THE HOMOLOGY AND HIGHER REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF A RIEMANN SURFACE

S. A. BROUGHTON

Abstract. The representations of the automorphism group of a compact Riemann surface on the first homology group and the spaces of \( q \)-differentials are decomposed into irreducibles. As an application it is shown that \( M_{24} \) is not a Hurwitz group.

1. Introduction. Let \( G \) be a finite group of orientation-preserving homeomorphisms of a Riemann surface \( S \) of genus \( \sigma \geq 2 \). We then have a representation of \( G \) on the first homology group \( H_1(S) = H_1(S, \mathbb{C}) \). If \( S \) has a conformal structure which is preserved under the \( G \)-action, then there are also representations of \( G \) on the various spaces of \( q \)-differentials \( \mathcal{H}^q(S) \) (\( \mathcal{H}^q(S) = \) holomorphic sections of \( T^*(S) \otimes \cdots \otimes T^*(S) \) \( q \) times), \( T^*(S) = \) cotangent bundle). In this note we give formulae (Propositions 1–2) for the decompositions of these representations into irreducibles.

The decompositions for \( H_1(S) = \mathcal{H}^1(S) \oplus \mathcal{H}^1(S)^* \) and \( \mathcal{H}^2(S) \) may be applied to the study of surfaces of genus \( \sigma \). From the decomposition of the homology representation it follows that the characters of \( G \) must satisfy certain inequalities (see (13) below). This is useful in showing that certain groups cannot occur as automorphism groups of a surface of a given genus \( \sigma \). In [S] L. L. Scott has given a formula equivalent to (13), though derived by a purely group-theoretic argument.

The decompositions of \( \mathcal{H}^2(S) \) may be used to locally describe the action of the Teichmüller modular group \( \text{Mod}_\sigma \) on Teichmüller space, \( \mathcal{T}_\sigma \) (see [R]). This was used by J. Lewittes [L] to compute the dimensions of the branch loci of the action of \( \text{Mod}_\sigma \) on \( \mathcal{T}_\sigma \).

The decompositions are derived in §2 from the Eichler Trace Formula and the Lefshetz Fixed Point Formula, using a simple character theory argument. In §3 we give an application showing that the Mathieu group \( M_{24} \) is not a Hurwitz group.

2. The decomposition formulae and their derivations. First we recall some facts about actions of a finite group \( G \) on a surface \( S \) (cf. [H, T]). The space \( T = S/G \) is a surface \( T \) of genus \( \tau \), and \( \pi : S \to T \) is branched over \( Q_1, \ldots, Q_\ell \in T \) with branching orders \( n_1, \ldots, n_\ell \). Call \( (\tau; n_1, \ldots, n_\ell) \) the branching data of \( G \) (write \( (n_1, \ldots, n_\ell) \) if \( \tau = 0 \)). The Riemann-Hurwitz formula [FK, p. 243] gives

\[
(2\sigma - 2)/|G| = 2\tau - 2 + \sum_{i=1}^{\ell} (1 - 1/n_i).
\]
We denote the right-hand side by $\kappa$. There are elements $a_1, \ldots, a_r, b_1, \ldots, b_r, c_1, \ldots, c_t$ generating $G$, such that

\[(2) \quad \prod_{i=1}^{r} [a_i, b_i] \prod_{j=1}^{t} c_j = 1,\]

and

\[(3) \quad o(c_i) = n_i.\]

If $P \in S$ is a point fixed by $g \in G$, then the induced map of tangent spaces $dg^{-1} : T_p(S) \to T_g(S)$ is multiplication by an $o(g)$th root of unity, denoted by $e(P, g)$. It is easy to show that we may pick the $c_i$ and $P_i \in \pi^{-1}(Q_i)$ such that $G_{P_i} = \{ g \in G \mid gP_i = P_i \} = \langle c_i \rangle$ and

\[(4) \quad e(P, c_i) = \exp(2\pi i / n_i).\]

Let $U_t \subseteq S^1$ be the group of $n$th roots of unity and let $\varphi_k : S^1 \to S^1$ be the character $z \mapsto z^k$, $k \in \mathbb{Z}$. Let $c_1, \ldots, c_t$ be as defined above and let $\nu_j : \langle c_i \rangle \to U_t$ be the isomorphism defined by $c_i \mapsto \exp(2\pi i / n_i)$. Let $\chi_0, \ldots, \chi_t$ be the irreducible characters of $G$ with $\chi_0$ principal character. Each $\chi_j$ defines a character of $U_t$ by means of the isomorphism $\nu_j$. Define $m^k_j(\chi_j), 0 \leq k \leq n_i - 1$, by

\[(5) \quad \chi_j |_{U_t} = \sum_{k=0}^{n_i - 1} m^k_j(\chi_j) \varphi_k |_{U_t},\]

and define $m^k_j(\chi_j)$ for all $k \in \mathbb{Z}$ by periodicity: $m^k_j(\chi_j) = m^{k+n_i}(\chi_j)$. Let $\text{ch}_{\mathcal{X}^g(S)}$ be the character of the representation of $G$ on $\mathcal{H}^g(S)$, and write

$\text{ch}_{\mathcal{X}^g(S)} = \mu_0^0 \chi_0 + \cdots + \mu_t^0 \chi_t$.  

Define the Poincaré series $P_{\chi_j}(z)$ by

\[P_{\chi_j}(z) = \sum_{q=0}^{\infty} \mu^q_j z^q.\]

We have the following propositions.

**Proposition 1.** Let $G$ be a group of conformal automorphisms of a Riemann surface $S$ of genus $\geq 2$ and let all notation be as above. Then:

(i) $P_{\chi_0}(z) = 1 + z + zR_{\chi_0}(z)$,
(ii) $P_{\chi_j} = zR_{\chi_j}(z), j \neq 0$, where
(iii)

\[R_{\chi_j} = \frac{(1 - \tau) \chi_j(1)}{1 - z} + \kappa \chi_j(1) \left( \frac{1}{1 - z^2} - \frac{1}{n_i} \sum_{i=1}^{t} \frac{\epsilon_i^0(j) + \epsilon_i^1(j) z + \cdots + \epsilon_i^{n_i-1}(j) z^{n_i-1}}{1 - z^{n_i}} \right),\]

and

\[\epsilon_i^r(j) = \sum_{k=0}^{n_i-1} k \cdot m^1_i + r + k(\chi_j).\]
Proposition 2. Let $G$ be a finite group of homeomorphisms of a Riemann surface $S$, $\text{ch}_{H_i(S)}$ the character of the homology representation, and other notation as above. Then the multiplicity of $X_j$ in $\text{ch}_{H_i(S)}$, $\langle X_j, \text{ch}_{H_i(S)} \rangle$, is given by

(i) $\langle X_0, \text{ch}_{H_i(S)} \rangle = 2\tau$, 

(ii) $\langle X_j, \text{ch}_{H_i(S)} \rangle = (2\tau - 2 + t)\chi_j(1) - \sum_{i=1}^{t} m_i^0(\chi_j), \quad j \neq 0.$

Let $p$ be the regular representation of $G$ and $p_i$ the permutation character determined by $G$ acting on the coset space $G/\langle c_i \rangle$. Then (i) and (iii) may be rewritten:

(iii) $\text{ch}_{H_i(S)} = 2\chi_0 + (2\tau - 2 + t)p - \sum_{i=1}^{t} p_i.$

Before proving Propositions 1–2, we recall the Eichler Trace Formula and the Lefschetz Fixed Point Formula. Let $\eta: G \to \mathbb{Z}$ be the class function on $G$ obtained by setting $\eta(g)$ equal to the negative of the Euler characteristic of the fixed point subset $S^g$ of $g$, i.e.

$\eta(1) = 2\sigma - 2, \quad \eta(g) = -|S^g|, \quad g \neq 1.$

By the Lefschetz Fixed Point Formula,

$\text{ch}_{H_i(S)}(g) = 2 + \eta(g), \quad g \in G.$

Define $\lambda_q: G \to \mathbb{C}, q \geq 0$, as follows:

$\lambda_0(g) = 1, \quad g \in G,$

$\lambda_q(1) = (\sigma - 1)(2q - 1), \quad q \geq 1,$

$\lambda_q(g) = \sum_{P \in S^g} \frac{(\epsilon(P, g))^q}{1 - \epsilon(P, g)}, \quad q \geq 1,$

where the last sum is zero if $S^g$ is empty. The Riemann-Roch Theorem and the Eichler Trace Formula state that the characters $\text{ch}_{\pi^*(S)}$ are given by

$\text{ch}_{\pi^*(S)}(g) = \lambda_q(g), \quad q \neq 1,$

$\text{ch}_{\pi^*(S)}(g) = 1 + \lambda_q(g).$

For proofs of (6)–(7) see [FK]. Observe [FK] that $\eta(g) = 2 \mathbb{R} \lambda_1(g)$.

Write

$\eta = \eta^0 + \cdots + \eta^i, \quad \lambda_q = \lambda_q^0 + \cdots + \lambda_q^i,$

where

$\eta^0(1) = 2\sigma - 2, \quad \eta^0(g) = 0, \quad g \neq 1,$

$\eta^i(1) = 0, \quad \eta^i(g) = -|S^g \cap \pi^{-1}(Q_i)|, \quad i > 0,$

$\lambda_q^0(1) = (\sigma - 1)(2q - 1), \quad \lambda_q^0(g) = 0, \quad g \neq 1, q \geq 1,$

$\lambda_q^i(1) = 0, \quad \lambda_q^i(g) = \sum_{P \in S^g \cap \pi^{-1}(Q_i)} \frac{(\epsilon(P, g))^q}{1 - \epsilon(P, g)}, \quad i > 0, q \geq 1.$
For \( g \neq e \in G \), \( S^g \cap \pi^{-1}(Q) \neq \emptyset \) if and only if the conjugacy class of \( g \), \( \text{Cl}(g) \), meets \( \langle c_i \rangle \). Assume \( g \in \langle c_i \rangle \), then since \( G_P \) is cyclic, \( S^g \cap \pi^{-1}(Q) \) is in 1-1 correspondence with \( N_G(\langle g \rangle)/\langle c_i \rangle \) by \( h \mapsto h \cdot P_i \). Furthermore, \( N_G(\langle g \rangle)/\text{Cent}(g) \) is in 1-1 correspondence with \( \text{Cl}(g) \cap \langle c_i \rangle \) by \( h \mapsto wh^{-1} \). From (4) and the definition of \( \nu_i \), \( \epsilon(P_i, g) = \nu_i(g) \), \( g \in \langle c_i \rangle \). It easily follows for \( i > 0 \) that

\[
\lambda_q'(g) = \frac{|\text{Cent}(g)|}{n_i} \sum_{h \in \text{Cl}(g) \cap \langle c_i \rangle} \frac{(\nu_i(h))^q}{1 - \nu_i(h)}.
\]

Since \( \lambda_q' \) is a class function, this holds for all \( 1 \neq g \in G \). Similarly, for \( 1 \neq g \in G \),

\[
\eta'(g) = -\frac{|\text{Cent}(g)|}{n_i} \frac{1}{|\text{Cl}(g) \cap \langle c_i \rangle|}.
\]

We now give proofs of the decompositions, first Proposition 2. Let \( 1 = g_0, \ldots, g_t \) be a set of representatives of conjugacy classes of \( G \). For \( j = 0, 1, \ldots, t \):

\[
\langle \eta, \chi_j \rangle = \sum_{i=0}^{t} \langle \eta, \chi_j \rangle = \sum_{i=0}^{t} \frac{1}{|G|} \sum_{g \in G} \eta'(g) \chi_j(g)
\]

\[
= \sum_{i=0}^{t} \sum_{k=0}^{t} \frac{\eta'(g_k) \chi_j(g_k)}{|\text{Cent}(g_k)|}
\]

\[
= \frac{2\alpha - 2}{|G|} \chi_j(1) - \sum_{i=1}^{t} \frac{1}{n_i} \sum_{1 \neq g \in \langle c_i \rangle} \chi_j(g),
\]

from (9) above. By the Riemann-Hurwitz Formula (1), (10) may be rewritten as

\[
(2\tau - 2 + t) \chi_j(1) - \sum_{i=1}^{t} \frac{1}{n_i} \sum_{g \in \langle c_i \rangle} \chi_j(g) = (2\tau - 2 + t) \chi_j(1) - \sum_{i=1}^{t} m_i(\chi_j).
\]

Since \( ch_{H_i(S)} = 2\chi_0 + \eta \), (i) and (ii) of Proposition 2 follow immediately; (iii) follows from (i)–(ii) and Frobenius reciprocity.

Let \( R_g(z) = \sum_{q=1}^{\infty} \lambda_q(g)z^q^{-1} \). To prove Proposition 1 it suffices by (7) to prove

\[
R_{\chi_j}(z) = \frac{1}{|G|} \sum_{g \in G} R_g(z) \chi_j(g).
\]

Using (8) and arguing as above, the right-hand side of (11) equals

\[
\sum_{q=1}^{\infty} \frac{(\alpha - 1) \chi_j(1)}{|G|} (2q - 1) z^{q-1} + \sum_{i=1}^{t} \sum_{q=1}^{\infty} \sum_{1 \neq g \in \langle c_i \rangle} \frac{1}{n_i} \frac{(\nu_i(g))^q}{1 - \nu_i(g)} \chi_j(g)z^{q-1}
\]

\[
= \kappa \chi_j(1)(1 - z)^{-2} \frac{\kappa \chi_j(1)(1 - z)^{-1}}{2}
\]

\[
+ \sum_{i=1}^{t} \frac{n_i^{-1}}{n_i} \sum_{1 \neq \omega \in \Gamma_n} \frac{\omega^{r+1}}{1 - \omega} \chi_j(\omega)z^{1 - z}^{-n_i}.
\]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
We calculate
\[
\sum_{1 \neq \omega \in U_n} \frac{\omega^s}{1 - \omega} \chi(\omega) = \lim_{x \to 1} \sum_{1 \neq \omega \in U_n} \frac{\omega^s}{1 - x\omega} \chi(\omega) = \lim_{x \to 1} \left( \sum_{q=0}^{\infty} \sum_{\omega \in U_n} \omega^{q+x} \chi(\omega) x^q - \sum_{q=0}^{\infty} \chi(1) x^q \right) \]
\[
= \lim_{x \to 1} \left( \frac{n}{1 - x^n} - \frac{\chi(1)}{1 - x} \right),
\]
where \( L_s(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \) and \( a_k = (1/n) \sum_{\omega \in U_n} \omega^{k+x} \chi(\omega) \). The limit is easily calculated by l'Hôpital's rule and equals \((n - 1)\chi(1)/2 - L'_s(1)\). Setting \( n = n_i, s = r + 1, \chi = \chi_j \), then \( a_k = m_i^{1+k+r}(\chi_j) \) and (11) now follows easily from (12) and the definition of \( R_{x_j} \).

3. Application. If \( \tau = 0 \), then \( G \) is generated by \( c_1, \ldots, c_t \) with \( c_1 \cdot c_2 \cdots c_t = 1 \), and from (ii) of Proposition 2 it follows that for a nonprincipal character \( \chi \)

\[
(t - 2)\chi_j(1) \geq \sum_{i=1}^t m_i^0(\chi_j).
\]
This is a reformulation of the inequality that L. L. Scott obtains in [S] by purely group theoretic means for arbitrary characteristic. The \( G \)-module he constructs on p. 475 of [S] may be identified with \( H(S) \). The inequality (13) may sometimes be used as a “Brauer trick” to show that a given group cannot occur as the automorphism group of a surface of given genus.

As an example of this let us verify that the Mathieu group \( M_{24} \) is not a Hurwitz group. The group \( G \) is a Hurwitz group if it occurs as the automorphism group of a surface \( S \) of genus \( \sigma \) with \( |G| = 84(\sigma - 1) \), Hurwitz’ upper bound for the order of an automorphism group. If \( G \) acts on \( S \) as above then the branching data is \((2, 3, 7)\) and \( G \) has a generating \((2, 3, 7)\)-vector \((c_1, c_2, c_3)\). In Table 1 we have copied a portion of the character table of \( M_{24} \) [Fr, p. 346], giving, for selected characters, the character values of all elements of order 1, 2, 3, or 7. The classes are given in cycle notation, \( M_{24} \) being realized as a permutation group of degree 24.

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( 1^{24} )</th>
<th>( 1^{8^2} )</th>
<th>( 1^{12^3} )</th>
<th>( 1^{3^3} )</th>
<th>( 1^{7^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>( 45 )</td>
<td>(-3)</td>
<td>( 5 )</td>
<td>( 0 )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>( 252 )</td>
<td>( 28 )</td>
<td>( 12 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

For \( c_i \) chosen from the classes in Table 1 all the nonidentity elements of \( \langle c_i \rangle \) are conjugate in \( M_{24} \) except for \( \langle c_3 \rangle \), where half lie in \( 1^{3^2} \) and the other half lie in \( 1^{7^3} \). Since \( k = 1/42 \), we obtain from (10), for any nonprincipal character \( \chi \) of \( M_{24} \),

\[
\frac{1}{42} (\chi(1) - 21\chi(c_1) - 28\chi(c_2) - 36 \text{Re} \chi(c_3)) = \langle \eta, \chi \rangle \geq 0,
\]
or
\[
\chi(1) \geq 21\chi(c_1) + 28\chi(c_2) + 36 \text{Re} \chi(c_3).
\]
(This is equivalent to (13) but slightly more convenient.) There is no possible choice of \( c_1, c_2, c_3 \) for which this inequality holds for both the characters \( \chi_1, \chi_2 \) above. It is interesting to note that for \( c_1 \in 2^{12}, c_2 \in 3^8, c_3 \in 1^{37}, \) or \( 1^{37} \), \( \chi_2 \) and its conjugate \( \overline{\chi}_2 \) are the only irreducible characters for which (13) fails, and that the standard Brauer trick \([I, p. 70]\) applied to any pair of \( \langle c_1 \rangle, \langle c_2 \rangle, \) or \( \langle c_3 \rangle \) fails.

REFERENCES


Department of Mathematics, University of Wisconsin - Madison, Madison, Wisconsin 53706

Current address: Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115