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NORMS OF HANKEL OPERATORS AND UNIFORM ALGEBRAS

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ABSTRACT. Two generalizations of the classical Hankel operators are defined on an abstract Hardy space that is associated with a uniform algebra. In this paper the norms of Hankel operators are studied. This has applications to weighted norm inequalities for conjugation operators, and invertible Topelitz operators. The results in this paper have applications to concrete uniform algebras, for example, a polydisc algebra and a uniform algebra which consists of rational functions.

0. Introduction. Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. For $\tau \in M_A$, the maximal ideal space of A, set $A_0 = \{f \in A\colon \tau(f) = 0\}$. Let m be a representing measure for τ on X.

The abstract Hardy space $H^p = H^p(m)$, $1 \le p \le \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak*-closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$. Suppose $H_0^p = \{f \in H^p\colon \int_X f\,dm = 0\}$, $K_0^p = \{f \in L^p\colon \int_X fg\,dm = 0$ for all g in $A\}$ and $K^p = \{f \in L^p\colon \int_X fg\,dm = 0$ for all g in $A_0\}$. Then $H_0^p \subset K_0^p$ and $H^p \subset K^p$. Moreover put $K_0 = K_0^p \cap C(X)$ and $K = K^p \cap C(X)$. Then $A_0 \subset K_0$ and $A \subset K$.

Let $Q^{(1)}$ be the orthogonal projection from L^2 to $\overline{K}_0^2$ and $Q^{(2)}$ the orthogonal projection from L^2 to $\overline{H}_0^2$. For a function ϕ in L^∞ we denote by M_ϕ the multiplication operator on L^2 that it determines. The two generalizations of the classical Hankel operators that we consider in this paper are defined as follows. For $\phi \in L^\infty$ and $f \in H^2$

$$H_\phi^{(j)} f = Q^{(j)} M_\phi f \qquad (j = 1, 2).$$

If A is a disc algebra and $\tau(f) = \tilde{f}(0)$ where $\tilde{f}$ denotes the holomorphic extension of $f \in A$, then τ is in M_A. Let m be a normalized Lebesgue measure on the unit circle; then m is a representing measure for τ. Then H^2 is the classical Hardy space and $H_0^2 = K_0^2$. Hence $H_\phi^{(1)} = H_\phi^{(2)}$. It is well known that $\|H_\phi^{(1)}\| = \|H_\phi^{(2)}\| = \|\phi + H^\infty\|$ where $\|\phi + H^\infty\| = \inf\{\|\phi + g\|_\infty\colon g \in H^\infty\}$. This is Nehari's theorem (cf. [**11**, Theorem 1.3]). However generalizations to uniform algebras are unknown except for Corollary 2.1.1 in [**4**]. This appears to be due to the lack of a factorization theorem of H^1, that is, if $h \in H^1$ and $\int_X |h|\,dm \le 1$, then $h = fg$, $f \in H^2$ and $g \in H^2$ where $\int_X |f|^2\,dm \le 1$ and $\int_X |g|^2\,dm \le 1$.

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In this paper we will study the relation between $\|H_\phi^{(1)}\|$ and $\|\phi + H^\infty\|$, and between $\|H_\phi^{(2)}\|$ and $\|\phi + K^\infty\|$ in general uniform algebras. The main idea of this paper is that we consider Hankel operators on vH^2 for every nonnegative invertible function v in L^∞. Two applications of our results to Hankel operators are given. One of them is an application to a Helson-Szegö theorem (cf. [**8**]) in a general uniform algebra. The other is an inversion theorem (cf. [**13**]) for Toepliz operators in a uniform algebra.

In §1, the norms of Hankel operators are studied in general abstract uniform algebras. In §2, the norms of Hankel operators are calculated more accurately in special uniform algebras. In §3, applications of results in §§1 and 2 to weighted norm inequalities for conjugation operators in uniform algebras are given. In §4, applications of results in §§1 and 2 to left invertible Toeplitz operators are given. In §5, we give concrete examples for which we can apply theorems in previous sections. That is, a uniform algebra which consists of rational functions on a multiply connected domain, a subalgebra of a disc algebra which contains the constants and which has finite codimension, and a polydisc algebra.

1. Hankel operators and general uniform algebras. Let v be a nonnegative function in L^∞ with $v^{-1} \in L^\infty$. Let $Q_v^{(1)}$ be the orthogonal projection from L^2 onto $(vH^2)^\perp = v^{-1}\overline{K}_0^2$ and $Q_v^{(2)}$ the orthogonal projection from L^2 onto $v^{-1}\overline{H}_0^2$. If v is a constant function then $Q_v^{(j)} = Q^{(j)}$ $(j = 1, 2)$. For $\phi \in L^\infty$ and $f \in vH^2$

$$H_\phi^{(j)v} f = Q_v^{(j)} M_\phi f \qquad (j = 1, 2).$$

If v is a nonzero constant then $H_\phi^{(j)v} = H_\phi^{(j)}$ $(j = 1, 2)$. Put $(L^\infty)_+^{-1} = \{v \in L^\infty : v^{-1} \in L^\infty \text{ and } v \geq 0\}$.

THEOREM 1. *Let ϕ be a function in L^∞, then*

$$\sup_{v \in (L^\infty)_+^{-1}} \|H_\phi^{(2)v}\| = \|\Phi + K^\infty\|.$$

If K^∞ is dense in K^1 then

$$\sup_{v \in (L^\infty)_+^{-1}} \|H_\phi^{(1)v}\| = \|\phi + H^\infty\|.$$

PROOF. We shall prove the theorem only for $H_\phi^{(1)v}$. For H^∞ is always dense in H^1 and so the proof for $H_\phi^{(2)v}$ is similar. By definition,

$$\begin{aligned}
\|H_\phi^{(1)v}\| &= \sup\{|(H_\phi^{(1)v}(vf), v^{-1}\overline{g})|; f \in H^2,\ g \in K_0^2, \|vf\|_2 \leq 1,\ \|v^{-1}g\|_2 \leq 1\} \\
&= \sup\left\{\left|\int_X \phi f g\, dm\right|; f \in H^2, g \in K_0^2, \|vf\|_2 \leq 1, \|v^{-1}g\|_2 \leq 1\right\} \\
&\leq \sup\left\{\left|\int_X \phi h\, dm\right| : h \in K_0^1, \|h\|_1 \leq 1\right\} \\
&= \|\phi + H^\infty\|.
\end{aligned}$$

It remains to show that $\sup_v \|H_\phi^{(1)v}\| \geq \|\phi + H^\infty\|$. Let $h \in K_0^\infty$ and set $E_n = \{x \in X;\ 0 < |h(x)| < 1/n\}$, so $m(E_n) \to 0$. Put $F_0 = \{x \in X; h(x) = 0\}$ and

$F_n = \{x \in X; |h(x)| \geq 1/n\}$. Define u_n by the formula

$$u_n(x) = \begin{cases} 1, & x \in E_n, \\ n, & x \in F_0, \\ |h(x)|^{-1/2}, & x \in F_n \end{cases}$$

(see [**4**, Lemma 2.1]). Clearly $u_n \in (L^\infty)_+^{-1}$.

$$\begin{aligned} \int_X |h|^2 u_n^2 \, dm &= \int_{E_n} |h|^2 u_n^2 \, dm + \int_{F_0} |h|^2 u_n^2 \, dm + \int_{F_n} |h|^2 u_n^2 \, dm \\ &= \int_{E_n} |h|^2 \, dm + \int_{F_n} |h| \, dm \\ &\leq \int_{E_n} |h| \, dm + \int_{F_n} |h| \, dm = \int_X |h| \, dm. \end{aligned}$$

On the other hand,

$$\int_X u_n^{-2} \, dm = m(E_n) + \left(\frac{1}{n^2}\right) m(F_0) + \int_{F_n} |h| \, dm.$$

If $v_n = u_n^{-1}$ then

$$\begin{aligned} \left| \int_X \phi h \, dm \right| &= |(\phi v_n, v_n^{-1} \bar{h})| \\ &\leq \|H_\phi^{(1)v_n}\| \left(\int_X |v_n|^2 \, dm \right)^{1/2} \left(\int_X |v_n^{-1} h|^2 \, dm \right)^{1/2}. \end{aligned}$$

Since $\lim_n \int_X |v_n|^2 \, dm = \int_X |h| dm$ and $\lim_n \int_X |v_n^{-1} h|^2 \, dm \leq \int_X |h| dm$,

$$\left| \int_X \phi h \, dm \right| \leq \sup_n \|H_\phi^{(1)v_n}\| \int_X |h| \, dm.$$

The annihilator of K_0^∞ in L^∞ is H^∞ because K_0^∞ is dense in K_0^1 by hypothesis. By duality,

$$\|\phi + H^\infty\| \leq \sup_{v \in (L^\infty)_+^{-1}} \|H_\phi^{(1)v}\|.$$

COROLLARY 1.1. *If $\phi = \bar{f}/f$ for some nonzero f in K_0^2 then $\|H_\phi^{(1)}\| = \|\phi + H^\infty\| = 1$. If $\phi = \bar{f}/f$ for some nonzero f in H_0^2 then $\|H_\phi^{(2)}\| = \|\phi + K^\infty\| = 1$ and so $\|H_\phi^{(1)}\| = \|H_\phi^{(2)}\|$.*

PROOF. If $\phi = \bar{f}/f$ and $f \in K_0^2$ is nonzero, set $g = f/\|f\|_2$. Then $\phi = \bar{g}/g$ and $|(H_\phi^{(1)} g, \bar{g})| = \int_X |g|^2 \, dm = 1$. Theorem 1 implies $\|H_\phi^{(1)}\| = \|\phi + H^\infty\| = 1$. The second assertion can be proved similarly.

2. Hankel operators and special uniform algebras. Let ∂_A denote the Shilov boundary of A and N_τ denote the set of representing measures for $\tau \in M_A$ whose support is contained in ∂_A. Suppose N_τ is finite dimensional and equal to n. Let m be a core point of N_τ and let N^∞ be the real annihilator of A in L_R^∞. Then $\dim N^\infty = n$ and $A + \overline{A}_0 + N_c^\infty$ is weak*-dense in L^∞ where $N_c^\infty = N^\infty + iN^\infty$ (cf. [**7**, p. 109]).

Set $\mathcal{E} = \exp N^\infty$; then $\mathcal{E}$ is a subgroup of $(L^\infty)_+^{-1}$. If $n = 0$ $\mathcal{E} = \{1\}$ and so the following theorem gives Corollary 2.1.1 in [**4**], that is, $\|H_\phi^{(1)}\| = \|H_\phi^{(2)}\| = \|\phi + H^\infty\|$.

THEOREM 2. *Let ϕ be a function in L^∞, then*

$$\sup_{v\in\mathcal{E}} \|H_\phi^{(1)v}\| = \|\phi + H^\infty\|$$

and

$$\sup_{v\in\mathcal{E}} \|H_\phi^{(2)v}\| = \|\phi + H^\infty + N_c^\infty\|.$$

PROOF. If $v \in (L^\infty)_+^{-1}$ then $\log v = u + u_0$ where $u_0 \in N^\infty$ and u is in the weak*-closure of $\operatorname{Re} A$ (cf. [**7**, p. 109]). Then $u = \log|h|$ for some invertible function h in H^∞ and so $v = |h|v_0$ where $v_0 = e^{u_0}$. Hence $vH^2 = b(v_0H^2)$, $v^{-1}\overline{H}_0^2 = b(v_0^{-1}\overline{H}_0^2)$ and $v^{-1}\overline{K}_0^2 = b(v_0^{-1}\overline{K}_0^2)$ where $b = |h|/h$. Then $Q_{v_0}^{(j)} = M_{\bar{b}}Q_v^{(j)}M_b$ and so $H_\phi^{(j)v_0} = M_{\bar{b}}Q_v^{(j)}M_bM_\phi$. Hence

$$\begin{aligned}\|H_\phi^{(j)v_0}\| &= \sup\{\|M_{\bar{b}}Q_v^{(j)}M_bM_\phi(v_0f)\|_2; v_0f \in v_0H^2 \text{ and } \|v_0f\|_2 \le 1\}\\ &= \sup\{\|Q_v^{(j)}M_\phi(bv_0f)\|_2; bv_0f \in vH^2 \text{ and } \|bv_0f\|_2 \le 1\}\\ &= \|H_\phi^{(j)v}\|.\end{aligned}$$

Thus for any $v \in (L^\infty)_+^{-1}$ there exists $v_0 \in \mathcal{E}$ such that for $j = 1, 2$

$$\|H_\phi^{(j)v}\| = \|H_\phi^{(j)v_0}\|.$$

Thus

$$\sup_{v\in(L^\infty)_+^{-1}} \|H_\phi^{(j)v}\| = \sup_{v\in\mathcal{E}} \|H_\phi^{(j)v}\| \quad \text{for } j = 1, 2.$$

Now the theorem follows from Theorem 1 because $K^\infty = H^\infty + N_c^\infty$ (cf. [**7**, pp. 106–109]).

We now consider a more special uniform algebra, that is, we assume that m is a unique logmodular measure. Then the linear span of $N^\infty \cap \log|(H^\infty)^{-1}|$ is N^∞ (cf. [**7**, p. 114]). Choose $h_1, \ldots, h_n \in (H^\infty)^{-1}$ so that $\{\log|h_j|\}_{j=1}^n$ is a basis for N^∞. Put $u_j = \log|h_j|$ $(1 \le j \le n)$ and $\mathcal{E}_0 = \{\exp(\sum_{j=1}^n s_ju_j) \colon 0 \le s_j \le 1\}$. Then $\mathcal{E}_0 \subset \mathcal{E}$.

THEOREM 3. *Suppose m is a unique logmodular measure for τ. If ϕ is a function in L^∞, then there exist t_1 and t_2 in $\mathcal{E}_0$ such that*

$$\|H_\phi^{(1)t_1}\| = \|\phi + H^\infty\| \quad \textit{and} \quad \|H_\phi^{(2)t_2}\| = \|\phi + H^\infty + N_c^\infty\|.$$

PROOF. If $v \in \mathcal{E}$ then $v = |h|v_0$ for some $h \in (H^\infty)^{-1}$ and some $v_0 \in \mathcal{E}_0$. Hence $vH^2 = b(v_0H^2)$ for $b = |h|/h$. Then as in the proof of Theorem 2

$$\sup_{v\in\mathcal{E}_0} \|H_\phi^{(j)v}\| = \sup_{v\in\mathcal{E}} \|H_\phi^{(j)v}\|$$

By Theorem 2 there exists a sequence $\{v_l\}$ in $\mathcal{E}_0$ such that $\|H_\phi^{(1)v_l}\| \le \|H_\phi^{(1)v_{l+1}}\|$ and $\lim_{l\to\infty}\|H_\phi^{(1)v_l}\| = \|\phi + H^\infty\|$. Since $v_l \in \mathcal{E}_0$, $v_l = \exp(\sum_{j=1}^n s_{jl}u_j)$ and $0 \le s_{jl} \le 1$ $(1 \le j \le n)$. By passing to a subsequence, if necessary, we can assume that s_{jl} converges to a function s_j for each j, and $|s_j| \le 1$ $(1 \le j \le n)$. Put $t_1 = \exp(\sum_{j=1}^n s_ju_j)$; then $t_1 \in \mathcal{E}_0$ and $\|H_\phi^{(1)t_1}\| = \|\phi + H^\infty\|$. For $H_\phi^{(2)v}$ the proof is similar.

3. Weighted norm inequalities. Let P be the orthogonal projection from L^2 to H^2. $P^{(1)}$ denotes P restricted to $A+\overline{K}_0$ and $P^{(2)}$ denotes P restricted to $A+\overline{A}_0$. We are interested in knowing when $P^{(j)}$ $(j=1,2)$ is bounded in $L^2(w)=L^2(w\,dm)$ where w is a nonnegative weight function in L^1. That is, we want to find the weight functions w for which there is a positive constant γ such that

$$\int_X |f|^2\,w\,dm \le \gamma \int_X |f+\bar{g}|^2\,w\,dm, \qquad f\in A,\ g\in K_0$$

or

$$\int_X |f|^2 w\,dm \le \gamma \int_X |f+\bar{g}|^2 w\,dm, \qquad f\in A,\ g\in A_0.$$

In case A is a disc algebra, such weight functions w are well known. Then $K_0=A_0$ and so $P^{(1)}=P^{(2)}$. $P^{(j)}$ is bounded in $L^2(w)$ if and only if $w=|h|^2$ for some outer function h in H^2 and $\|\phi+H^\infty\|<1$ where $\phi=|h|^2/h^2$. This result is called the Helson-Szegö theorem. The Helson-Szegö theorem was generalized by [**5**, **9** and **10**] in case $N_\tau=\{m\}$. However in case $N_\tau\neq\{m\}$ it has not been generalized. This is probably due to the lack of a factorization of H^1 functions similar to norms of Hankel operators.

In this section we shall study weighted norm inequalities in general uniform algebras. These can be obtained easily using the theorems in §§1 and 2. Let $v\in(L^\infty)_+^{-1}$ and let P_v be the orthogonal projection from L^2 onto vH^2. $P_v^{(1)}$ denotes P_v restricted to $vA+v^{-1}\overline{K}_0$ and $P_v^{(2)}$ denotes P_v restricted to $vA+v^{-1}\overline{A}_0$.

PROPOSITION 4. *Suppose K is dense in K^1. Let $w=|h|^2$ for some outer function h in H^2, i.e. hA is dense in H^2. Moreover suppose hK is dense in K^2. Let $\phi=|h|^2/h^2$. $P_v^{(2)}$ is uniformly bounded in $L^2(w)$ with respect to $v\in(L^\infty)_+^{-1}$ if and only if $\|\phi+H^\infty\|<1$. $P_v^{(1)}$ is uniformly bounded in $L^2(w)$ with respect to $v\in(L^\infty)_+^{-1}$ if and only if $\|\phi+K^\infty\|<1$.*

The proof follows easily from Theorem 1.

PROPOSITION 5. *Suppose $\dim N_\tau=n<\infty$ and m is a core point of N_τ. Let $w=|h|^2$ for some outer function h in H^2 and $\phi=|h|^2/h^2$. $P_v^{(2)}$ is uniformly bounded in $L^2(w)$ with respect to v in $\mathcal{E}$ if and only if $\|\phi+H^\infty\|<1$ and $P_v^{(1)}$ is uniformly bounded in $L^2(w)$ with respect to $v\in\mathcal{E}$ if and only if $\|\phi+H^\infty+N_c^\infty\|<1$.*

The proof follows easily from Theorem 2 or Proposition 4.

Suppose m is a unique logmodular measure in N_τ. Then any k in H^2 with $\log|k|\in L^1$ has the form: $k=Fh$ where F is an inner function and h is an outer function [**2**, p. 138]. We call a function F in H^p an inner function if $|F|\in\mathcal{E}$. If $\log w\in L^1$ then $w=|k|^2$ for some k in H^2 [**2**, Theorem 10.3].

THEOREM 6. *Let m be a unique logmodular measure in N_τ.*

(1) *Suppose $N^\infty\subset C(X)$. $P_v^{(2)}$ is bounded in $L^2(w)$ for every v in $\mathcal{E}_0$ if and only if $w^{-1}\in L^1$, $w=|k|^2$ for some k in H^2 and*

$$\||k|^2/k^2+H^\infty/F^2\|<1$$

where F is an inner part of k.

(2) *$P_v^{(1)}$ is bounded in $L^2(w)$ for every v in $\mathcal{E}_0$ if and only if $\log w\in L^1$, $w=|k|^2$ for some k in H^2 and $\|\,|k|^2/k^2+K^\infty/F^2\|<1$ where F is an inner part of k.*

PROOF. (1) If $P_v^{(2)}$ is bounded in $L^2(w)$ for every v in $\mathcal{E}_0$ then there is a positive constant γ such that

$$\int_X |f|^2 w dm \le \gamma \int_X |f + \bar{g}|^2 w dm, \qquad f \in A,\ g \in A_0 + N_c^\infty.$$

Let $f = 1$ and $g \in A_0$. Then

$$\int_X |1 + g|^2\, w dm \ge \frac{1}{\gamma} \int w dm > 0.$$

Since m is a unique logmodular measure, $\log w$ belongs to L^1 [**2**, p. 145]. Hence for f_0 in A_0

$$\int_X |1 + f_0|^2 w dm \ge \exp \int_X \log w dm$$

by Jensen's inequality. Thus

$$\int_X |1 + f_0 + \bar{g}|^2\, w dm \ge \frac{1}{\gamma} \exp \int_X \log w dm > 0$$

for any $f_0 \in A_0$ and $g \in A_0 + N_c^\infty$. It is easy to see that $w^{-1} \in L^1$ (cf. [**12**, Theorem 4.3.1]). Since $\log w \in L^1$, $w = |k|^2$ for some $k \in H^2$ and so $k = Fh$ where F is an inner function and h is an outer function. It is easy to see that for every $v \in \mathcal{E}_0$

$$\cos(vA, v^{-1}(A_0 + N_c^\infty)) = \sup \left\{ \left| \int_X fgw dm \right| ; f \in A,\ g \in A_0 + N_c^\infty, \right.$$
$$\left. \int_X |vf|^2\, w\, dm \le 1, \int_X |v^{-1} g|^2\, w\, dm \le 1 \right\} < 1$$

because $P_v^{(2)}$ is bounded in $L^2(w)$ for every $v \in \mathcal{E}_0$. As in the proof of Theorem 3 we can show that there exists a $v_0 \in \mathcal{E}_0$ such that

$$\sup_{v \in \mathcal{E}_0} \cos(vA, v^{-1}(A_0 + N_c^\infty)) = \cos(v_0 A, v_0^{-1}(A_0 + N_c^\infty)).$$

Thus setting $\phi = |k|^2/k^2$ we get

$$\sup \left\{ \left| \int_X F^2 g \phi\, dm \right| : \int_X |F^2 g|\, dm \le 1,\ g \in K_0^1 \right\} < 1.$$

By duality $\|\phi + (F^2)^{-1} H^\infty\| < 1$. The converse is easy to show.

The proof of (2) is similar to that of (1).

4. Toeplitz operators. For $\phi \in L^\infty$ let T_ϕ be the operator on H^2 defined by $T_\phi f = P(M_\phi f)$. The operator T_ϕ will be called a Toeplitz operator. Suppose ϕ is a unimodular function in L^∞. We want to know when T_ϕ is left invertible. In case A is a disc algebra, Widom (cf. [**13**]) showed that T_ϕ is left invertible if and only if $\|\phi + H^\infty\| < 1$. In this section we shall study the inversion theorem in general uniform algebras.

For $\phi \in L^\infty$ and $v \in (L^\infty)_+^{-1}$ let T_ϕ^v be the operator on vH^2 defined by $T_\phi^v f = P^v(M_\phi f)$. Let I^v be the identity operator on vH^2. Then $(T_\phi^v)^* T_\phi^v + (H_\phi^{(1)v})^* H_\phi^{(1)v} = I^v$. Hence if K^∞ is dense in K^1, by Theorem 1,

$$\sup_{v \in (L^\infty)_+^{-1}} \|(T_\phi^v)^* T_\phi^v - I^v\| = \|\phi + H^\infty\|.$$

PROPOSITION 7. *Suppose ϕ is a unimodular function in L^∞. If K^∞ is dense in K^1, then there exists a nonzero positive constant $\varepsilon(v)$ such that for any $v \in (L^\infty)_+^{-1}$*

$$\|T_\phi^v f\|_2 \geq \varepsilon(v)\|f\|_2, \qquad f \in vH^2,$$

and $\inf_{v\in(L^\infty)_+^{-1}} \varepsilon(v) > 0$ if and only if $\|\phi + H^\infty\| < 1$. If $\dim N_\tau = n < \infty$, it is sufficient that v ranges only over $\mathcal{E}$ instead of $(L^\infty)_+^{-1}$.

The proof follows from Theorems 1 and 2.

THEOREM 8. *Let m be a unique logmodular measure in N_τ and ϕ a unimodular function in L^∞. T_ϕ^v is left invertible for every v in $\mathcal{E}_0$ if and only if $\|\phi + H^\infty\| < 1$.*

The proof follows from Theorem 3.

5. Concrete examples. All results in this paper were known in the disc algebra. We shall now apply them to some other concrete examples.

(I) Let Y be a compact subset of the plane, and let $R(Y)$ be the uniform closure of the rational functions in $C(Y)$. We regard $R(Y)$ as a uniform algebra on its Shilov boundary, the topological boundary X of Y. Suppose the complement of Y^c of Y has a finite number n of components and the interior Y^0 of Y is a nonempty connected set. Let $A = R(Y)|X$; then $M_A = Y$. If $\tau \in M_A$ is in Y^0 and m is harmonic measure, then m is a unique logmodular measure of N_τ and $\dim N_\tau = n < \infty$ [**7**, p. 116]. Then $N^\infty \subset C(X)$. Theorems 3, 6, and 8 apply to this situation. Then Theorem 8 is essentially a theorem of Abrahamse [**1**, Theorem 4.1].

(II) Let $\mathcal{A}$ be the disc algebra and let A be a subalgebra of $\mathcal{A}$ which contains the constants and which has finite codimension in $\mathcal{A}$. Two examples of such subalgebras are $\{f \in \mathcal{A},\ f'(0) = 0\}$ and $\{f \in \mathcal{A};\ f(0) = f(\frac{1}{2})\}$. Anderson and Rochberg [**3**, p. 815] described A. Let b be a finite Blaschke product. $b\mathcal{A}$ is a closed ideal in $\mathcal{A}$ and hence $\mathcal{A}/b\mathcal{A}$ is a finite dimensional algebra. Let H be any subspace of $\mathcal{A}$ with the property that the image of H in $\mathcal{A}/b\mathcal{A}$ is a subalgebra of $\mathcal{A}/b\mathcal{A}$ which contains the identity. The set $A = \{a \in \mathcal{A}\colon f = h + bg$ for some $h \in H,\ g \in \mathcal{A}\}$ is a subalgebra of $\mathcal{A}$ and has finite codimension. By a theorem of Gamelin [**6**], these are the only such subalgebras. If $\tau(f) = f(0)$ for $f \in A$ and m is the normalized Lebesgue measure on the circle T, then it is easy to check that m is a core point of N_τ and $N^\infty \subset C(T)$. Hence we can apply Theorem 2, and Propositions 5 and 7. Anderson and Rochberg [**2**] studied when T_ϕ is left invertible and they got a theorem. Their characterization is different from ours. If μ is a finite complex measure which annihilates $A + N_c^\infty$ (A resp.) then $\mu = kdm$ for some k in H_0^1 ($H_0^1 + N_c^\infty$ resp.). Thus $H_0^1 = (C(T)/A + N_c^\infty)^*$ and $H_0^1 + N_c^\infty = (C(T)/A)^*$. For $\phi \in L^\infty$ set $K_\phi^{(1)}(f) = \int_T f\phi\, dm$ ($f \in H_0^1 + N_c^\infty$) and $K_\phi^{(2)}(f) = \int_T f\phi\, dm$ ($f \in H_0^1$). Then $K_\phi^{(1)} \in (H_0^1 + N_c^\infty)^*$ and $K_\phi^{(2)} \in (H_0^1)^*$. By duality $\|K_\phi^{(1)}\| = \|\phi + H^\infty\|$ and $\|K_\phi^{(2)}\| = \|\phi + H^\infty + N_c^\infty\|$. Hence if $\phi \in C(T)$ there exists $F_1 \in H_0^1 + N_c^\infty$ and $F_2 \in H_0^1$ such that $\|F_j\|_1 = 1$ and $\|K_\phi^{(j)}\| = K_\phi^{(j)}(F_j)$ for $j = 1, 2$. Since $\log|F_j| \in L^1$, $\log|F_j| = u_j + u_{j0}$, where $u_{j0} \in N^\infty$ and u_j is in the annihilator of N^∞. This gives the factorization of F_j such that $F_j = b_j h_j^2$, where h_j is an outer function in H^2 with $|h_j|^2 = e^{u_j}$ and b_j is a function with $|b_j| = e^{u_{j0}}$. Then

$b_1 \in H_0^1 + N_c^\infty$ and $b_2 \in H_0^1$. $F_j = h_j \cdot b_j h_j$ and

$$\int_T |F_j|\, dm = \int_T |h_j|^2 v_j\, dm = \int_T |b_j h_j|^2 v_j^{-1} dm$$

where $v_j = |b_j| \in \mathcal{E}$. Thus $\|H_\phi^{(1)v_1}\| = \|\phi + H^\infty\|$ and $\|H_\phi^{(2)v_2}\| = \|\phi + H^\infty + N_c^\infty\|$. For $\phi \in C(T)$ this shows Theorem 3. Unfortunately, we could not show this for all $\phi \in L^\infty$. Similar ideas give some versions of Theorems 6 and 8.

(III) The unit polydisc U^n and the torus T^n are cartesian products of n copies of the unit disc U and of the unit circle T, respectively. $A(U^n)$ is the class of all continuous complex functions on the closure $\overline{U}^n$ of U^n with holomorphic restrictions to U^n is holomorphic there. Let $A = A(U^n)|X$ and $X = T^n$. This is the so-called polydisc algebra. For simplicity we assume $n = 2$. Let m be normalized Lebesgue measure; then m is a representing measure for τ on X where $\tau(f) = f(0)$ and $0 \in U^2$. Suppose $1 \le p \le \infty$ and $Z_+^2 = \{(n,m) \in Z^2;\ n \ge 0 \text{ and } m \ge 0\}$. Then $H^p = \{f \in L^p; \hat{f}(n,m) = 0 \text{ if } (n,m) \notin Z_+^2\}$ and $K^p = \{f \in L^p; \hat{f}(n,m) = 0 \text{ if } (-n,-m) \in Z_+^2\}$. K^∞ is dense in K^p. We can apply Theorem 1, and Propositions 5 and 7.

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REFERENCES

1. M. B. Abrahamse, *Toeplitz operators in multiply connected regions*, Amer. J. Math. **96** (1974), 261–297.
2. P. R. Ahern, and D. Sarason, *The H^p spaces of a class of function algebras*, Acta Math. **117** (1967), 123-163.
3. J. M. Anderson, and R. H. Rochberg, *Toeplitz operators associated with subalgebras of the disc algebra*, Indiana Univ. Math. J. **30** (1981), 813–820.
4. R. E. Curto, P. S. Muhly, T. Nakazi and J. Xia, *Hankel operators and uniform algebras*, Arch. Math. **43** (1984), 440–447.
5. A. Devinatz, *Conjugate function theorems for Dirichlet algebras*, Rev. Un. Mat. Argentina **23** (1966), 3–30.
6. T. Gamelin, *Embedding Riemann surfaces in maximal ideal spaces*, J. Funct. Anal. **2** (1968), 123–146.
7. ——, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
8. H. Helson and G. Szegö, *A problem in prediction theory*, Ann. Mat. Pura Appl. **51** (1960), 107–138.
9. I. I. Hirschman, Jr., and R. Rochberg, *Conjugate function theory in weak* Dirichlet algebras*, J. Funct. Anal. **16** (1974), 359–371.
10. Y. Ohno, *Remarks on Helson-Szegö problems*, Tôhoku Math. J. **18** (1965), 54–59.
11. S. Power, *Hankel operators on Hilbert space*, Pitman, Boston, Mass., 1982.
12. T. P. Srinivasan and J. K. Wang, *Weak-* Dirichlet algebras*, Function Algebras, Proc. Internat. Sympos. on Function Algebras (Tulane University, 1965), F. T. Birtel (ed.), Scott, Foresman, Chicago, Ill., 1966, pp. 216–249.
13. H. Widom, *Inversion of Toeplitz matrices*, Illinois J. Math. **4** (1960), 88–89.

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