NONHARMONIC FOURIER SERIES AND SPECTRAL THEORY

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ABSTRACT. We consider the problem of using functions $g_n(x) := \exp(i\lambda_n x)$ to form biorthogonal expansions in the spaces $L^p(\pi, \pi)$, for various values of $p$. The work of Paley and Wiener and of Levinson considered conditions of the form $|\lambda_n - n| \leq \Delta(p)$ which insure that $\{g_n\}$ is part of a biorthogonal system and the resulting biorthogonal expansions are pointwise equiconvergent with ordinary Fourier series. Norm convergence is obtained for $p = 2$. In this paper, rather than imposing an explicit growth condition, we assume that $\{\lambda_n - n\}$ is a multiplier sequence on $L^p(\pi, \pi)$. Conditions are given insuring that $\{g_n\}$ inherits both norm and pointwise convergence properties of ordinary Fourier series. Further, $\lambda_n$ and $g_n$ are shown to be the eigenvalues and eigenfunctions of an unbounded operator $\Lambda$ which is closely related to a differential operator, $i\Lambda$ generates a strongly continuous group and $-\Lambda^2$ generates a strongly continuous semigroup. Half-range expansions, involving $\cos \lambda_n x$ or $\sin \lambda_n x$ on $(0, \pi)$ are also shown to arise from linear operators which generate semigroups. Many of these results are obtained using the functional calculus for well-bounded operators.

1. Introduction. For $n$ an integer, let $\{\lambda_n\}$ be a sequence of pairwise distinct complex numbers. For $-\pi \leq x \leq \pi$ let

$$g_n(x) = e^{i\lambda_n x}, \quad \varphi_n(x) = e^{inx},$$

and for integrable functions $f, g$ let

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g}(x) \, dx.$$

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For fixed $p$, assume there exists a sequence $\{h_n\}$ in $L^q (= L^q(\pi, \pi))$ such that

$$(g_n, h_m) = \delta_{nm}.$$

Then for $f$ in $L^p$, define the partial sum operator

$$S_N(x; f) = \sum_{n=-N}^{N} (f, h_n) g_n(x).$$

The partial sum operator for ordinary Fourier series is

$$S_N(x; f) = \sum_{n=-N}^{N} \hat{f}_n \varphi_n(x), \quad \hat{f}_n = (f, \varphi_n).$$
The problem of nonharmonic Fourier series is to find conditions on \( \{ \lambda_n \} \) so that for some \( p \), the dual sequence \( \{ h_n \} \) exists in \( L^q \), and for all \( f \) in \( L^p \), the partial sum operators \( S_N(x; f) \) have the same properties as the operators \( S_N(x; f) \), with respect to norm behavior, pointwise behavior, or both.

In this paper we shall consider these questions, subject to the basic assumption that the sequence \( \{ \delta_n \} \), defined by

\[
\delta_n = \lambda_n - n,
\]

is a multiplier sequence on \( L^p \) for some fixed but arbitrary \( p \), \( 1 \leq p < \infty \). This means that there is a bounded linear operator \( \mathcal{M} : L^p \to L^p \) such that for each \( f \) in \( L^p \),

\[
(\mathcal{M} f)_n = \delta_n f_n.
\]

Another significant property of the sequences \( \{ n \} \) and \( \{ q_n \} \) is that they contain the eigenvalues and eigenfunctions of the differential operator \( \Lambda_0 \) defined by

\[
\Lambda_0 u = -iu', \quad (u' = du/dx),
\]

with domain \( \mathcal{D}(\Lambda_0) \) consisting of all absolutely continuous functions \( u \) such that \( u' \) is in \( L^p \) and such that

\[
\quad u(-\pi) = u(\pi).
\]

Thus

\[
\quad \Lambda_0 q_n = n q_n.
\]

For \( p = 2 \) the operator \( \Lambda_0 \) is selfadjoint. For \( 1 < p < \infty \) the spectral theory of \( \Lambda_0 \) is embodied in the statement that for some complex number \( \lambda \) in the resolvent set of \( \Lambda_0 \), the resolvent operator \( R(\lambda, \Lambda_0) \) is well-bounded. See [2] for the definition and applications to differential operators. We shall give conditions under which there exists a linear operator \( \Lambda \) such that

\[
\Lambda g_n = \lambda_n g_n,
\]

and such that the resolvent operator is well-bounded, \( 1 < p < \infty \). This is then used to study the properties of half-range expansions, i.e., expansions on \( L^p(0, \pi) \) (or on \( L^p(-\pi, 0) \)) using the sequence \( \{ \cos \lambda_n x \} \) or \( \{ \sin \lambda_n x \} \). In particular, we show that the operators associated with these expansions generate strongly continuous semigroups.

The study of nonharmonic Fourier series was initiated by Paley and Wiener [8] and by Levinson [7]. Paley and Wiener showed that for \( p = 2 \) and \( \lambda_n \) real, if \( |\delta_n| \leq 1/\pi^2 \), then \( \{ h_n \} \) exists and for any \( f \) in \( L^2(-\pi, \pi) \), the partial sums \( S_N(x; f) \) and \( S_n(x; f) \) have the same behavior with respect to pointwise convergence:

\[
\lim_{N \to \infty} \left[ S_N(x; f) - S_n(x; f) \right] = 0,
\]

uniformly on each closed subinterval interior to \( (-\pi, \pi) \). With respect to convergence in the norm of \( L^2(-\pi, \pi) \), Paley and Wiener also showed that \( \{ g_n \} \) is a Riesz basis: there exists a bounded and invertible linear operator \( A \) on \( L^2 \) such that

\[
A q_n = q_n,
\]

and thus \( \{ g_n \} \) has the same norm convergence properties in \( L^2 \) as does \( \{ q_n \} \).
The above result on pointwise convergence was generalized by Levinson, who showed that if \( 1 < p \leq 2 \) and if
\[
|\delta_n| \leq L < (p - 1)/2p,
\]
then \( \{h_n\} \) exists and for any \( f \) in \( L^p(-\pi, \pi) \) the partial sums \( S_N(x; f) \) and \( \mathcal{S}_N(x; f) \) are uniformly equiconvergent on closed intervals interior to \((-\pi, \pi)\). Levinson did not give any results on the norm convergence of \( \mathcal{S}_N \).

The question of norm convergence was considered by Pollard in [10]. There it was shown that for \( 1 < p < \infty \), if \( r = 2p/(2 - p) \) and if \( \{\delta_n\} \) is in \( l^r \), with
\[
\|\{\delta_n\}\|_r < (\ln 2)/\pi,
\]
then \( \{g_n\} \) is a basis for \( L^p \) and there exists a bounded invertible operator \( A : L^p \rightarrow L^p \) such that (1.13) holds. If \( p = 2 \) then \( r = \infty \) and (1.15) becomes
\[
|\delta_n| \leq L < (\ln 2)/\pi.
\]
This result for \( p = 2 \) had been obtained earlier by Duffin and Eachus [4].

All of these conditions on \( \{\delta_n\} \), whether for pointwise convergence, norm convergence, or both, impose a limitation on \( \{\delta_n\} \): in none of these conditions is \( |\delta_n| \) allowed to be greater than \( \frac{1}{2} \). Consider the example \( \delta_n = \delta \) for all \( n \), where \( \delta \) is an arbitrary complex number. Then
\[
g_n(x) = e^{i\delta x} \varphi_n(x).
\]
It is a simple matter to see that even if \( \delta \) is selected so that none of the above conditions are satisfied, the resulting \( \{g_n\} \) satisfies all of the conclusions of the above theorems, and in fact more is true: the pointwise equiconvergence theorem holds in the larger class \( L'(-\pi, \pi) \), and \( \{g_n\} \) is the set of eigenfunctions of an unbounded linear operator which generates a strongly continuous bounded group of transformations on \( L^p \), \( 1 < p < \infty \), and whose square generates a strongly continuous semigroup.

The conditions given by Paley and Wiener and by Pollard imply that \( \{\delta_n\} \) is a multiplier sequence, and the same clearly holds for the above example. Thus the assumption that \( \{\delta_n\} \) is a multiplier sequence contains all of the previous norm results, frees the theory from explicit growth conditions, and allows the association to each sequence \( \{g_n\} \) of an unbounded linear operator whose spectral theory incorporates the norm properties of \( \{g_n\} \). Further, if \( \{\delta_n\} \) is a multiplier sequence and if \( \{g_n\} \) is a basis for \( L^p \) equivalent to \( \{\varphi_n\} \), then pointwise equiconvergence is also obtained. Levinson's results are not included in this theory.

A survey of nonharmonic Fourier series is in [13] and other recent results on norm behavior can be found in [14, 15].

2. Norm convergence.

2.1. Definition. The sequences \( \{g_n\} \), \( \{\varphi_n\} \) are equivalent in \( L^p \) if there exists a bounded linear operator \( A : L^p \rightarrow L^p \), with bounded inverse, such that
\[
A\varphi_n = g_n.
\]
Note that the definition applies for \( p = 1 \), where \( \{ \varphi_n \} \) is not a basis. The invertibility of \( A \) is sufficient for the existence of the dual sequence \( \{ h_n \} \) in \( L^q \):

\[
h_n = A^{-1} \varphi_n.
\]

2.4. LEMMA. If \( \{ g_n \} \) and \( \{ \varphi_n \} \) are equivalent, then

\[
\mathcal{S}_N = A S_N A^{-1}.
\]

**PROOF.** From (2.2) and (2.3) we have \((f, h_n)g_n = A(A^{-1}f, \varphi_n)\varphi_n\).

2.6. THEOREM. If \( \{ g_n \} \) is equivalent to \( \{ \varphi_n \} \) in \( L^p \), \( 1 < p < \infty \), then

\[
\lim_{N \to \infty} ||\mathcal{S}_N f - f||_p = 0.
\]

If \( \{ g_n \} \) is equivalent to \( \{ \varphi_n \} \) in \( L^1 \), then the arithmetic means of \( \mathcal{S}_N f \) converge to \( f \) in the norm of \( L^1 \).

**PROOF.** We have \( \mathcal{S}_N - I = A[S_N - I]A^{-1} \) and

\[
\frac{1}{N + 1} \sum_{n=0}^{N} \mathcal{S}_N - I = A \left[ \frac{1}{N + 1} \sum_{n=0}^{N} S_n - I \right]A^{-1}.
\]

Thus \( \mathcal{S}_N \) inherits the properties of \( S_N \).

Let \( X: L^p \to L^p \) be the linear operator defined by

\[
(X f)(x) = x f(x).
\]

Note that \( ||X|| = \pi \).

2.8. THEOREM. If \( \{ \delta_n \} \) is a multiplier sequence for some \( L^p \), \( 1 \leq p < \infty \), and if \( A \) is the linear operator defined by

\[
A = \sum_{k=0}^{\infty} \frac{(iX)^k \mathcal{M}^k}{k!},
\]

then \( A\varphi_n = g_n \). (It is not claimed that \( A \) is invertible.)

**PROOF.** If there exists an operator \( A \) such that \( A\varphi_n = g_n \), then for any trigonometric polynomial

\[
t(x) = \sum_{n=-N}^{N} \hat{t}_n \varphi_n(x)
\]

we must have

\[
A t = \sum_{n=-N}^{N} \hat{t}_n g_n.
\]

Now \( g_n(x) = \varphi_n(x) e^{ixx} \), so

\[
A t = \sum_{n=-N}^{N} \hat{t}_n \varphi_n \sum_{k=0}^{\infty} \frac{(ix)^k \delta^k}{k!} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \sum_{n=-N}^{N} \delta^k \hat{t}_n \varphi_n.
\]

Since \( \mathcal{M}^k t = \sum_{n=-N}^{N} \delta^k \hat{n} \varphi_n \), we have

\[
A t = \sum_{k=0}^{\infty} \frac{(ix)^k \mathcal{M}^k}{k!} t, \quad ||A t|| \leq e^{\pi ||\mathcal{M}|| ||t||}.
\]
Since the trigonometric polynomials are dense in \( L^p \) for \( 1 \leq p < \infty \), the extension to all of \( L^p \) of the operator defined by (2.10) is the operator defined in (2.9).

2.11. **Theorem.** If for some \( p \), \( 1 \leq p < \infty \),

\[
\|a\|_p < \frac{\ln 2}{\pi},
\]

then \( \{ g_n \} \) is equivalent to \( \{ \varphi_n \} \) in \( L^p \).

**Proof.** It suffices to show that \( \| A - I \| < 1 \). From (2.9),

\[
\| A - I \| \leq \sum_{k=1}^{\infty} \frac{\| X \|^k \| \mathcal{M} \|^k}{k!} = e^{\pi\| \mathcal{M} \|} - 1.
\]

Then (2.12) follows from the condition \( e^{\pi\| \mathcal{M} \|} - 1 < 1 \).

This theorem contains the theorems of Duffin and Eachus and of Pollard. Using the Fredholm alternative to invert operators of the form \( I - K \), where \( K \) is compact, along with a representation of the dual sequence given by Levinson [7, Lemma 16.2], condition (1.15) of Pollard’s theorem can be eliminated.

2.13. **Theorem.** Let \( 1 \leq p < \infty \), \( p \neq 2 \), and let \( r = 2p/(2-p) \). Then \( \{ g_n \} \) and \( \{ \varphi_n \} \) are equivalent if

(i) \( \lambda_n \neq \lambda_m \) for \( n \neq m \);

(ii) \( \delta_n \) is in \( l^r \).

The proof follows some preliminary material.

2.14. **Lemma.** Let \( \{ u_n \} \) be a sequence in a Banach space \( B \) and let \( \{ v_n \} \) be a sequence in a dual space \( B^* \) such that \( (u_n, v_m) = \delta_{nm} \). Let \( \{ g_n \} \) be a sequence in \( B \) such that

1. \( g_n = u_n \) except for \( n \) in a finite set \( S \);
2. \( (g_n, v_m)_{n,m} \neq 0 \).

Then there exists a bounded, invertible operator \( A : B \to B \) such that \( Au_n = g_n \).

**Proof.** For \( f \) in \( B \) define an operator \( K \) by

\[
Kf = \sum_{n \in S} (f, v_n)(u_n - g_n),
\]

and let \( A = I - K \). Then \( K \) is compact and \( Au_n = g_n \) for all \( n \). To show that \( A \) is invertible it suffices to show (by the Fredholm alternative) that \( Af = 0 \) implies \( f = 0 \). We have \( Af = 0 \) if and only if \( f = Kf \):

\[
f = \sum_{n \in S} (f, v_n)(u_n - g_n).
\]

Then for \( m \) in \( S \),

\[
\sum_{n \in S} (f, v_n)(g_n, v_m) = 0.
\]

From condition (2), \( (f, v_n) = 0 \) for all \( n \) in \( S \), and from (2.15), \( f = 0 \).

For Levinson’s representation of \( h_n \), we need the Fourier transform \( \mathcal{F} \) and its inverse \( \mathcal{F}^{-1} \) defined by

\[
\mathcal{F} f (\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} \, dx \quad (f \text{ defined on } (-\infty, \infty)),
\]

\[
\mathcal{F}^{-1} \mathcal{F} f (x) = f(x).
\]
For the sequence \( \{ \lambda_n \} \), let

\[
G(\lambda) = (\lambda - \lambda_0) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \left(1 - \frac{\lambda}{\lambda - \lambda_n}\right).
\]

Questions of convergence will be considered below. For \( 1 < p < \infty \), let \( p^{-1} + q^{-1} = 1 \), \( s = \min(p, q) \), \( s^{-1} + t^{-1} = 1 \). \( L^p \) refers to the interval \((-\pi, \pi)\) and \( L^p(\mathbb{R}) \) refers to \((-\infty, \infty)\).

2.19. THEOREM (Levinson [7; pp. 48–58]). Let \( 1 < p < \infty \). Assume

\[
|\delta_n| \leq L < (s - 1)/2s.
\]

Then the infinite product (2.18) converges to an entire function \( G(\lambda) \) such that if

\[
H_n(\lambda) = G(\lambda)/\left[(\lambda - \lambda_n)G'(\lambda_n)\right],
\]

then

(i) \( H_n \) is in \( L^s(\mathbb{R}) \) for \( \lambda \) restricted to \( \mathbb{R} \),

(ii) \( (\mathcal{F}^{-1}H_n)(x) \) is in \( L^s(\mathbb{R}) \), and its support is contained in \((-\pi, \pi)\),

(iii) the dual sequence \( \{ h_n \} \) is given by

\[
\bar{h}_n(x) = 2\pi(\mathcal{F}^{-1}H_n)(x), \quad -\pi < x < \pi,
\]

and

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda x} \bar{h}_n(x) \, dx = H_n(\lambda), \quad \lambda \in \mathbb{C}.
\]

2.24. REMARKS. Levinson’s theorems are stated for \( 1 < p \leq 2 \), but using the containment relations for \( L^p \) spaces on finite intervals, the above extension of the range of \( p \) holds. Also, what is denoted by \( h_n \) in Levinson’s work is \( 2\pi \bar{h}_n \) in our notation.

2.25. LEMMA. For a finite set \( S \) of indices, let \( \{ \lambda_n \}, \{ \mu_n \}, n \in S \), be two sets of complex numbers such that no two numbers are the same. Then

\[
M := \det((\lambda_n - \mu_m)^{-1})_{n,m \in S} \neq 0.
\]

PROOF. Let \( p(\lambda) = \prod_{m \in S} (\lambda - \mu_m) \) and let \( p_i(\lambda) = p(\lambda)/(\lambda - \mu_i) \). Then

\[
\frac{1}{\lambda_n - \mu_m} = \frac{p_m(\lambda_n)}{p(\lambda_n)}, \quad (p(\lambda_n) \neq 0).
\]

Thus

\[
\left[ \prod_{n=1}^{\infty} p(\lambda_n) \right] M = \det(p_m(\lambda_n)).
\]

Now each \( p_m \) is a polynomial of degree \(|S| - 1\), where \(|S|\) is the cardinality of \( S \), and all zeros of \( p_m(\lambda) \) are accounted for by \( \lambda = \mu_i \), where \( i \in S, \ i \neq m \). Since \( \lambda_n \neq \mu_i \), we have \( M \neq 0 \).
PROOF OF THEOREM 2.13. There exists a finite set $S$ of indices $n$ such that $\delta_n \neq n$ for $n \in S$, $|\delta_n| \leq L < (s - 1)/2s$, and

$$\left( \sum_{n \in S} |\delta_n|^s \right)^{1/r} < (\ln 2)/\pi.$$ 

Let $\mu_n = m$ for $n \in S$, $\mu_n = \lambda_n$ for $n \not\in S$, and let $u_n(x) = e^{i\mu_n x}$. Since $\{\mu_n - m\}$ satisfies Pollard's theorem (or Theorem 2.11), we see that $\{u_n\}$ is equivalent to $\{\varphi_n\}$. Let $\{v_n\}$ denote the dual sequence. Since $\{\mu_m - n\}$ also satisfies Levinson's condition (2.20), we have

$$\frac{1}{2\pi} \int_{-\pi}^\pi e^{i\lambda x} \delta_n(x) dx = H_m(\lambda).$$

Since $\{g_n\}$ and $\{u_n\}$ differ only for $n \in S$, to show that $\{g_n\}$ and $\{u_n\}$ are equivalent it suffices to show that

$$\det((g_n, v_m))_{n, m \in S} \neq 0.$$ 

Using (2.26), this becomes

$$\det(H_m(\lambda_n))_{n, m \in S} \neq 0.$$ 

Using (2.21), this becomes

$$\left[ \prod_{n \in S} G(\lambda_n) \right] \left[ \det((\lambda_n - \mu_m)^{-1})_{n, m \in S} \right] \neq 0.$$ 

Recall that $G(\lambda)$ is formed with zeros at $\{\mu_m\}$, so $G(\lambda_n) \neq 0$ for $n \in S$. Since the set $\{\lambda_n\}$ is disjoint from the set $\{\mu_n\}$ for $n \in S$, the determinant in (2.27) is not zero.

2.28 REMARK. The analogue of Theorem 2.13 for $p = 2$ is that $|\delta_n| \leq L < (\ln 2)/\pi$ for $|n|$ sufficiently large, and, for the finitely many remaining $\delta_n$'s, that they are pairwise distinct.

For $p \neq 2$, Theorem 2.13 requires that $\delta_n \to 0$ as $|n| \to \infty$. Using the theory of well-bounded operators, a general class of multipliers can be given for which $\delta_n \to 0$ is not necessary. A special case will yield a proof of a theorem of Kadec [6]:

THEOREM (KADEC). Let $\{\delta_n\}$ be real and assume $|\delta_n| \leq L < \frac{1}{4}$. Then $\{g_n\}$ is a basis for $L^2$ equivalent to $\{\varphi_n\}$.

Some of the details of this theory are now presented.

2.29. DEFINITION. An arc $C$ in the complex plane is admissible if it is simple, nonclosed and rectifiable.

Let $S$ denote the length of $C$ and let $\rho:[0, S] \to C$ denote the arc-length parameterization of $C$, with $b = \rho(S)$. A function $f:C \to C$ is said to be absolutely continuous on $C$ if $f \circ \rho$ is absolutely continuous on $[0, S]$, and for such functions $f$, we define

$$\|f\|_C = |f(b)| + \int_C |df| dz.$$ 

2.31. DEFINITION (RINGROSE [12, p. 634]). An operator $T$ on a Banach space is well-bounded on $C$ if there exists a constant $K > 0$ such that if $p(z)$ is any
polynomial, then
\[ \|p(T)\| \leq K \|p\|_C. \]

2.33. **Theorem** [12, p. 636]. If \( T \) is well-bounded on \( C \), then for each absolutely continuous function \( f \) on \( C \), there is a bounded linear operator \( f(T) \) such that the mapping \( f \to f(T) \) is a homomorphism of \( AC(C) \) into the algebra of bounded linear operators, and
\[ \|f(T)\| \leq K \|f\|_C. \]

If the underlying Banach space is reflexive, then there exists a family of projections \( \{E(\lambda) : \lambda \in C\} \) a spectral family for \( T \), which can be used to express \( f(T) \) as a modified Riemann-Stieltjes integral [3, Chapter 17]. See also [2, Proposition 2.3], where we see that the constant \( K \) of (2.34) can be chosen to be \( \sup\{\|E(\lambda)\| : \lambda \in C\} \).

For \( \Delta > 0 \), let
\[ T_\Delta = \Delta T, \quad C_\Delta = \{\Delta z : z \in C\}, \quad E_\Delta(\lambda) = E(\lambda/\Delta), \quad \lambda \in C_\Delta. \]

2.36. **Theorem.** \( T_\Delta \) is a well-bounded operator on \( C_\Delta \) with spectral family \( E_\Delta \), and for any function \( f \) which is absolutely continuous on \( C_\Delta \),
\[ \|f(T_\Delta)\| \leq K \|f\|_{C_\Delta}, \]
where
\[ K = \sup\{E_\Delta(\lambda) : \lambda \in C_\Delta\} = \sup\{E(\lambda) : \lambda \in C\}. \]

For the proof of this theorem, see [1] for a general discussion of functions of well-bounded operators. We emphasize that the constant \( K \) for \( T_\Delta \) in (2.37) is computable from the spectral family for \( T \).

Multiplier transforms which are well-bounded have been studied by D. J. Ralph [11].

2.39. **Definition.** A real sequence \( \{\delta_n\}, -\infty < n < \infty \), is piecewise monotone if \( \{\delta_n\} \) is monotone for \( |n| \) sufficiently large. A complex sequence \( \{\delta_n\} \) lying on an admissible arc \( C \) is piecewise monotone if \( \{\rho^{-1}(\delta_n)\} \) is piecewise monotone.

Note that the sense of monotonicity does not have to be the same for the two tails of \( \{\delta_n\} \).

2.40. **Theorem** [11, Corollary 3.2.6]. If \( \{\delta_n\} \) is a piecewise monotone sequence on an admissible arc \( C \), then \( \{\delta_n\} \) is a multiplier sequence for \( L^p \), \( 1 < p < \infty \), and the associated multiplier transform \( M \) is well-bounded on \( C \). Moreover, if \( f \) is absolutely continuous on \( C \), then for any \( g \) in \( L^p \)
\[ \left( (f(T)g)_n^* = f(\delta_n)\hat{g}_n. \right. \]

The proof of the next theorem depends upon the expression (due to Kadec [6]) of \( 1 - e^{i\lambda x} \) in the orthonormal system \( \{1, \cos nx, \sin(n - \frac{1}{2})x\} \) for \( n \geq 1 \):
\[ 1 - e^{i\lambda x} = \left( 1 - \frac{\sin \pi \delta}{\pi \delta} \right) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \delta \sin \pi \delta}{k^2 - \delta^2} \cos kx \]
\[ + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \delta \cos \pi \delta}{(k - \frac{1}{2})^2 - \delta^2} \sin \left( k - \frac{1}{2} \right)x. \]
2.43. **Theorem.** Let $M$ be a well-bounded multiplier transform on some $L^p$, $1 < p < \infty$, with multiplier sequence $\{\delta_n\}$. Then there exists $\Delta = \Delta(M, p) > 0$ such that if $\lambda_n = n + \Delta \delta_n$, then $\{g_n\}$ is a basis for $L^p$ equivalent to $\{\varphi_n\}$.

**Proof.** Let $f$ be a trigonometric polynomial in $L^p$: $f = \sum_{-N}^{N} \hat{f}_n \varphi_n$. For such $f$, $Af = \sum_{-N}^{N} \hat{f}_n g_n$ exists, and $B = I - A$ is defined:

$$Bf = \sum_{-N}^{N} \hat{f}_n \varphi_n [1 - e^{i\Delta x}].$$

Using (2.42) with $\delta = \Delta \delta_n$ and then interchanging the order of summation, we have

$$(2.44) \quad Bf = \sum_{n=-N}^{N} \left(1 - \frac{\sin \pi \Delta \delta_n}{\pi \Delta \delta_n}\right) \hat{f}_n \varphi_n$$

Define functions

$$\alpha(\delta) = 1 - \frac{\sin \pi \delta}{\pi \delta}, \quad \beta_k(\delta) = \frac{\delta \sin \pi \delta}{k^{2} - \delta^2},$$

$$\gamma_k(\delta) = \frac{\delta \cos \pi \delta}{(k - \frac{1}{2})^2 - \delta^2}, \quad k = 1, 2, \ldots.$$  

These functions are absolutely continuous on any admissible arc, and by Theorem 2.40, for any $f$ in $L^p$,

$$\alpha(\mathcal{M}_\Delta)f = \sum_{-\infty}^{\infty} \alpha(\Delta \delta_n) \hat{f}_n \varphi_n, \quad \beta_k(\mathcal{M}_\Delta)f = \sum_{-\infty}^{\infty} \beta_k(\Delta \delta_n) \hat{f}_n \varphi_n,$$

$$\gamma_k(\mathcal{M}_\Delta)f = \sum_{-\infty}^{\infty} \gamma_k(\Delta \delta_n) \hat{f}_n \varphi_n.$$  

Thus, using the density in $L^p$ of the trigonometric polynomials, we see that the operators $A, B$ have continuous extensions to all of $L^p$, and

$$Bf = \alpha(\mathcal{M}_\Delta)f + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \cos kx \beta_k(\mathcal{M}_\Delta)f$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \sin (k - \frac{1}{2})x \gamma_k(\mathcal{M}_\Delta)f.$$  

Let $K > 0$ be selected as in (2.38), where $E(\lambda)$ is the spectral family of $\mathcal{M}$. Using the triangle inequality,

$$\|Bf\| \leqslant \|f\| K \left\{ \|\alpha\|_{C_\Delta} + \frac{2}{\pi} \sum_{k=1}^{\infty} \|\beta_k\|_{C_\Delta} + \frac{2}{\pi} \sum_{k=1}^{\infty} \|\gamma_k\|_{C_\Delta} \right\}.$$
Note that 
\[ |\beta_k(\delta)| = \mathcal{O}(\delta/k^2), \quad \text{uniformly as } \delta \to 0, \ k \to \infty, \]
\[ |\beta'_k(\delta)| = \mathcal{O}(\delta/k^2), \quad \text{also uniformly,} \]
so that 
\[ \|\beta_k\|_{C_0} = \mathcal{O}(\Delta/k^2) \quad \text{as } k \to \infty. \]

Using similar estimates for \( \alpha, \gamma_k \), we see that for \( \Delta \) sufficiently small, \( \|B\| < 1 \), and then \( A \) is invertible.

If \( \{\delta_n\} \) is real, then more precision in estimating \( \|\beta_k\| \), etc., can be obtained. For \( |\delta| \leq L \leq \frac{1}{4} \) we see that \( \text{var} \beta_k = 2\beta_k(L) \) so that 
\[ \|\beta_k\|_{[-L,L]} = 3\beta_k(L) \]
and similarly for \( \alpha, \gamma_k \). Thus
\[ \|Bf\| \leq \|f\|3K \left( 1 - \frac{\sin \pi \Delta L}{\pi \Delta L} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\Delta L \sin \pi \Delta L}{k^2 - (\Delta L)^2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\Delta L \cos \pi \Delta L}{(k - \frac{1}{2})^2 - (\Delta L)^2} \right). \]

Again using Kadec [6], we note that
\[ \frac{2\Delta L}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 - (\Delta L)^2} = \frac{1}{\Delta \pi L} - \cot \pi \Delta L, \]
\[ \frac{2\Delta L}{\pi} \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2 - (\Delta L)^2} = \tan \pi \Delta L, \]
so
\[ \|Bf\| \leq \|f\|3K \left( 1 - \frac{\sin \pi \Delta L}{\pi \Delta L} + \frac{\sin \pi \Delta L}{\pi \Delta L} - \cos \pi \Delta L + \sin \pi \Delta L \right), \]
\[ \|B\| \leq 3K \left[ 1 - \cos \pi \Delta L + \sin \pi \Delta L \right]. \]

Then for \( \Delta \) sufficiently small, \( \|B\| < 1 \).

To say how small \( \Delta \) should be, it is necessary to know \( K \). Let \( H \) denote the conjugate function mapping on \( L^p \). For \( 1 < p < \infty \), let \( s = \min(p, q) \). Then [9]
\begin{equation}
(2.45) \quad \|H\|_p = \tan(\pi/2r). \end{equation}

Using the representation of the spectral family of \( \mathcal{M} \) [11, Theorem 3.2.4] we have:

2.46. LEMMA. If \( \{\delta_n\} \) is real and piecewise monotone, and if \( m \) is the number of intervals (of integers) on which \( \{\delta_n\} \) is monotone, then
\[ K \leq m \left[ 1 + \tan \pi/2r \right]. \]

Kadec’s theorem was based on Parseval’s equality. A spectral-theoretic proof can be given, since \( \mathcal{M} \) is then selfadjoint and \( \|\beta_k(\mathcal{M})\| = \sup(|\beta_k(\delta_n)|), \) etc.
3. Pointwise convergence.

3.1. THEOREM. Let \( p \) be fixed, \( 1 \leq p < \infty \). Assume \( \{ \delta_n \} \) is a multiplier sequence in \( L^p \) and that \( \{ g_n \}, \{ \varphi_n \} \) are equivalent. Then for each \( f \) in \( L^p \),
\[
\lim_{N \to \infty} \left[ \mathcal{S}_N(x; f) - S_N(x; f) \right] = 0,
\]
uniformly on each interval \([-\pi + d, \pi - d] \), \( d > 0 \).

3.2. REMARK. Note that this theorem includes the case \( p = 1 \), even though \( \{ \varphi_n \}, \{ g_n \} \) are not bases in \( L^1 \). Theorem 3.1 contains as a special case a result of Duffin and Schaeffer [5, §4] for \( L^2 \).

Proof of Theorem 3.1. Since \( \{ g_n \}, \{ \varphi_n \} \) are equivalent, we have \( \mathcal{S}_N = A S_N A^{-1} \). Using the expression (2.9) for \( A \), we have
\[
\mathcal{S}_N = \sum_{k=0}^{\infty} X^k \frac{(i\mathcal{M})^k}{k!} S_N A^{-1},
\]
but since \( \mathcal{M} \) and \( S_N \) commute,
\[
(3.3) \quad \mathcal{S}_N = \sum_{k=0}^{\infty} X^k S_N \frac{(i\mathcal{M})^k}{k!} A^{-1}.
\]
Since \( S_N = S_N A A^{-1} \), we have
\[
(3.4) \quad S_N = \sum_{k=0}^{\infty} S_N X^k \frac{(i\mathcal{M})^k}{k!} A^{-1},
\]
and then
\[
\mathcal{S}_N f - S_N f = \sum_{k=1}^{\infty} \left( X^k S_N - S_N X^k \right) \frac{(i\mathcal{M})^k}{k!} A^{-1} f.
\]
Let \( D_N \) denote the Dirichlet kernel
\[
D_N(x - t) = \frac{\sin(N + \frac{1}{2})(x - t)}{2 \sin((x - t)/2)}.
\]
For any function \( g \) in \( L^p \),
\[
( X^k S_N - S_N X^k ) g(x) = \int_{-\pi}^{\pi} D_N(x - t)(x^k - t^k) g(t) \, dt,
\]
where
\[
D_N(x - t)(x^k - t^k)
= \sin(N + \frac{1}{2})(x - t) \frac{x - t}{2 \sin((x - t)/2)} [x^{k-1} + x^{k-2}t + \cdots + t^{k-1}].
\]
Given \( d > 0 \), there exists \( K = K(d) > 0 \) such that if \( |x| \leq \pi - d \), then
\[
|D_N(x - t)(x^k - t^k)| \leq Kk\pi^k, \quad |x| \leq \pi - d, \quad |t| \leq \pi.
\]
Thus
\[
|(X^k S_N - S_N X^k) g(x)| \leq 2\pi K k \pi^k ||g||.
\]
Let $\varepsilon > 0$ be given. Then there exists $J = J(\varepsilon, f)$ such that
\[
\left| \sum_{k=1}^{\infty} (X^k S_N - S_N X^k) \frac{(i \mathcal{M})^k}{k!} A^{-1} f \right| < \frac{\varepsilon}{2}
\]
for all $N$, $|x| \leq \pi - d$. For the finitely many remaining terms, it is easily seen that the Riemann-Lebesgue lemma holds uniformly in $x$, $|x| \leq \pi - d$, so for $N$ sufficiently large,
\[
\left| \sum_{k=1}^{J-1} (X^k S_N - S_N X^k) \frac{(i \mathcal{M})^k}{k!} A^{-1} f \right| < \frac{\varepsilon}{2}.
\]

4. Eigenfunction expansions. In this section we assume $\{\delta_n\}$ is a multiplier sequence for $L^p$, for some $p$, $1 \leq p < \infty$, and that the corresponding $\{g_n\}$ is equivalent to $\{\varphi_n\}$. Let $\Lambda_0$ be the differential operator defined in (1.8), (1.9), and let $\Lambda$ be defined by

\[(4.1) \quad \Lambda = A(\Lambda_0 + \mathcal{M}) A^{-1}, \quad \mathcal{D}(\Lambda) = A \mathcal{D}(\Lambda_0).\]

4.2. Theorem. $\Lambda$ is a closed, densely defined operator on $L^p$,

\[(4.3) \quad \Lambda g_n = \lambda_n g_n,
\]
and $i\Lambda$ is the infinitesimal generator of the uniformly bounded, strongly continuous group

\[(4.4) \quad U(t) = A U_0(t) e^{i \mathcal{M} t} A^{-1}, \quad t \in \mathbb{R},
\]
where $U_0(t)$ is the translation group generated by $i\Lambda_0$.

Proof. This is a direct consequence of (4.1), noting that $\Lambda_0$ and $\mathcal{M}$ commute.

For the further study of $\Lambda$, let $1 < p < \infty$. Then

\[(4.5) \quad (\lambda I - \Lambda)^{-1} f = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_n)^{-1} (f, h_n) g_n, \quad f \in L^p.
\]
Since $\{(\lambda - \lambda_n)^{-1}\}$ is in $l^r$ for all $r$, $1 < r < \infty$, it follows that $\{(\lambda - \lambda_n)^{-1}\}$ is a multiplier sequence in $L^p$ for $1 < p < \infty$. For $(\lambda I - \Lambda)^{-1}$ to be well-bounded, it suffices to have $\{(\lambda - \lambda_n)^{-1}\}$ piecewise monotone. If $\{\delta_n\}$ is real, this is the case if $|\delta_n| \leq L < 1/2$.

4.6. Lemma. Let $\delta_n = \alpha_n + i \beta_n$ where

\[(4.7) \quad |\alpha_n| < L < \frac{1}{2}, \quad \beta_n = o(1)
\]
for $n$ sufficiently large. Let $\lambda$ be a real number distinct from the $\lambda_n$. Then $\{(\lambda_n - \lambda)^{-1}\}$ lies on an admissible arc $C$ and is piecewise monotone.

Proof. If suffices to show that $\{\text{Re}(\lambda_n - \lambda)^{-1}\}$ is piecewise monotone and $\{\text{Im}(\lambda_n - \lambda)^{-1}\}$ is of bounded variation, since then the arc formed by joining successive points $\{(\lambda_n - \lambda)^{-1}\}$ with straight lines is admissible. A computation yields

\[
\text{Re}(\lambda_n - \lambda)^{-1} = \frac{1}{n} - \frac{\alpha_n - \lambda}{n^2} + \frac{\gamma_n}{n^3}, \quad \gamma_n = o(1),
\]
and then the difference of two successive ones is

\[ \frac{1}{n(n+1)} \left[ 1 - (\alpha_n - \alpha_{n+1}) + \frac{\gamma_n}{n} \right]. \]

For \(|n|\) sufficiently large this is positive, since \(\alpha_n - \alpha_{n+1} < 1\). Clearly \(\text{Im}(\lambda_n - \lambda)^{-1} = \mathcal{O}(n^{-2})\), so this sequence is of bounded variation.

4.8. THEOREM. If \(\{\delta_n\}\) satisfies (4.7), then \(R(\lambda, \Lambda)\) is well-bounded.

PROOF. By the above lemma, \(\{\lambda - \lambda_n\}^{-1}\) satisfies the conditions of [11, Corollary 3.2.6] (see also Theorem 2.40), so \(R(\lambda, \Lambda_0 + \mathcal{M})\) is well-bounded. Well-boundedness is preserved by similarity transforms.

We have \(\Lambda^2 = A(\Lambda_0 + \mathcal{M})^2A^{-1}\) with domain \(\mathcal{D}(\Lambda_0^2)\).

4.9. THEOREM. If \(\lambda\) does not coincide with any \(\lambda_n^2\) and if (4.7) holds, then \(R(\lambda, \Lambda^2)\) is well-bounded on \(L^p\), \(1 < p < \infty\).

The proof is similar to that for \(\Lambda\).

4.10. COROLLARY. For \(1 < p < \infty\), \(-\Lambda^2\) is the infinitesimal generator of a semigroup in \(L^p\).

PROOF. Since the admissible arc \(C\) containing \(\{\lambda - \lambda_n^2\}^{-1}\) enters the origin with bounded slope, the conditions of [2, Theorem 5.15] are satisfied. (See also [2, Lemma 5.48].)

5. Half-range expansions. Assuming the sequence \(\{\lambda_n\}\) is odd:

\(\lambda_{-n} = -\lambda_n\),

we consider expansions for \(0 < x < \pi\) (or for \(-\pi < x < 0\)) in \(\{\cos \lambda_n x\}\), \(n \geq 0\) and in \(\{\sin \lambda_n x\}\), \(n \geq 1\). We give conditions assuring that these functions are eigenfunctions of linear operators which generate strongly continuous semigroups in \(L^p(0, \pi)\).

We assume throughout this section that (5.1) holds and \(\{g_n\}, \{\varphi_n\}\) are equivalent in some space \(L^p\).

5.2. LEMMA. \(g_{-n}(x) = g_n(-x), h_{-n}(x) = h_n(-x)\).

PROOF. Since \(\{g_n\}\) is given explicitly, this is an immediate consequence of (5.1).

For \(h_n\), let \(m\) be fixed and let \(w(x) = h_m(-x)\). Then for all \(n\), and using the above property of \(g_n\), we have \((g_n, w - h_{-m}) = 0\) all \(n, m\). Since \(\{g_n\}\) is complete we have \(w = h_{-m}\).

For the remainder of this section we consider cosine expansions. Thus let

\[ G_n(x) = \left[ g_n(x) + g_{-n}(x) \right]/2, \quad H_n(x) = \left[ h_n(x) + h_{-n}(x) \right]/2, \quad c_n(x) = \cos nx. \]

Clearly

\[ G_n = Ac_n, \quad H_n = A^{-1}c_n. \]

For an even function \(f\) on \([-\pi, \pi]\), let

\[ F(x) = f(x), \quad 0 < x < \pi, \]

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and for two functions \( u, v \) on \((0, \pi)\), let

\[
\langle u, v \rangle = \frac{2}{\pi} \int_0^\pi u(x) \bar{v}(x) \, dx.
\]

5.7. **Lemma.** If \( f \) is an even function, then for \( 0 < x < \pi \),

\[
(f, h_0) g_0 = \frac{1}{2} \langle F, H_0 \rangle G_0,
\]

\[
(f, h_n) g_n + (f, h_{-n}) g_{-n} = \langle F, H_n \rangle G_n,
\]

\[
\sum_{n=-N}^N (f, h_n) g_n = \frac{1}{2} \langle F, H_0 \rangle G_0 + \sum_{n=1}^N \langle F, H_n \rangle G_n := T_N(x; F).
\]

**Proof.** Computational.

Let

\[
T_N(x; F) = \frac{1}{2} \langle F, c_0 \rangle c_0(x) + \sum_{n=1}^N \langle F, c_n \rangle c_n(x), \quad 0 < x < \pi.
\]

5.11. **Theorem.** If \((5.1)\) holds and if \( \{g_n\}, \{q_n\} \) are equivalent in \( L^p \) for some \( p, 1 < p < \infty \), then for all \( F \) in \( L^p(0, \pi) \),

\[
\lim_{N \to \infty} \left[ T_N(x; F) - T_N(x; F) \right] = 0,
\]

uniformly on \([0, \pi - d]\) for each \( d > 0 \). If \( 1 < p < \infty \), then

\[
\lim_{N \to \infty} T_N(x; F) = F
\]

in the norm of \( L^p(0, \pi) \).

**Proof.** These are direct consequences of the relations

\[
\mathcal{S}_N(x; f) = \mathcal{S}_N(x; F), \quad S_N(x; f) = T_N(x; F), \quad 0 < x < \pi,
\]

and the analogous theorems for \( \mathcal{S}_N, S_N \).

If \( f \) is an even function in \( \mathcal{D}(\Lambda^2) \), then for \( 0 < x < \pi \),

\[
\Lambda^2 f = \sum_{n=1}^\infty \lambda_n^2 (F, H_n) G_n := \Gamma^2 F,
\]

where \( \mathcal{D}(\Gamma^2) \) consists of all \( F \) in \( L^p(0, \pi) \) such that the even extension to \([-\pi, \pi]\) is in \( \mathcal{D}(\Lambda^2) \). For \( \lambda \neq \lambda_n^2 \), and for any polynomial \( P \),

\[
P \left( R(\lambda, \Gamma^2) \right) F = \frac{1}{2} P(\lambda^{-1}) \langle F, H_0 \rangle G_0 + \sum_{n=1}^\infty P \left( \lambda - \lambda_n^2 \right)^{-1} \langle F, H_n \rangle G_n
\]

\[
= P \left( R(\lambda_n^2, \Lambda^2) \right) f, \quad 0 < x < \pi.
\]

5.13. **Theorem.** If \( \{\delta_n\} \) satisfies \((4.7)\), along with the other assumptions of this section, then for \( 1 < p < \infty \), \( R(\lambda, \Gamma^2) \) is well bounded and \(-\Gamma^2 \) generates a strongly continuous semigroup on \( L^p(0, \pi) \).

**Proof.** Since for any function \( F \) and its even extension \( f \) we have

\[
\| f \|^p = 2\| F \|^p.
\]
from (5.12) and the well-boundedness of $R(\lambda, \Lambda^2)$,
\[
\|P(R(\lambda, \Gamma^2))f\| = 2^{-1/p}\|P(R(\lambda, \Lambda^2))f\| \\
\leq 2^{-1/p}K\|\|P\|\|f\| = K\|\|P\|\|F\|,
\]
where $\|\|P\|\|$ is computed on the piecewise linear admissible arc containing $\{(\lambda - \lambda_n^2)^{-1}\}$. Thus $R(\lambda, \Gamma^2)$ is well-bounded and the proof of Corollary 4.10 applies.

References


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