_n})\} \) is a Douglas algebra. By (1), we have \( \|\overline{I_n} \pm g + B_n\| \leq 1 \). By [13, Theorem 3], \( \overline{I_n} + B_n \) is an extreme point of ball\((L^\infty/B_n)\). Thus \( g \in B_n \), that is,

\[ (2) \quad g|N(\overline{I_n}) \subseteq H^\infty|N(\overline{I_n}) \quad \text{for each } n. \]

To show \( g \in H^\infty + C \), let \( x \in M(H^\infty + C) \). If \( |I_n(x)| = 1 \) for every \( n \), then \( x \in M(B) \) and \( g|\supp \mu_x \subseteq H^\infty|\supp \mu_x \). If \( |I_n(x)| < 1 \) for some \( n \), then \( \supp \mu_x \subseteq N(\overline{I_n}) \). By (2), \( g|\supp \mu_x \subseteq H^\infty|\supp \mu_x \). By [20], we get \( g \in H^\infty + C \). Thus \( \{\overline{I_n} + H^\infty + C\}_{n=1}^\infty \) is an extreme family.

Next suppose that \( B \) is not countably generated. Let \( \{f_n\}_{n=1}^\infty \) be a sequence in \( B \) with \( \|f_n + H^\infty + C\| = 1 \). Since \( H^\infty + C \) has the best approximation property [2], we may assume \( \|f_n\| = 1 \). By Lemma 2.2, there is a function \( F \) in \( L^\infty \) such that

\[ (3) \quad [H^\infty, \{f_n\}_{n=1}^\infty] = [H^\infty, F] \subset B. \]

Since \([H^\infty, F]\) is countably generated by Lemma 2.2, there is an interpolating Blaschke product \( I \) with \( \overline{I} \in B \) and \( \overline{I} \notin [H^\infty, F] \). By Corollary 2.4, we have
By Corollary 2.1, there is a QC-level set \( Q \) such that \( Q \cap N(F) = \emptyset \) and \( Q \subset N(\bar{I}) \). Then there is a function \( q \) in QC such that

\[
0 \leq q \leq 1 \quad \text{and} \quad q = 1 \quad \text{on} \quad Q,
\]

\[
q = 0 \quad \text{on} \quad N(F).
\]

By Lemma 3.1, we get \( \bar{I}q \in B \) and \( \bar{I}q \notin H^\infty + C \). By (3) and (5), \( qf_n \in H^\infty + C \). Then

\[
\|f_n \pm \bar{I}q + H^\infty + C\| \leq \|f_n \pm \bar{I}q - qf_n\|
\]

\[
\leq \|1 - q\| + |q| = 1 \quad \text{by} \quad \|f_n\| = 1 \quad \text{and} \quad (4).
\]

Thus \( \{f_n + H^\infty + C\} \) is not an extreme family, and this completes the proof.

To prove Theorem 3.2, we need lemmas.

**Lemma 3.2.** Let \( f \in L^\infty \) and \( f \notin H^\infty + C \). Then \( N(f) \) contains uncountably many QC-level sets.

**Proof.** By Chang-Marshall’s theorem, there is an interpolating Blaschke product \( I \) with \( \bar{I} \in [H^\infty, f] \). Then \( N(\bar{I}) \subset N(f) \). Let \( \{z_n\}_{n=1}^\infty \) be the zero sequence of \( I \). Take a sparse subsequence \( \{w_n\}_{n=1}^\infty \) of \( \{z_n\}_{n=1}^\infty \), and let \( b \) be the sparse Blaschke product with zeros \( \{w_n\}_{n=1}^\infty \). Then \( Z(b) \subset Z(\bar{I}) \). By [13, Lemma 5], \( Q_x \neq Q_y \) for \( x, y \in Z(\bar{I}) \) and \( x \neq y \). Since \( Z(b) = \text{cl}\{w_n\}_{n=1}^\infty \setminus \{w_n\}_{n=1}^\infty \) and \( \text{cl}\{w_n\}_{n=1}^\infty \) is homeomorphic to the Stone-Čech compactification of \( \{w_n\}_{n=1}^\infty \), \( Z(b) \) is an uncountable set.

The following lemma is a key to prove Theorem 3.2.

**Lemma 3.3.** Let \( I \) be an interpolating Blaschke product. Let \( \mu \) be a probability measure on \( N(\bar{I}) \). Then \( \supp \mu \subset \subset N(\bar{I}) \), and there is a sparse Blaschke product \( b \) such that \( Ib \in H^\infty \) and \( N(b) \subset N(\bar{I}) \setminus \supp \mu \).

**Proof.** By Lemma 3.2, \( N(\bar{I}) \) contains uncountably many QC-level sets. Then there is a QC-level set \( Q \) such that \( Q \subset N(\bar{I}) \) and \( \mu(Q) = 0 \). Since \( Q \) is a weak peak set for \( QA \), there is a peak set \( E \) for \( QA \) such that

\[
Q \subset E \subset X \quad \text{and} \quad \mu(E) = 0.
\]

Let \( f \) be a peaking function in \( QA \) for \( E \), that is,

\[
f = 1 \quad \text{on} \quad E \quad \text{and} \quad \|f\| < 1 \quad \text{on} \quad X \setminus E.
\]

We put

\[
K_n = \{x \in X; |f(x)| \leq 1 - 1/n\}.
\]

Then

\[
\mu(K_n \cap N(\bar{I})) = \mu(K_n) \to 1 \quad \text{as} \quad n \to \infty.
\]

By Lemma 3.1, there is a point \( x_0 \in Z(I) \) such that \( Q = Q_{x_0} \). Take an open and closed subset \( U_n \) of \( Z(I) \) such that

\[
\{x \in Z(I); |f(x)| \leq 1 - 1/n\} \subset U_n \subset \{x \in Z(I); |f(x)| \leq 1 - 1/n + 1\}.
\]

Then \( U_n \cup \{Q_x; x \in U_n\} \subset K_{n+1} \), because \( f \in QA \) is constant on each QC-level set. Since \( U_n \) is an open and closed subset of \( Z(I) \), there is an interpolating Blaschke product \( I_n \) with \( \bar{I}I_n \in H^\infty \) and \( Z(I_n) = U_n \) [12, Corollary 1]. By Lemma 3.1,

\[
N(\bar{I}_n) \subset K_{n+1}.
\]
Moreover we have

\[ K_n \cap N(\bar{I}) \subset N(\bar{I}_n). \]

To show (6), let \( y \in K_n \cap N(\bar{I}). \) By Lemma 3.1, there is a point \( x_1 \in Z(I) \) such that \( y \in Q_{x_1}. \) Since \( |f(y)| < 1 - 1/n, \) \( |f(x_1)| < 1 - 1/n. \) Thus \( x_1 \in U_n \) and \( y \in N(\bar{I}_n). \) Since \( Q_{x_0} \cap N(\bar{I}_n) = \emptyset \) by (1), (2), (3) and (5), we have \( |I_n(x_0)| = 1. \) Since \( I(x_0) = 0, I \not\in [H^\infty, \{\bar{I}_n\}]_{n=1}. \) By the proof of Corollary 2.4, there is a sparse Blaschke product \( b \) such that

\[ I_b \in H^\infty \quad \text{and} \quad N(b) \cap \text{cl} \left( \bigcup \{N(\bar{I}_n); n = 1, 2, \ldots \} \right) = \emptyset. \]

Since \( I_b \in H^\infty, N(b) \subset N(\bar{I}). \) By equations (4) and (6), we have \( \text{supp } \mu \subset \text{cl} \left( \bigcup \{N(\bar{I}_n); n = 1, 2, \ldots \} \right). \) Thus we get our assertions.

**Theorem 3.2.** Let \( B \) be a Douglas algebra with \( B \supsetneq H^\infty + C. \) Then there are no exposed points in \( \text{ball}(B/H^\infty + C). \)

**Proof.** By [14] and Lemma 2.1, we may assume \( B = [H^\infty, \{\bar{I}_n\}] \) for some interpolating Blaschke product \( I. \) Let \( f \in B \) with \( \|f + H^\infty + C\| = 1. \) Since \( H^\infty + C \) has the best approximation property [2], we may assume \( \|f\| = 1. \) Let \( \mu \) be a measure on \( X \) such that \( \|\mu\| = 1, \mu \perp H^\infty + C, \) and \( \int_X f d\mu = 1. \) By [13, Lemma 9], \( \text{supp } \mu \subset N(\bar{I}). \) By Lemmas 3.1 and 3.3, there is a QC-level set \( Q \) with \( Q \subset N(\bar{I}) \) and \( Q \cap \text{supp } \mu = \emptyset. \) This fact is the key point to prove a special case of Theorem 3.2 [13, Theorem 3]. We can go the same way as in [13], and we can show the existence of \( g \) in \( B \) such that \( \|g + H^\infty + C\| = 1, \int_X g d\mu = 1 \) and \( f + H^\infty + C \neq g + H^\infty + C. \) This completes the proof.

### 4. Sarason’s three functions problem

In [22], Sarason showed that if \( f \) and \( g \) in \( L^\infty \) satisfy \( f|\text{supp } \mu_x \in H^\infty |\text{supp } \mu_x \) or \( g|\text{supp } \mu_x \in H^\infty |\text{supp } \mu_x \) for every \( x \in M(H^\infty + C), \) then \( f|Q \in H^\infty |Q \) or \( g|Q \in H^\infty |Q \) for every QC-level set \( Q. \) The following problem occurs from the above fact [22]; is it still true for three functions in \( L^\infty ? \) In this section, we shall show

**Theorem 4.1.** Let \( \{f_n\}_{n=1}^N \) be a finite subset of \( L^\infty. \) Suppose that for each point \( x \in M(H^\infty + C), \) there exists \( n \) such that \( f_n| \text{supp } \mu_x \in H^\infty |\text{supp } \mu_x. \) Then \( \bigcap_{n=1}^N N(f_n) = \emptyset. \)

We note that Corollary 2.1 and Theorem 4.1 give an affirmative answer for the above problem. To show Theorem 4.1, we need some lemmas.

**Lemma 4.1.** Let \( B \) be a Douglas algebra. Then \( B \) is countably generated if and only if \( M(B) \) is a \( G_\delta \)-subset of \( M(H^\infty). \)

**Proof.** Let \( B = [H^\infty, \{\bar{I}_n\}]_{n=1}^\infty \) for a sequence of inner functions \( \{I_n\}_{n=1}^\infty. \) Then

\[
M(B) = \{x \in M(H^\infty); |I_n(x)| = 1 \text{ for every } n\}
= \bigcap_{n=1}^\infty \{x \in M(H^\infty); |I_n(x)| = 1\}.
\]

It is easy to see that \( M(B) \) is a \( G_\delta \)-subset of \( M(H^\infty). \)
Suppose that $M(B)$ is a $G_\delta$-subset of $M(H^\infty)$. Then there is a sequence of open subsets $\{U_n\}_{n=1}^\infty$ of $M(H^\infty)$ with $\bigcap_{n=1}^\infty U_n = M(B) = \bigcap_{I \in B} \{ x \in M(H^\infty) ; |I(x)| = 1 \}$, where $I$ runs through all inner functions with $I \in B$. Since $U_n^c \subset M(H^\infty) \setminus M(B)$ and $U_n^c$ is a compact subset of $M(H^\infty)$, there is an inner function $I_n$ such that $I_n \in B$ and $U_n^c \subset \{ x \in M(B) ; |I_n(x)| < 1 \}$. Then $M(B) = M(H^\infty, \{I_n\}_{n=1}^\infty)$. By Chang-Marshall’s theorem, we obtain $B = [H^\infty, \{I_n\}_{n=1}^\infty]$.

**Lemma 4.2** (Sarason’s unpublished result, see [7, Theorem 3.4]). Let $\{B_\alpha\}_{\alpha \in \Lambda}$ be a family of Douglas algebras. Then $M(\bigcap_{\alpha \in \Lambda} B_\alpha)$ coincides with the closure of $\bigcup_{\alpha \in \Lambda} M(B_\alpha)$ in $M(H^\infty)$.

**Lemma 4.3.** For functions $f$ and $g$ in $L^\infty$, there is a function $h$ in $L^\infty$ with $[H^\infty, f] = [H^\infty, f] \cap [H^\infty, g]$.

**Proof.** By Lemma 4.2,

$$M([H^\infty, f] \cap [H^\infty, g]) = M([H^\infty, f]) \cup M([H^\infty, g]).$$

By Lemma 2.2 and 4.1, $M([H^\infty, f]) \cup M([H^\infty, g])$ is a $G_\delta$-subset of $M(H^\infty)$, so is $M([H^\infty, f] \cap [H^\infty, g])$. By Lemmas 2.2 and 4.1 again, there is $h \in L^\infty$ with $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, g]$.

**Lemma 4.4.** Let $f$, $g$ and $h$ be functions in $L^\infty$ with $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, g]$. Then $N(h) = N(f) \cap N(g)$.

**Proof.** By our assumption, we have easily $N(h) \subset N(f) \cap N(g)$. Suppose that $N(h) \subsetneq N(f) \cap N(g)$. By Corollary 2.1, there is a QC-level set $Q$ with $Q \subset N(f) \cap N(g)$ and $Q \cap N(h) = \emptyset$. Take a function $q$ in QC such that

1. $0 \leq q \leq 1$ on $X$ and $q = 0$ on $N(h)$,
2. $q = 1$ on some open neighborhood of $Q$.

By Lemma 4.2,

$$M([H^\infty, h]) = M([H^\infty, f]) \cup M([H^\infty, g]).$$

By (1) and (3), we have $f_q|\text{supp } \mu_x \in H^\infty|\text{supp } \mu_x$ or $g_q|\text{supp } \mu_x \in H^\infty|\text{supp } \mu_x$ for every $x \in M(H^\infty + C)$. By Theorem 2.1, we get $N(f_q) \cap N(g_q) = \emptyset$. By (2), $Q \cap N(f(1-q)) = \emptyset$. Since $N(f) = N(f_q) \cup N(f(1-q))$, $Q \subset N(f_q)$. Also we obtain $Q \subset N(gq)$. These contradict $N(f_q) \cap N(gq) = \emptyset$.

**Proof of Theorem 4.1.** By Lemmas 4.3 and 4.4, there is $F \in L^\infty$ such that $[H^\infty, F] = \bigcap_{n=1}^N [H^\infty, f_n]$ and $N(F) = \bigcap_{n=1}^N N(f_n)$. By Lemma 4.2 and our assumption, $F \in H^\infty + C$. Thus $N(F) = \emptyset$. This completes the proof.

We note that there is a sequence of functions $\{f_n\}_{n=1}^\infty$ in $L^\infty$ such that

(a) for each $x$ in $M(H^\infty + C)$, there exists $n$ such that $f_n|\text{supp } \mu_x \in H^\infty|\text{supp } \mu_x$,

(b) $\bigcap_{n=1}^\infty N(f_n) \neq \emptyset$.

**Example.** Let $\lambda_n \in \partial D$ with $\lambda_n \to 1$ ($n \to \infty$), and let $S_n$ be the singular inner function associated with the singular measure $\sum_{k=1}^\infty (1/2)^k \delta_{\lambda_k}$. We put $f_n = (z-1)S_n$, then $\{f_n\}_{n=1}^\infty$ satisfies (a). Since $N(f_n) \supset N(f_{n+1})$, we get $\bigcap_{n=1}^\infty N(f_n) \neq \emptyset$. We note that if $Q \subset \bigcap_{n=1}^\infty N(f_n)$ then $f_n|Q \in H^\infty |Q$ for every $n$, hence for each QC-level set $Q$ there is $n$ such that $f_n|Q \in H^\infty |Q$.

In the last part of this section, we give a result which relates to Corollary 2.6.
PROPOSITION 4.1. For every \( f \in L^\infty \) with \( N(f) \neq \emptyset \), there is a Douglas algebra \( B \) such that

(i) \( N(B) = N(f) \), and

(ii) \( B \) is not countably generated.

PROOF. Let \( Q \) be a QC-level set with \( f|Q \notin H^\infty|Q \). Put

\[ B = [H^\infty, I; I \text{ is an inner function with } I \in [H^\infty, f] \text{ and } I|Q \in H^\infty|Q] \]

Then \( B \subset [H^\infty, f] \), so \( N(B) \subset N(f) \).

CLAIM. Put \( E = \bigcup \{ \text{supp } \mu_x; x \in M(H^\infty + C), f|\text{supp } \mu_x \notin H^\infty|\text{supp } \mu_x \text{ and supp } \mu_x \cap Q = \emptyset \} \). Then \( E \) is dense in \( N(f) \).

To show our claim, suppose not. Then \( \text{cl } E \nsubseteq \text{supp } \mu_y \) for some \( y \in M(H^\infty + C) \) such that \( f|\text{supp } \mu_y \notin H^\infty|\text{supp } \mu_y \) and \( \text{supp } \mu_y \subset Q \). Hence there is an open and closed subset \( U \) of \( X \) such that \( E \cap U = \emptyset \). By Lemma 2.6, \( Q_y \subset E(U) \). Thus \( N(f) \cap N(U) = \emptyset \). By Lemma 4.3, there is \( h \in L^\infty \) such that \( [H^\infty, h] = [H^\infty, f] \cap [H^\infty, \chi_U] \). By Lemma 4.4, \( h \notin H^\infty + C \). By Lemma 3.2, there is \( \zeta \in M(H^\infty + C) \) with \( h|\text{supp } \mu_\zeta \notin H^\infty|\text{supp } \mu_\zeta \) and \( \text{supp } \mu_\zeta \cap Q = \emptyset \). By Lemma 4.2, both \( f|\text{supp } \mu_\zeta \) and \( \chi_U|\text{supp } \mu_\zeta \) are not contained in \( H^\infty|\text{supp } \mu_\zeta \). This contradicts the definitions of \( E \) and \( U \). Hence we get our claim.

To show (i), it is sufficient to prove \( E \subset N(B) \) by our claim. To prove this, let \( x \in M(H^\infty + C) \) such that \( f|\text{supp } \mu_x \notin H^\infty|\text{supp } \mu_x \) and \( \text{supp } \mu_x \cap Q = \emptyset \). Take a function \( q \in QC \) with \( q = 0 \) on \( Q \) and \( q = 1 \) on \( \text{supp } \mu_x \). Hence \( \text{supp } \mu_x \subset N(B) \), so \( E \subset N(B) \).

To show (ii), suppose not. Then \( B = [H^\infty, F] \) for some \( F \in [H^\infty, f] \). By Corollary 2.6, \( [H^\infty, F] = [H^\infty, f] \). Since \( F|Q \in H^\infty|Q \), \( f|Q \in H^\infty|Q \). But this is a contradiction.

5. Discrete sequences in \( M(QC) \). A sequence \( \{y_n\}_{n=1}^\infty \) in a topological space \( Y \) is called discrete if there is a sequence of open subsets \( \{V_n\}_{n=1}^\infty \) of \( Y \) such that \( Y_n \in V_n \) and \( V_n \cap \text{cl } (\bigcup_{m \neq n} V_n) = \emptyset \). In this section, we study discrete sequences in \( M(QC) \) and show three theorems as applications of §2. The first one, Theorem 5.1, gives properties of a sequence of QC-level sets. In Theorem 5.2, we shall show the existence of a certain function in \( H^\infty \), which is motivated by [11]. Using them, we shall prove a theorem which is more precise than the one proved in [8, Theorem 2.1].

A QC-level set is called simple if it consists of only one point. It is not known whether there is a simple QC-level set or not. It is easy to see that a QC-level set \( Q \) is not simple if and only if there is \( x \in M(H^\infty + C) \) such that \( \text{supp } \mu_x \subset Q \). We note that every QC-level set in \( N(f) \), \( f \in L^\infty \), is not simple. Because, for a given \( f \in L^\infty \), there is an inner function \( I \) such that \( N(f) \subset N(I) \) (see the proof of Corollary 7 in [13]). By Lemma 3.1, \( N(I) \) does not contain any simple QC-level sets.

A discrete sequence \( \{y_n\}_{n=1}^\infty \) in \( M(QC) \) is called strongly discrete if each \( \pi^{-1}(y_n) \) is not simple.

THEOREM 5.1. Let \( \{y_n\}_{n=1}^\infty \) be a strongly discrete sequence in \( M(QC) \), and let \( y_0 \in M(QC) \) be its cluster point. Then

(i) \( \pi^{-1}(y_0) \) is not simple.

(ii) \( \pi^{-1}(y_0) \subset \text{cl } (\bigcup_{n=1}^\infty \pi^{-1}(y_n)) \).
(iii) If \( \{a_n\}_{n=1}^{\infty} \) is a bounded sequence of complex numbers, there is \( h \in QA \) such that \( h(y_n) = a_n \) for every \( n \).

PROOF. By our assumption, there is a sequence of open subsets \( \{V_n\}_{n=1}^{\infty} \) of \( M(QC) \) satisfying \( y_n \in V_n \) and

\[
V_n \cap \text{cl} \left( \bigcup_{m \neq n} V_m \right) = \emptyset.
\]

Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( M(H^{\infty} + C) \setminus X \) with \( \text{supp } \mu_{x_n} \subset \pi^{-1}(y_n) \). By [10, p. 177], there is an inner function \( \eta_n \) with

\[
\text{cl} \{y_n\}_{n=1}^{\infty} \subset \pi(N(F)).
\]

Thus \( \pi(N(F)) \) is a compact subset of \( M(QC) \), \( y_0 \in \pi(N(F)) \). By Corollary 2.1, \( \pi^{-1}(y_0) \subset N(F) \). By the remark before Theorem 5.1, we get (i).

To show (ii), suppose that \( \pi^{-1}(y_0) \not\subset \text{cl} \left( \bigcup_{n=1}^{\infty} \pi^{-1}(y_n) \right) \). There is an open and closed subset \( U \) of \( X \) such that \( U \cap \text{cl} \left( \bigcup_{n=1}^{\infty} \pi^{-1}(y_n) \right) = \emptyset \) and \( U \cap \pi^{-1}(y_0) \neq \emptyset \). Hence we may take a sequence of open subsets \( \{V_n\}_{n=1}^{\infty} \) satisfying moreover

\[
\pi^{-1}(U) \cap V_n = \emptyset.
\]

By the same way as (i), we have a function \( F = \sum_{n=1}^{\infty} (1/2)^n \tilde{T}_n q_n \). Let \( x \in M(H^{\infty} + C) \) with \( F|_{\text{supp } \mu_x} \notin H^{\infty}|_{\text{supp } \mu_x} \). Hence \( \text{supp } \mu_{x_n} \subset N(F) \), so \( y_n \in \pi(N(F)) \) for every \( n \). Thus \( \text{cl} \{y_n\}_{n=1}^{\infty} \subset \pi(N(F)) \). Since \( \pi(N(F)) \) is a compact subset of \( M(QC) \), \( y_0 \in \pi(N(F)) \). By Corollary 2.1, \( \pi^{-1}(y_0) \subset N(F) \). By the remark before Theorem 5.1, we get (i).

To show (ii), suppose that \( \pi^{-1}(y_0) \not\subset \text{cl} \left( \bigcup_{n=1}^{\infty} \pi^{-1}(y_n) \right) \). There is an open and closed subset \( U \) of \( X \) such that \( U \cap \text{cl} \left( \bigcup_{n=1}^{\infty} \pi^{-1}(y_n) \right) = \emptyset \) and \( U \cap \pi^{-1}(y_0) \neq \emptyset \). Hence we may take a sequence of open subsets \( \{V_n\}_{n=1}^{\infty} \) satisfying moreover

\[
\pi^{-1}(U) \cap V_n = \emptyset.
\]

By the same way as (i), we have a function \( F = \sum_{n=1}^{\infty} (1/2)^n \tilde{T}_n q_n \). Let \( x \in M(H^{\infty} + C) \) with \( F|_{\text{supp } \mu_x} \notin H^{\infty}|_{\text{supp } \mu_x} \). Then \( \tilde{T}_n q_n |_{\text{supp } \mu_x} \notin H^{\infty}|_{\text{supp } \mu_x} \) for some \( n \). By (3), \( \text{supp } \mu_x \subset N(F) \). Hence we have \( N(F) \subset X \setminus U \). Since \( \pi^{-1}(y_0) \subset N(F) \) by the proof of (i), we get \( \pi^{-1}(y_0) \cap U = \emptyset \). But this is a contradiction, so we get (ii).

(iii) Let \( F \) be a function in the proof of (i). By [23, Lemmas 2.2 and 2.3], \( m_0(\pi(N(F))) = 0 \) and \( \pi(N(F)) \) is an interpolation set for \( QA \), that is, \( QA|\pi(N(F)) = C(\pi(N(F))) \). Since \( \text{cl} \{y_n\}_{n=1}^{\infty} \subset \pi(N(F)) \), \( \text{cl} \{y_n\}_{n=1}^{\infty} \) is an interpolation set for \( QA \). To prove (iii), it is sufficient to show that \( \text{cl} \{y_{n_k}\}_{k=1}^{\infty} \cap \text{cl} \{\{y_n\}_{n=1}^{\infty} \setminus \{y_{n_k}\}_{k=1}^{\infty}\} = \emptyset \) for every subset \( \{y_{n_k}\}_{k=1}^{\infty} \) of \( \{y_n\}_{n=1}^{\infty} \) (see [10, p. 205]). To show this, put \( G = \sum_{k=1}^{\infty} (1/2)^k \tilde{T}_{n_k} q_{n_k} \) and \( H = F - G \). By our construction, \( G \) and \( H \) satisfy the assumption of Theorem 2.1, so we get \( N(G) \cap N(H) = \emptyset \). Thus \( \text{cl} \{y_{n_k}\}_{k=1}^{\infty} \subset \pi(N(G)) \) and \( \{y_n\}_{n=1}^{\infty} \setminus \{y_{n_k}\}_{k=1}^{\infty} \subset \pi(N(H)) \), we have

\[
\text{cl} \{y_{n_k}\}_{k=1}^{\infty} \cap \text{cl} \{\{y_n\}_{n=1}^{\infty} \setminus \{y_{n_k}\}_{k=1}^{\infty}\} \subset \pi(N(G)) \cap \pi(N(H)) = \emptyset.
\]

This completes the proof.

In [11], Hoffman showed that a discrete sequence \( \{y_n\}_{n=1}^{\infty} \) in \( X \) is an \( l^\infty \)-interpolation set for \( H^{\infty} \), that is, for every bounded sequence of complex numbers \( \{a_n\}_{n=1}^{\infty} \) there is \( h \in H^{\infty} \) such that \( h(y_n) = a_n \) for \( n = 1, 2, \ldots \). Using his technique, we shall show the existence of a certain function in \( H^{\infty} \).
LEMMA 5.1 [11]. Let $K$ be a closed subset of $X$ with $m(K) = 0$. Let $g$ be a bounded continuous function on $X\setminus K$. Suppose that there is a bounded sequence \( \{f_n\}_{n=1}^{\infty} \) in $H^\infty$ such that $f_n$ converges to $g$ uniformly on each compact subset of $X\setminus K$. Then there is $f \in H^\infty$ with $f|X\setminus K = g$.

THEOREM 5.2. Let $\{y_n\}_{n=1}^{\infty}$ be a strongly discrete sequence in $M(QC)$. Let $\{h_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^\infty + C$. Then there exists a function $F$ in $H^\infty$ such that $F|\pi^{-1}(y_n)$ for every $n$.

PROOF. Suppose that $\|h_n\| < M$, where $M$ is an absolute constant. Since $\pi^{-1}(y_n)$ is a weak peak set for $H^\infty$ and $H^\infty + C|\pi^{-1}(y_n) = H^\infty|\pi^{-1}(y_n)$, there is $f_n \in H^\infty$ such that $f_n|\pi^{-1}(y_n) = h_n|\pi^{-1}(y_n)$ and $\|f_n\| < M$. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open subsets of $M(QC)$ such that $y_n \in V_n$ and $V_n \cap \text{cl} \left( \bigcup_{m \neq n} V_m \right) = \emptyset$. Let $W_0$ be the interior of $\pi^{-1}(M(QC) \setminus \bigcup_{n=1}^{\infty} V_n)$. By [4, p. 18], $m(W_0) = m(\pi^{-1}(M(QC) \setminus \bigcup_{n=1}^{\infty} V_n))$. Put $K = (X\setminus W_0) \setminus \bigcup_{n=1}^{\infty} \pi^{-1}(V_n)$. Then $K$ is a compact subset of $X$ and $m(K) = 0$, because

\[
m(K) = 1 - m(\pi^{-1}(V_n)) \leq 1 - m(\bigcup_{n=1}^{\infty} \pi^{-1}(V_n)) = 1 - m(\pi^{-1}(M(QC) \setminus \bigcup_{n=1}^{\infty} V_n)) = 0.
\]

We may take a function $q_n$ in $QA$ satisfying

(1) $\|q_n\| = 1$, $q_n(y_n) = 1$ and $|q_n| < (1/2)^n$ on $M(QC) \setminus V_n$.

By (iii) of Theorem 5.1, we may assume that

(2) $q_{m}(y_n) = 0$ if $m \neq n$.

Put $G_N = \sum_{k=1}^{N} f_k q_k$. Then $G_N \in H^\infty$. We shall show that $\{G_N\}_{N=1}^{\infty}$ satisfies the assumption of Lemma 5.1 for $K$. By (1), we have

\[
|G_N| \leq |f_n| + \sum_{k \neq n} |f_k| |q_k| \leq M \left( 1 + \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k \right) \leq 2M,
\]

on $\pi^{-1}(V_n)$,

\[
|G_N| \leq M \left( \pi^{-1}(M(QC) \setminus \bigcup_{n=1}^{\infty} V_n) \right) \leq M \left( \pi^{-1}(M(QC) \setminus \bigcup_{n=1}^{\infty} V_n) \right) \leq M.
\]

Hence $\{G_N\}_{N=1}^{\infty}$ is a bounded sequence in $H^\infty$. Let $E$ be a compact subset of $X\setminus K$. Then $K \subset W_0 \cup \bigcup_{n=1}^{n_0} \pi^{-1}(V_k)$ for some $n_0$. For $n_0 \leq i < j$, we have

\[
|G_j - G_i| = \left| \sum_{k=i+1}^{j} f_k q_k \right| \leq M \left( \sum_{k=i+1}^{j} \left( \frac{1}{2} \right)^k \right) \leq M \left( \frac{1}{2} \right)^i \text{ on } W_0 \cup \bigcup_{k=1}^{n_0} \pi^{-1}(V_k).
\]
Hence \( \{ G_N \}_{N=1}^{\infty} \) converges to \( \sum_{k=1}^{\infty} f_k q_k \) uniformly on \( E \). By Lemma 5.1, there is a function \( G \) in \( H^\infty \) such that \( F = \sum_{k=1}^{\infty} f_k q_k \) on \( X \setminus K \). By (1) and (2), we get \( F|^{-1}(y_n) = f_n|^{-1}(y_n) = h_n|^{-1}(y_n) \).

A closed subset \( E \) of \( X \) is called antisymmetric for \( H^\infty \) if \( H^\infty|E \) does not contain any nonconstant real functions. An antisymmetric set is called maximal if there are no antisymmetric sets which contain \( E \) properly.

**Theorem 5.3 (Cf. [8, Theorem 2.1]).** Let \( \{ y_n \}_{n=1}^{\infty} \) be a strongly discrete sequence in \( M(QC) \). Let \( \{ \lambda_n \}_{n=1}^{\infty} \) be a sequence in \( X \) with \( \lambda_n \in \pi^{-1}(y_n) \). If \( \lambda_0 \) is a cluster point of \( \{ \lambda_n \}_{n=1}^{\infty} \) in \( X \), then \( \{ \lambda_0 \} \) is a maximal antisymmetric set for \( H^\infty \) and it is not a QC-level set.

**Proof.** Let \( \lambda_0 \) be a cluster point of \( \{ \lambda_n \}_{n=1}^{\infty} \) with \( \lambda_n \in \pi^{-1}(y_n) \). There is a QC-level set \( Q_0 \) with \( Q_0 \supseteq \lambda_0 \). Since \( \pi(Q_0) \in \text{cl} \{ y_n \}_{n=1}^{\infty} \), there is \( y_0 \in \text{cl} \{ y_n \}_{n=1}^{\infty} \) such that \( Q_0 = \pi^{-1}(y_0) \). By Theorem 5.1(i), \( Q_0 \) is not simple. We note that the maximal antisymmetric set containing \( y_0 \) is contained in \( Q_0 \). To show our assertion, let \( E \) be a closed subset with \( \{ y_0 \} \subseteq E \subseteq Q_0 \). We shall show that \( E \) is not antisymmetric. Take an open and closed subset \( U \) of \( X \) satisfying \( \lambda_0 \in U \) and \( E \subsetneq U \). By [1], there is \( h \in H^\infty + C \) such that \( |h| = \chi_U \) on \( X \). Using a function \( h \), we shall construct a function \( F \) in \( H^\infty \) such that

1. \( F = 1 \) on \( U \cap \{ \lambda_n \}_{n=1}^{\infty} \),
2. \( F = 0 \) on \( U^c \cap \pi^{-1}(y_n) \) for every \( n \), and
3. the sequence of ranges \( F(\pi^{-1}(y_n)) \) converges in \([-1, 1] \), that is, for every open subset \( W \) in the complex plane with \([-1, 1] \subset W \) there is \( n_0 \) such that \( F(\pi^{-1}(y_n)) \subset W \) for every \( n \geq n_0 \).

We let \( D_n \) denote the open ellipse with major axis \([-1, 1] \) and minor axis \([-i/n, i/n] \). Let \( \psi_n \) be a conformal mapping of \( D \) onto \( D_n \) such that \( \psi_n(0) = 0 \) and \( \psi_n(h(\lambda_n)) = 1 \) for every \( n \) with \( |h(\lambda_n)| = 1 \). We note that \( \psi_n \circ h \in H^\infty + C \) and \( \| \psi_n \circ h \| = 1 \) for every \( n \). By Theorem 5.2, there exists a function \( F \) in \( H^\infty \) such that \( F|^{-1}(y_n) = \psi_n \circ h|^{-1}(y_n) \). It is easy to see that \( F \) satisfies (1) and (2) and (3). Since \( \lambda_0 \in \text{cl} \{ \lambda_n \}_{n=1}^{\infty} \), \( F(\lambda_0) = 1 \) by (1). By Theorem 5.1(ii) and (2), \( F = 0 \) on \( U^c \cap \pi^{-1}(y_0) \). Also by Theorem 5.1(ii) and (3), \( F \) is a real function on \( \pi^{-1}(y_0) \). Thus \( F|E \in H^\infty|E \) is not a nonconstant real function. Hence \( E \) is not antisymmetric.

**Remark.** It is not true that a cluster point of discrete sequence \( \{ \lambda_n \}_{n=1}^{\infty} \) in \( X \) is a maximal antisymmetric set for \( H^\infty \). For, let \( x \in M(H^\infty + C) \setminus X \), then supp \( \mu_x \) is an antisymmetric set for \( H^\infty \). We may choose a sequence \( \{ \lambda_n \}_{n=1}^{\infty} \) in supp \( \mu_x \) which is discrete in \( X \). Then a cluster point of \( \{ \lambda_n \}_{n=1}^{\infty} \) is continued in supp \( \mu_x \).

6. **Singly generated Douglas algebras.** In this section, we answer the following problem given in [6, 19]; when is \([H^\infty, f], f \in L^\infty \), singly generated? The characterization of singly generated Douglas algebras in [14] does not answer the above problem explicitly. We want to know conditions on \( f \) satisfying that \([H^\infty, f] \) is singly generated.

For a point \( y \in M(QC) \) and \( f \in L^\infty \), we put

\[
\|f + H^\infty\|_y = \inf_{h \in H^\infty} \{ \sup_{x \in \pi^{-1}(y)} |f(x) + h(x)|; x \in \pi^{-1}(y) \}.
\]

By [23], the set \( \{ \|f + H^\infty\|_y; y \in M(QC) \} \) contains 0. First we shall prove the following proposition, which is interesting in its own right.

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PROPOSITION 6.1. For a given $f \in L^\infty$, the map $QC \ni y \mapsto \|f + H^\infty\|_y$ is upper semicontinuous.

PROOF. Let $r$ be a real number. Let $\{y_\alpha\}_{\alpha \in \Lambda}$ be a net in $M(QC)$ such that

\begin{enumerate}
  \item $y_\alpha \to y_0 \in M(QC)$, \label{eq:1}
  \item $\|f + H^\infty\|_{y_\alpha} \geq r$ for every $\alpha \in \Lambda$. \label{eq:2}
\end{enumerate}

We shall show that $\|f + H^\infty\|_{y_0} \geq r$. Since $\pi^{-1}(y_\alpha)$ is a weak peak set for $H^\infty$, by (2) there is a measure $\mu_\alpha$ such that

\begin{enumerate}
  \item $\|\mu_\alpha\| = 1$ and $\text{supp} \mu_\alpha \subset \pi^{-1}(y_\alpha)$, \label{eq:3}
  \item $\int_X f \, d\mu_\alpha = \|f + H^\infty\|_{y_\alpha}$ and $\mu_\alpha \perp H^\infty$. \label{eq:4}
\end{enumerate}

Let $\mu_0$ be a weak*-cluster point of $\{\mu_\alpha\}_{\alpha \in \Lambda}$, that is, $\int_X g \, d\mu_\alpha \to \int_X g \, d\mu_0$ for every $g \in C(X)$. Then $\|\mu_0\| \leq 1$. By (2) and (4), we have

$$\int_X f \, d\mu_0 \geq \inf_\alpha \int_X f \, d\mu_\alpha = \inf_\alpha \|f + H^\infty\|_{y_\alpha} \geq r.$$

We note that $\pi^{-1}(y_0)$ is also a weak peak set for $QA$. Let $h \in QA$ be any peaking function such that $\pi^{-1}(y_0) \subset \{x \in X; h(x) = 1\}$. Since $h$ is constant on each $QC$-level set, we have $\int_X h \, d\mu_\alpha = h(y_\alpha) \int_X f \, d\mu_\alpha$ by (3). Thus

$$\int_X f \, d\mu_0 = \lim_\alpha h(y_\alpha) \int_X f \, d\mu_\alpha = \int_X f \, d\mu_0 \quad \text{by (1)}.$$ 

This shows that $\int_{\pi^{-1}(y_0)} f \, d\mu_0 = \int_X f \, d\mu_0 \geq r$. By [4, p. 58] and (4), we have $\mu_0|\pi^{-1}(y_0) \perp H^\infty$. Since $\|\mu_0\| \leq 1$, $\|f + H^\infty\|_{y_0} \geq r$. This completes the proof.

Our theorem is

THEOREM 6.1. Let $f \in L^\infty$. Then the following assertions are equivalent.

\begin{enumerate}
  \item $[H^\infty, f]$ is singly generated. \label{eq:i}
  \item $Q(f)$ is a closed subset of $X$, consequently $Q(f) = N(f)$. \label{eq:ii}
  \item In the set $\{\|f + H^\infty\|_y; y \in M(QC)\}$, 0 is an isolated point. \label{eq:iii}
\end{enumerate}

PROOF. (i) $\Rightarrow$ (ii) follows from Lemma 3.1.

(ii) $\Rightarrow$ (iii) Suppose that 0 is not isolated in the set $\{\|f + H^\infty\|_y; y \in M(QC)\}$. Then there is a sequence $\{y_n\}_{n=1}^\infty$ in $M(QC)$ such that $0 < \|f + H^\infty\|_{y_n} < 1/n$ for $n = 1, 2, \ldots$. Taking a subsequence, we may assume that $\{y_n\}_{n=1}^\infty$ is discrete in $M(QC)$. Since $0 < \|f + H^\infty\|_{y_n}$, $\pi^{-1}(y_n)$ is not simple. Thus $\{y_n\}_{n=1}^\infty$ is strongly discrete. Let $h_n \in H^\infty$ with

$$\sup\{|(f + h_n)(x)|; x \in \pi^{-1}(y_n)\} < 1/n. \quad \text{(1)}$$

Since $\pi^{-1}(y_n)$ is a weak peak set for $H^\infty$, we may assume that $\{h_n\}_{n=1}^\infty$ is a bounded sequence in $H^\infty$. By Theorem 5.2, there is $F \in H^\infty$ such that $F = h_n$ on $\pi^{-1}(y_n)$. By (1),

$$\sup\{|(f + F)(x)|; x \in \pi^{-1}(y_n)\} < 1/n. \quad \text{(2)}$$

Let $y_0 \in M(QC)$ be a cluster point of $\{y_n\}_{n=1}^\infty$. By Theorem 5.1, $\pi^{-1}(y_0) \subset \text{cl} (\bigcup_{n=1}^\infty \pi^{-1}(y_n)) \subset N(f)$. By (2), $(f + F)(x) = 0$ for $x \in \pi^{-1}(y_0)$. Thus $f|\pi^{-1}(y_0) \in H^\infty|\pi^{-1}(y_0)$, so $Q(f) \subset N(f)$. 

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(iii) ⇒ (i) Suppose that 0 is isolated in the set \( \{ \| f + H^\infty \|_y; y \in M(QC) \} \). Then there is \( \varepsilon > 0 \) such that

\[
\{ y \in M(QC); \| f + H^\infty \|_y \neq 0 \} = \{ y \in M(QC); \| f + H^\infty \|_y \geq \varepsilon \}.
\]

By Proposition 6.1, \( \{ y \in M(QC); \| f + H^\infty \|_y \neq 0 \} \) is a closed subset of \( M(QC) \). Hence \( \pi^{-1}\{ y \in M(QC); \| f + H^\infty \|_y \neq 0 \} = N(f) \) by Corollary 2.1. Let \( I \) be an inner function such that \( I \in [H^\infty, f] \) and \( \| If + H^\infty \| < \varepsilon \). We note that if \( y \in M(QC) \) satisfies \( \| If + H^\infty \|_y = 0 \), then \( \pi^{-1}(y) \subset N(I) \). For, if \( \pi^{-1}(y) \cap N(I) = \emptyset \) then \( \varepsilon \leq \| f + H^\infty \|_y = \| If + H^\infty \|_y < \varepsilon \). Hence \( N(f) \subset N(I) \). By Corollary 2.5, \( \pi^{-1}(y) = N(I) \).

The following corollary was proved by Marshall [19].

COROLLARY 6.1. \([H^\infty, \chi_U]\) is singly generated for every open and closed subset \( U \) of \( X \).

PROOF. We shall show that for \( y \in M(QC) \) either \( \| \chi_U + H^\infty \|_y = 1/2 \) or \( \| \chi_U + H^\infty \|_y = 0 \). It is easy to see that \( \| \chi_U + H^\infty \|_y \leq 1/2 \). Suppose \( \| \chi_U + H^\infty \|_y < 1/2 \). There is \( h \in H^\infty \) such that \( \sup_{x \in \pi^{-1}(y)} \| \chi_U(x) + h(x) \| < 1/2 \). Then there is a sequence of analytic polynomials \( \{ p_n \}_{n=1}^\infty \) such that \( p_n \circ h \to \chi_U \) uniformly on \( \pi^{-1}(y) \). Thus \( \| \chi_U + H^\infty \|_y = 0 \). By Theorem 6.1, we get our assertion.

We shall give an example concerning countable valued functions.

EXAMPLE. There exist two functions \( f \) and \( g \) in \( L^\infty \) such that

(a) \( f(X) = g(X) = \{ 0, 1/n; n = 1, 2, \ldots \} \),

(b) \([H^\infty, g]\) is not singly generated, and

(c) \([H^\infty, f]\) is singly generated.

PROOF. Let \( \{ O_n \}_{n=1}^\infty \) be a sequence of open arcs such that \( O_n = \{ e^{i\theta}; 1/n + 1 < \theta < 1/n \} \). Put \( U_n = \{ x \in X; \chi_{O_n}(x) = 1 \} \). Then \( U_n \) is an open and closed subset of \( X \). Put

\[
g = \begin{cases} \sum_{n=1}^\infty \frac{1}{n} \chi_{O_n} & \text{on } \bigcup_{n=1}^\infty O_n, \\ 0 & \text{on } \partial D \setminus \bigcup_{n=1}^\infty O_n. \end{cases}
\]

By the same way as the proof of Corollary 6.1,

\[
\{ \| g + H^\infty \|_y; y \in M(QC) \} = \{ 0, 1/2 \} \cup \{ (1/n - 1/n + 1)/2; n = 1, 2, \ldots \}.
\]

By Theorem 6.1, \( g \) satisfies (a) and (b). Put

\[
f = \begin{cases} \sum_{n=1}^\infty \frac{1}{n} \chi_{O_n} & \text{on } \bigcup_{n=1}^\infty O_{2n}, \\ 1 & \text{on } \partial D \setminus \bigcup_{n=1}^\infty O_{2n}. \end{cases}
\]

Then \( \{ \| f + H^\infty \|_y; y \in M(QC) \} = \{ (1 - 1/n)/2; n = 1, 2, \ldots \} \). By Theorem 6.1, \( f \) satisfies (a) and (c).

7. M-ideals. Let \( F \) be a weak peak subset of \( X \) for \( H^\infty + C \). We put \( (H^\infty + C)_F = \{ f \in L^\infty; f|F \in H^\infty + C|F \} \). Then \( (H^\infty + C)_F \) is a Douglas algebra. In [18], Luecking and Younis gave the following conjecture: Let \( B \) be a Douglas algebra such that \( B/H^\infty \) is an M-ideal of \( L^\infty/H^\infty \). Is \( B = (H^\infty + C)_F \) for some weak peak set \( F \) for \( H^\infty + C \)? We shall give a negative answer.
THEOREM 7.1. Let $E \subseteq X$ be a peak set for QC. Put

$$B = \{H^\infty, \{\overline{I}; I \text{ is an inner function with } N(\overline{I}) \subset E\}\}.$$ 

Then

(i) $B/H^\infty$ is an $M$-ideal of $L^\infty/H^\infty$.

(ii) $B \neq (H^\infty + C)_F$ for every weak peak set $F$ for $H^\infty + C$.

To show this, we need some lemmas.

**LEMMA 7.1 [16, COROLLARY 5.1].** Let $B$ be a Douglas algebra with $B \supseteq H^\infty + C$. Then $B/H^\infty$ is an $M$-ideal of $L^\infty/H^\infty$ if and only if $B/H^\infty + C$ is an $M$-ideal of $L^\infty/H^\infty + C$.

The following lemma is a characterization of $M$-ideals of $L^\infty/H^\infty + C$, which is obtained by [5] essentially. For a Douglas algebra $B$, we denote by $B_\perp$ the space of annihilating measures on $X$ for $B$.

**LEMMA 7.2 (SEE [16, THEOREM 5.1]).** Let $B$ be a Douglas algebra with $B \supseteq H^\infty + C$. Then $B/H^\infty + C$ is an $M$-ideal of $L^\infty/H^\infty + C$ if and only if for each $\overline{I} \in (H^\infty + C)_\perp$ there exists $f_\mu \in L^1(|\mu|)$ such that

(a) $\int f_\mu^2 d\mu = f_\mu$ a.e. $d|\mu|$, 

(b) $\mu - f_\mu \perp B_\perp$, and

(c) $f_\mu \in B_\perp$.

For a subset $E$ of $X$, we put $\Lambda_E = \{I; I \text{ is an inner function with } N(\overline{I}) \subset E\}$. As applications of Lemma 2.5 and Theorem 2.1, we get the following lemma.

**LEMMA 7.3.** Let $E \subseteq X$ be a peak set for QC. For a sequence of inner functions $\{I_n\}_{n=1}^\infty$ in $\Lambda_E$, there exists $I \in \Lambda_E$ such that $N(I_n) \subset N(I)$ for all $n$.

**PROOF.** Let $h \in QC$ be a peaking function for $E$. By Lemma 2.1, we may assume that each $I_n$ is an interpolating Blaschke product with zeros $\{z_{n,k}\}_{k=1}^\infty$ and $\sum_{k=1}^\infty \sum_{n=1}^\infty (1 - |z_{n,k}|) < \infty$. Then $\prod_{n=1}^\infty I_n$ is a Blaschke product. Put $\psi = \prod_{n=1}^\infty I_n$ and $g = \psi(1 - h)$. Then

(1) $N(\psi) \setminus E \subset N(g)$.

To prove that $\{I_n\}_{n=1}^\infty$ and $g$ satisfy the assumptions of Lemma 2.5, let $x \in M(H^\infty + C)$. Since $E$ is a union set of some QC-level sets, $\mathrm{supp}\mu_x \subset E$ or $\mathrm{supp}\mu_x \cap E = \emptyset$. If $\mathrm{supp}\mu_x \subset E$, we get $0 = g|\mathrm{supp}\mu_x \subset H^\infty|\mathrm{supp}\mu_x$. If $\mathrm{supp}\mu_x \cap E = \emptyset$, then $I_n|\mathrm{supp}\mu_x \subset H^\infty|\mathrm{supp}\mu_x$ for all $n$, because $I_n \in \Lambda_E$. By Lemma 2.5, there is a Blaschke product $\overline{I}$ such that

(2) $N(\overline{I}) \subset N(\psi)$,

(3) either $\overline{I}|\mathrm{supp}\mu_x \subset H^\infty|\mathrm{supp}\mu_x$ or $g|\mathrm{supp}\mu_x \subset H^\infty|\mathrm{supp}\mu_x$ for every $x \in M(H^\infty + C)$, and

(4) $N(\overline{I}_n) \subset N(\overline{I})$ for all $n$.

By (3), applying Theorem 2.1, we get $N(\overline{I}) \cap N(g) = \emptyset$. Hence, by (1) and (2), $N(\overline{I}) \subset E$, so $I \in \Lambda_E$. 

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LEMMA 7.4 [13, LEMMA 3]. Let $E$ be a closed $G_δ$-subset of $X$. Then there is an inner function $I$ with $φ \neq N(\overline{I}) \subset E$.

LEMMA 7.5. Let $ν$ be a measure on $X$ with $ν \in (H^\infty + C)^\perp$. If $J$ is an inner function with $Jν \not\in (H^\infty + C)^\perp$, then $|ν|(N(\overline{J})) \neq 0$.

PROOF. Suppose that $|ν|(N(\overline{J})) = 0$. Then there is a sequence of compact subsets $\{K_n\}_{n=1}^\infty$ of $X$ such that $\lim_{n \to \infty} |ν|(K_n) = |ν|$ and $K_n \cap N(\overline{J}) = \emptyset$. Since $π^{-1}(π(N(\overline{J}))) = N(\overline{J})$, moreover we may assume $π^{-1}(π(K_n)) = K_n$. Then $K_n$ is a weak peak set for $QC$ and for $H^\infty + C$. Hence $ν|K_n \in (H^\infty + C)^\perp$ [4, p. 58]. Since $K_n \cap N(\overline{J}) = \emptyset$, $J|K_n \in (H^\infty + C)|K_n$. Hence $J\nu|K_n \in (H^\infty + C)^\perp$. Since $lim_{n \to \infty} |ν|(K_n) = |ν|$, $J\nu \in (H^\infty + C)^\perp$. But this is a contradiction.

LEMMA 7.6 [16, THEOREM 2.1]. Let $B$ be a Douglas algebra with $B \supset H^\infty + C$. Let $λ$ be a measure on $X$ with $λ \in B^\perp$. If $ν$ is a measure with $ν \ll λ$, then there is an inner function $I$ such that $Iν \in B^\perp$.

PROOF OF THEOREM 7.1. (i) By Lemma 7.4, $H^\infty + C \subsetneq B$. We shall show that $B/H^\infty + C$ is an $M$-ideal of $L^\infty/H^\infty + C$, then we get (i) by Lemma 7.1. To show the above fact, we use Lemma 7.2. Let $μ \in (H^\infty + C)^\perp$ with $|μ| = 1$. Put $α = \sup\{μ|N(\overline{I})|; I \in \Lambda_E\}$. Then there is a sequence $\{I_n\}_{n=1}^\infty$ in $\Lambda_E$ such that $lim_{n \to \infty} |μ|(N(\overline{I}_n)) = α$. By Lemma 7.3, there is $I_0 \in \Lambda_E$ such that $N(\overline{I}_n) \subset N(\overline{I}_0)$. Hence $|μ|(N(\overline{I}_0)) = α$. Put $f_μ = 1 - \chi_{N(\overline{I}_0)}$. Then $f_μ$ satisfies (a) of Lemma 7.2. Also by Lemma 7.3, 

\[ |f_μ(\overline{I})| = 0 \quad \text{for every } I \in λ_E. \]

To show $f_μμ \in B^\perp$, suppose that $f_μμ \not\in B^\perp$. Since $B$ coincides with the closed linear span of $\{\overline{I}(H^\infty + C); I \in \Lambda_E\}$, there is $J \in \Lambda_E$ such that $Jf_μμ \not\in (H^\infty + C)^\perp$. We note that $f_μμ \in (H^\infty + C)^\perp$, because $N(\overline{I}_0)$ is a weak peak set for $H^\infty + C$ by Corollary 2.1. By Lemma 7.5, $|f_μμ|(N(\overline{J})) \neq 0$. But this contradicts (1). Thus we get (c) of Lemma 7.2.

To prove (b), we shall show

(2) $λ | N(\overline{I}) = 0 \quad \text{for every } λ \in B^\perp \text{ and } I \in \Lambda_E$. Fix $λ \in B^\perp$ and $I \in \Lambda_E$. By Lemma 7.6, there is an inner function $Ψ$ such that

\[ Ψ|λ| | N(\overline{I}) \in B^\perp. \]

Let $h \in QC$ be a peaking function for $E$. We note that for $x \in M(H^\infty + C)$, either $\overline{I} | supp μ_x \in H^\infty | supp μ_x$ or $Ψ(1-h) | sup supp μ_x \in H^\infty | supp μ_x$, because $N(\overline{I}) \subset E$. By Theorem 2.1, $N(\overline{I}) \cap N(Ψ(1-h)) = \emptyset$. By Corollary 2.1, there is a function $q$ in $QC$ such that $0 \leq q \leq 1$,

\[ q = 1 \quad \text{on } N(\overline{I}) \quad \text{and} \quad q = 0 \quad \text{on } N(Ψ(1-h)). \]

If $qΨ|supp μ_x \not\in H^\infty |supp μ_x$ for $x \in M(H^\infty + C)$, then $q(x) \neq 0$ and $Ψ|supp μ_x \not\in H^\infty |supp μ_x$. Since $h \in QC$, $h(x) = 1$ by (4). Hence $supp μ_x \subset E$, so $N(qΨ) \subset E$. By Lemma 2.2, there is a sequence of inner functions $\{Ψ_n\}_{n=1}^\infty$ such that $[H^\infty, qΨ] = [H^\infty, \{Ψ_n\}_{n=1}^\infty]$. Since $N(Ψ_n) \subset N(qΨ) \subset E$, we get $[H^\infty, qΨ] \subset B$. By (3) and (4),

\[ 0 = \int_{N(\overline{I})} qΨΨd|λ| = \int_{N(\overline{I})} d|λ|. \]
Hence \( \lambda |N(\overline{I}) = 0 \). Thus we get (2). Consequently \( \mu - f_\mu \mu = \mu |N(\overline{I}_0) \perp B_\perp \), so we get (b) of Lemma 7.2.

Applying Lemma 7.2, \( B/H^\infty + C \) is an \( M \)-ideal of \( L^\infty /H^\infty + C \). This completes the proof of (i).

(ii) Suppose that \( B = (H^\infty + C)_F \) for a weak peak subset of \( X \) for \( H^\infty + C \). To show \( F \supset X \setminus E \), suppose not. Then there exists an open and closed subset \( U \) of \( X \) with \( U \cap (E \cup F) = \emptyset \). By Lemma 7.4, there is an inner function \( I \) with \( \emptyset \neq N(\overline{I}) \subset U \). Then there is \( x \in M(H^\infty + C) \backslash X \) such that \( \text{supp} \mu_x \subset U \). Since \( \text{supp} \mu_x \cap E = \emptyset \), \( B|\text{supp} \mu_x = H^\infty|\text{supp} \mu_x \). Since \( \text{supp} \mu_x \cap F = \emptyset \), \( (H^\infty + C)_F|\text{supp} \mu_x \) coincides with the space of continuous functions on \( \text{supp} \mu_x \). This is a contradiction, so we have \( F \supset X \setminus E \).

Let \( V \) be the closure of \( X \setminus E \). By [4, p. 18], \( V \) is an open and closed subset of \( X \) and \( V \subset F \). Since \( QC \) does not have nontrivial idempotents, \( E \cap V \neq \emptyset \). Since \( E \cap V \) is a closed \( G_\delta \)-set, again by Lemma 7.4 there is an inner function \( J \) with \( \emptyset \neq N(\overline{J}) \subset E \cap V \). By the definition of \( B \), \( J \in B \). Since \( N(\overline{J}) \subset V \subset F \), \( J \notin (H^\infty + C)_F \). This contradicts \( B = (H^\infty + C)_F \). Hence we get (ii).

REMARK. Let \( B \) be a Douglas algebra given in Theorem 7.1. By the above proof and [4, p. 59], \( N(I) \) is an interpolation set for \( B \) for every \( I \in \Lambda_E \), that is, \( B|N(\overline{I}) = C(N(\overline{I})) \).

COROLLARY 7.1. Let \( B \) be a Douglas algebra given in Theorem 7.1. Then \( B \) has the best approximation property, that is, for each \( f \in L^\infty \) there is \( g \in B \) such that \( \| f + B \| = \| f - g \| \).

As a special case, we get Proposition 2 in [18]. Let \( F \) be an open subset of \( \partial D \). Put \( L^\infty_F = \{ f \in L^\infty ; f \text{ is continuous at each point of } F \} \), and \( E = \{ x \in X ; z(x) \in X \setminus F \} \). Then \( E \) is a peak set for \( QC \), and it is easy to see that \( H^\infty + L^\infty_F = [H^\infty, \{ \overline{I} ; I \text{ is an inner function with } N(\overline{I}) \subset E \}] \). Hence \( H^\infty + L^\infty_F \) has the best approximation property.

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DEPARTMENT OF MATHEMATICS, KANAGAWA UNIVERSITY, YOKOHAMA 221, JAPAN