A GLOBAL APPROACH TO THE RANKIN-SELBERG CONVOLUTION FOR GL(3, Z)

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Abstract. We discuss the Rankin-Selberg convolution on GL(3, Z) in the ‘classical’
language of symmetric spaces and automorphic forms.

Introduction. Study of the Rankin-Selberg convolution of two automorphic forms
on GL(2) has yielded many interesting number theoretic applications. Recently, this
construction has been extended to automorphic forms on GL(n) by Jacquet,
Piatetskii-Shapiro, and Shalika (see [8] for the local results). Their methods general-
ize those of Jacquet-Langlands; in particular, they make considerable use of the
representation theory over local fields, as developed by Bernstein, Zelevinsky, and
others. In this paper we shall give a discussion of their results in a classical,
nonadelic, language in the simplest higher rank situation: that of two automorphic
forms on GL(3, Z).

We must emphasize that most of the ideas in this paper are not really new, but
simply restatements in this more classical language of those of Jacquet, Piatetskii-
Shapiro, and Shalika; further, a classical sketch of the Rankin-Selberg method has
been given by Jacquet [7]. However, we are able to sharpen these results at the
archimedean place by giving the precise gamma factors at infinity, and also the
behavior of the convolution at its poles (see also [10]). These are of particular
interest because of two applications which require this precise version of the
convolution.

The first, noted by Moreno-Shahidi [30] and Serre [17] (see also [14]), obtains the
coefficient bound

\[ |a(p)| \leq p^{1/5} + p^{-1/5} \]

for the size of the Fourier coefficient of a GL(2) Maass wave form which is also a
Hecke eigenform (here the normalization is such that the Ramanujan-Petersson
conjecture predicts \(|a(p)| < 2\)). This follows by combining the convolution with the
Gelbart-Jacquet lifting [3] from GL(2) to GL(3) and a result of Chandrasekharan-
Narasimhan [2] on the order of partial \(L\) series. The second is given in the thesis of

Received by the editors February 25, 1985 and, in revised form, April 10, 1986.
1980 Mathematics Subject Classification (1985 Revision). Primary 10D05, 10D20, 10D24; Secondary
12A67.

Key words and phrases. Automorphic forms, Eisenstein series, Rankin-Selberg convolution, Hecke
operators.

Research supported in part by the NSF.
G. Gilbert [4], who obtains multiplicity one theorems based on knowledge of the $a(p)$ for $p$ in a Frobenius class of an extension; the explicit convolution allows him to give effective results in low rank cases (see also Moreno [12, 13]).

Now, let us describe the Rankin-Selberg Dirichlet series for GL(3). Let $\mathcal{O}$ be the center of $GL(3, \mathbb{R})$, and

$$H = GL(3, \mathbb{R})/\mathcal{O}(3).$$

Then from the Iwasawa decomposition one sees that the symmetric space $H$ may be regarded as the set of (cosets)

$$(0.1) \quad \tau = \left( \begin{array}{ccc} y_1 y_2 & y_1 x_2 & x_3 \\ y_1 & x_1 & 1 \end{array} \right)$$

with $x_1, x_2, x_3 \in \mathbb{R}$, $y_1, y_2 \in \mathbb{R}_{>0}$; we shall use this parametrization throughout. Left matrix multiplication induces an action of $GL(3, \mathbb{R})$ on $H$, which we write as $\circ$.

An automorphic form on $GL(3, \mathbb{Z})$ is a left-$GL(3, \mathbb{Z})$ invariant function $\phi: H \to \mathbb{C}$ satisfying certain differential equations and growth conditions. Such a form has a Fourier expansion; in the case of a cusp form, it is given by

$$\phi(\tau) = \sum_{g \in \Gamma_\infty \setminus \Gamma^2} \sum_{m,n=1}^{\infty} a_{m,n} (mn)^{-s} W\left( \begin{pmatrix} mn \\ m \\ 1 \end{pmatrix} g \circ \tau \right),$$

where

$$\Gamma^2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\},$$

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \right\},$$

$$\Gamma_\infty^2 = \Gamma^2 \cap \Gamma_\infty,$$

and $W(\tau)$ is a certain $GL(3)$ Whittaker function, which depends only on the differential equations for $\phi$ (see §1 for details). The numbers $a_{m,n}$ are called the Fourier coefficients of $\phi$; the factor $(mn)^{-s}$ is included for convenience.

Now let $\phi$ be as above, and $\varphi$ be another $GL(3, \mathbb{Z})$ form, with Fourier coefficients $b_{m,n}$. Then the Rankin-Selberg Dirichlet series is given by

$$D(s, \phi, \varphi) = \sum_{m,n=1}^{\infty} a_{m,n} b_{m,n} m^{-s-1} n^{-s}.$$

It converges absolutely for $\text{Re}(s)$ sufficiently large.

The main properties of this series are described in Theorems 3.2, 3.4 (meromorphic continuation and functional equation under $s \to 1 - s$, $(m, n) \to (n, m)$), and 4.5 (Euler product) below. First, in §1, we review the basic information about $GL(3, \mathbb{Z})$ automorphic forms which is needed in the sequel—Fourier expansions, Hecke operators, and the like—and also discuss briefly the standard minimal parabolic Eisenstein series. This section is based on the thesis of Daniel Bump, and I would like to thank him for providing me with prepublication access to this work.
[1]. Then, in §2, the standard maximal parabolic Eisenstein series of type (2,1), $E(s, \tau)$, is introduced, and its properties are summarized. Next, in §3, we show how the Dirichlet series $D(s, \phi, \varphi)$ arises by integrating against $E(s, \tau)$, and obtain the basic properties of the Rankin-Selberg convolution from this. Finally, in §4 the Euler product for $D(s, \phi, \varphi)$ is derived from $GL(3)$ Hecke theory, and used to explicitly evaluate the gamma factors ‘at infinity.’ This is accomplished modulo one hypothesis—that the Rankin-Selberg method extends to noncusp forms—which is not verified here.

Though we have confined ourselves to the case of $GL(3, \mathbb{Z})$ in this note, both for simplicity, and to take advantage of the results of [1], it is possible to extend this approach to the other cases considered in [7,8].

I would like to warmly thank Daniel Bump, Dorian Goldfeld, Carlos Moreno, and Peter Sarnak for several helpful conversations.

1. Automorphic forms on $GL(3, \mathbb{Z})$. Let $\Gamma = GL(3, \mathbb{Z}) = \{ \gamma \in M(3, \mathbb{Z}) \mid \det \gamma = \pm 1 \}$, $\Gamma_1 = SL(3, \mathbb{Z})$, and $v_1, v_2$ be complex numbers. Also, let $\mathcal{O}$ denote the center of the universal enveloping algebra of $GL(3, \mathbb{R})$, acting as an algebra of $GL(3, \mathbb{R})$-invariant differential operators on $H$.

**Definition 1.1.** An automorphic form on $\Gamma$ of type $(v_1, v_2)$ is a function $\psi: H \to \mathbb{C}$ such that

1. $\psi(\gamma \circ \tau) = \psi(\tau)$ for all $\gamma \in \Gamma$, $\tau \in H$,
2. $\psi$ is an eigenfunction of $\mathcal{O}$ with the same eigenvalues as the function $I_{v_1, v_2}(\tau) = y_1^{2v_1} y_2^{v_1 + 2v_2}$,
3. $\psi$ grows at most polynomially in $y_1$, $y_2$ as $y_1, y_2 \to \infty$, uniformly in $x_1, x_2, x_3$.

**Definition 1.2.** A cusp form is an automorphic form which satisfies

$$
\int_0^1 \int_0^1 \phi \left( \begin{array}{ccc}
1 & \xi_3 \\
1 & \xi_1 \\
1 & \xi_2
\end{array} \right) d\xi_1 d\xi_3 = 0,
$$

$$
\int_0^1 \int_0^1 \phi \left( \begin{array}{ccc}
1 & \xi_3 \\
1 & \xi_2 \\
1 & 1
\end{array} \right) d\xi_2 d\xi_3 = 0.
$$

**Example 1.3.** The Eisenstein series of type $(v_1, v_2)$ associated to a minimal parabolic is given by

$$
E_{v_1, v_2}(\tau) = \frac{1}{4} \zeta(3v_1) \zeta(3v_2) \zeta(3v_1 + 3v_2 - 1) \cdot \sum_{\gamma \in \Gamma_1 \backslash \Gamma} I_{v_1, v_2}(\gamma \circ \tau)
$$

(where $\zeta$ is the Riemann zeta function) if $\text{Re}(\nu_1), \text{Re}(\nu_2) > \frac{3}{2}$, and by analytic continuation in $\nu_1, \nu_2$ (based on the action of the Weyl group) for other values of $\nu_1, \nu_2$ (see [1] for details). It is an automorphic form for $\Gamma$ of type $(\nu_1, \nu_2)$, but not a cusp form. We shall give another example of a $\Gamma$-automorphic form, the maximal parabolic Eisenstein series, in §2 below.
Now set

$$\Gamma_1^2 = \Gamma^2 \cap \Gamma_1, \quad e[x] = \exp(2\pi i x), \quad w_1 = \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ \xi_1 & 1 \\ -1 & -1 \end{pmatrix},$$

and put

$$W_{\nu_1, \nu_2}(\tau) = \int_{\mathbb{R}^2} I_{\nu_1, \nu_2}(w_1 \circ \tau) \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ \xi_1 & 1 \\ -1 & -1 \end{pmatrix} e[-\xi_1 - \xi_2] d\xi_1 d\xi_2 d\xi_3$$

if $\text{Re}(\nu_1), \text{Re}(\nu_2) > \frac{1}{2}$. Explicitly,

$$I_{\nu_1, \nu_2}(w_1 \circ \tau) = I_{\nu_1, \nu_2}(\tau) \left( (x_1^2 + x_2^2 + y_1^2 + y_2^2)^{-3\nu_1/2} \right) \cdot \left( (x_1 x_2 - x_3)^2 + x_1^2 y_2^2 + y_1^2 y_2^2 \right)^{-3\nu_2/2}.$$

As shown in [1], $W_{\nu_1, \nu_2}$ too has a meromorphic continuation to all values of $(\nu_1, \nu_2)$, which we again write $W_{\nu_1, \nu_2}$, based on the action of the Weyl group of $GL(3)$ on $(\nu_1, \nu_2)$ (see also [20]).

**Proposition 1.4 (Shalika).** Let $\phi$ be an automorphic form of type $(\nu_1, \nu_2)$ on $\Gamma$. Then $\phi$ has a Fourier expansion given by

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} \phi_{0,n}(\tau) + \frac{1}{2} \sum_{\gamma \in \Gamma_1^2 \setminus \Gamma_1^2} \sum_{m,n=-\infty}^{\infty} \phi_{m,n}(\gamma \circ \tau),$$

where

$$\phi_{m,n}(\tau) = \int_0^1 \int_0^1 \int_0^1 \phi \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ \xi_1 & 1 \\ -1 & -1 \end{pmatrix} e[-m\xi_1 - n\xi_2] d\xi_1 d\xi_2 d\xi_3.$$

If $\phi$ is a cusp form, this may be simplified to

$$\phi(\tau) = \sum_{\gamma \in \Gamma_1^2 \setminus \Gamma_2} \sum_{m,n=1}^{\infty} a_{m,n}(mn)^{-1} W_{\nu_1, \nu_2}(\begin{pmatrix} mn & m \\ 1 & \gamma \circ \tau \end{pmatrix}).$$

The proof of this result, due to Shalika, may be found in Bump [1, Chapter 4] (we have also used his formula (3.16) and changed the notation slightly) and Shalika [18, Theorem 5.9]. A similar simplification may be given in the case of noncusp forms as well.

Given an automorphic form $\phi$ of type $(\nu_1, \nu_2)$ on $\Gamma$, one can construct a form $\tilde{\phi}$ of type $(\nu_2, \nu_1)$ as follows. Define an involution $\iota$ of $GL(3, \mathbb{R})$ by

$$\iota g = w_1 g^{-1} w_1;$$

$\iota$ induces an involution of $H$:

$$\iota \tau = \begin{pmatrix} y_2 y_1 & -y_2 x_1 & x_1 x_2 - x_3 \\ y_2 & -x_2 & x_3 \\ 1 & & \end{pmatrix}.$$
Define the function \( \tilde{\phi} \) by

\[
\tilde{\phi}(\tau) = \phi(\tau);
\]

if \( \phi \) is an automorphic form of type \((v_1, v_2)\), with Fourier coefficients \( a_{m,n} \), then one can verify that \( \tilde{\phi} \) is an automorphic form of type \((v_2, v_1)\), with Fourier coefficients \( a_{n,m} \). The involution \( \tilde{\cdot} \) will play a key role in the functional equation for the convolution.

Let us conclude this section with a brief discussion of \( L \)-series and \( GL(3) \) Hecke theory. For \( \text{Re}(s) \) sufficiently large, put

\[
L(s, \phi) = \sum_{n=1}^{\infty} a_{1,n} n^{-s},
\]

so that

\[
L(s, \tilde{\phi}) = \sum_{m=1}^{\infty} a_{m,1} m^{-s}.
\]

**Proposition 1.5.** Suppose that \( \phi \) is an eigenfunction of the Hecke algebra, normalized so that \( a_{1,1} = 1 \). Then

1. \( L(s, \phi) \), \( L(s, \tilde{\phi}) \) have Euler products given by

\[
L(s, \phi) = \prod_{p} \left( 1 - a_{1,p} p^{-s} + a_{p,1} p^{-2s} - p^{-3s} \right)^{-1},
\]

\[
L(s, \tilde{\phi}) = \prod_{p} \left( 1 - a_{p,1} p^{-s} + a_{1,p} p^{-2s} - p^{-3s} \right)^{-1},
\]

where \( \prod_{p} \) denotes the product over all primes \( p \).

2. 

\[
\sum_{\substack{u \mid n \atop t \mid m}} a_{mv/u, nu/t} = \sum_{\substack{nu = r \atop u \mid m \atop t \mid n}} a_{m,v/u, n/u/t}.
\]

For a proof, see [1]. Additional references for Hecke theory are [23, 24, 25, 26].

For example, in the case of the Eisenstein series \( E_{v_1, v_2} \) of Example 1.3 above, one has

**Proposition 1.6.**

\[
L(s, E_{v_1, v_2}) = \xi(s + 1 - 2v_1 - v_2) \xi(s + v_1 - v_2) \xi(s - 1 + v_1 + 2v_2).
\]

There is also a meromorphic continuation of \( L(s, \phi) \) to the entire \( s \) plane, as well as a functional equation relating \( L(s, \phi) \) to \( L(1 - s, \tilde{\phi}) \) (see [1] for details). Also, for an adelic, representation theoretic approach to automorphic forms on \( GL(3) \), see [9].

2. **Maximal parabolic Eisenstein series.** In this section we briefly give the properties of the Eisenstein series associated to the standard maximal parabolic of type \((2, 1)\), \( E(s, \tau) \). Let \( \hat{\Gamma} \) be the group

\[
\hat{\Gamma} = \left\{ \begin{pmatrix} * & * & * \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \right\}.
\]
Note that for $g$ in $\Gamma$,
\begin{equation}
\det(g \circ \tau) = \det(\tau);
\end{equation}
this is actually implied by the proof of Lemma 2.2 below, but is also easy to see
directly. $E(s, \tau)$ is given by
\begin{equation*}
E(s, \tau) = \sum_{\gamma \in \Gamma \setminus \Gamma_1} \left[ \det(\gamma \circ \tau) \right]^s;
\end{equation*}
by (2.1), this is well defined. It is clearly a $\Gamma$-automorphic form.

**Lemma 2.1.** The cosets of $\hat{\Gamma} \setminus \Gamma_1$ are in one-to-one correspondence with the relatively
prime triples of integers via the map
\begin{equation*}
\hat{\Gamma} \gamma \rightarrow \text{last row of } \gamma.
\end{equation*}

**Proof.** The map is clearly well defined and injective, and is surjective since every
relatively prime triple can be completed to a matrix in $\text{SL}(3, \mathbb{Z})$. □

**Lemma 2.2.** Let
\begin{equation*}
\gamma = \begin{pmatrix}
* & * & * \\
* & * & * \\
a & b & c
\end{pmatrix}
\end{equation*}
be a representative for a coset of $\hat{\Gamma} \setminus \Gamma_1$. Then
\begin{equation*}
\det(\gamma \circ \tau) = \frac{\det(\tau)}{|y_1^2|az_2 + b|^2 + (ax_3 + bx_1 + c)^2|^{3/2}},
\end{equation*}
where $z_2 = x_2 + iy_2$, and one takes the positive square root.

**Proof.** Write
\begin{equation}
\gamma \tau = \tau'k(rI_3),
\end{equation}
where $\tau'$ in $\text{GL}(3, \mathbb{R})$ is of shape (0.1), $k$ is in $O(3)$, $r$ is real, and $I_3$ denotes the
$3 \times 3$ identity. Then, by comparing the bottom rows of both sides of (2.2), and using
$k \in O(3)$, one sees that
\begin{equation*}
r^2 = (ay_1y_2)^2 + (ay_1x_2 + by_1)^2 + (ax_3 + bx_1 + c)^2.
\end{equation*}
But taking determinants in (2.2) gives
\begin{equation*}
\det(\tau) = \det(\tau')|r|^3,
\end{equation*}
so the result follows. □

**Proposition 2.3.** The Eisenstein series $E(s, \tau)$ converges absolutely for $\Re(s) > 1$.

**Proof.** By Lemmas 2.1 and 2.2,
\begin{equation}
E(s, \tau) = \sum \frac{(y_1^2y_2)^s}{|y_1^2|az_2 + b|^2 + (ax_3 + bx_1 + c)^2|^{3s/2}},
\end{equation}
where the sum is over all relatively prime triples of integers $(a, b, c)$. Application of
the integral test then gives the result. □
For convenience, introduce the series

\[ E^*(s, \tau) = \xi(3s) E(s, \tau); \]

\( E^*(s, \tau) \) is given by the right-hand side of (2.3), with the sum taken over all triples of integers \((a, b, c) \neq (0, 0, 0)\).

**Theorem 2.4.** The Eisenstein series \( E^*(s, \tau) \) \((\text{Re}(s) > 1)\) has a Fourier expansion given by (1.3), with

\[
\begin{align*}
\phi_{m,n}(\tau) &= \begin{cases} 
\frac{4(y_1^{1/2}y_2^{1/2})^3}{\Gamma(3s/2)} y_1^{1/2}y_2^{1/2}s^{3s/2} |m|^{|3s-1|/2} \sigma_1(m) \\
\cdot K_{(3s-1)/2} \left( 2\pi |m|y_1 \right) e(mx_1), & m \neq 0, n = 0, \\
\frac{4y_1^{1-s}y_2^{1-s}(s/2)}{\Gamma(3s/2)} \pi^{3s/2} |n|^{|3s-1|/2} \sigma_2(n) \\
\cdot K_{(3s-2)/2} \left( 2\pi |n|y_2 \right) e(nx_2), & m = 0, n \neq 0, \\
2y_1^2y_2^2 \xi(3s) + 2y_1^{1-s}y_2^{1-s}(3s-1) \Gamma \left( \frac{3s-1}{2} \right) \pi^{1/2} \Gamma \left( \frac{3s}{2} \right) \\
\cdot + 2y_1^{1-s}y_2^{1-s} \xi(3s-2) \pi \Gamma \left( \frac{3s}{2} - 1 \right) \Gamma \left( \frac{3s}{2} \right), & m = n = 0,
\end{cases}
\end{align*}
\]

and \( \phi_{m,n}(\tau) \) identically zero when \( mn \neq 0 \) (here \( K \) denotes the modified \( K \)-Bessel function of the third kind,

\[
K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+1/t)/2} t^{s-1} dt \quad (y > 0),
\]

and \( \sigma \) denotes the divisor function \( \sigma(n) = \sum_{0 < d \mid n} d^s \)).

**Proof.** This follows by breaking the sum (2.3) for \( E^*(s, \tau) \) into two pieces (the terms \( a = 0, a \neq 0 \)), using the Fourier expansion of a GL(2) Eisenstein series to compute the \( a = 0 \) piece, and [5, formulas 3.276(2), 8.432(5)] and Poisson summation to evaluate the \( a \neq 0 \) terms (see Terras [28] for details). \( \square \)

**Corollary 2.5.** The Eisenstein series \( E(s, \tau) \) can be meromorphically continued to the entire \( s \) plane, and satisfies the functional equation

\[
\pi^{-3s/2} \Gamma \left( \frac{3s}{2} \right) \xi(3s) E(s, \tau) = \pi^{-3(1-s)/2} \Gamma \left( \frac{3(1-s)}{2} \right) \xi(3(1-s)) E(1-s, \tau).
\]

Further,

\[
\pi^{-3s/2} \Gamma \left( \frac{3s}{2} \right) \xi(3s) E(s, \tau)
\]

is holomorphic except for simple poles at \( s = 0, 1 \), of residues \( -\frac{4}{3}, \frac{2}{3} \) respectively.

The analytic continuation and functional equation of Eisenstein series has been established in far greater generality by Selberg [22] and Langlands [11]. It can also be easily derived here since \( E(s, \tau) \) is an Epstein zeta function (cf. Terras [26, 27, 28]). Additional references for the Fourier expansions of Eisenstein series include [6, 21, 28, 29].
3. The Rankin-Selberg integral. In this section we use the Rankin-Selberg method \cite{15,16} to study the properties of the Dirichlet series $D(s, \phi, \varphi)$. To do this, we first need

**Lemma 3.1.** The $GL(3, \mathbb{R})$-invariant (Haar) measure on $H$ is given by

$$d^H \tau = dx_1 dx_2 dx_3 dy_1 dy_2 / (y_1 y_2)^3.$$ 

Now let $\phi$ and $\varphi$ be two cusp forms for $\Gamma$, of types $(\nu_1, \nu_2)$ and $(\mu_1, \mu_2)$, and with Fourier coefficients $a_{m,n}$, $b_{m,n}$ respectively. By Lemma 3.1, the Rankin-Selberg integral

$$\int_{\Gamma \backslash H} \phi(\tau) \overline{\varphi(\tau)} E(s, \tau) d^H \tau$$

is well defined. For later use, let us examine the more general integral

$$I(s, \phi, \varphi, \xi_1, \xi_2) = \int_{\Gamma \backslash H} \phi(\tau) \overline{\varphi(\tau)} \left( \begin{array}{ccc} 1 & \xi_2 & 0 \\ \xi_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) E(s, \tau) d^H \tau,$$

where $\xi_1, \xi_2$ are real. This integral is evaluated by

**Theorem 3.2.** For $\Re(s)$ sufficiently large,

$$I(s, \phi, \varphi, \xi_1, \xi_2) = G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2) \sum_{m,n=1}^{\infty} a_{m,n} b_{m,n} m^{-2s} n^{-s},$$

where

(3.1)

$$G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2) = \int_0^\infty \int_0^\infty W_{\nu_1, \nu_2} \left( \begin{array}{ccc} y_1 y_2 & \nu_1 & 0 \\ y_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \cdot \overline{W_{\mu_1, \mu_2}} \left( \begin{array}{ccc} y_1 y_2 & \nu_1 & 0 \\ y_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) e^{-\xi_1 y_1 - \xi_2 y_2} (y_1^2 y_2^2)^s dy_1 dy_2 / (y_1 y_2)^3.$$

**Proof.** First, since

$$E(s, \tau) = \frac{1}{2} \sum_{\gamma \in \Gamma \backslash \Gamma} [\det(\gamma \circ \tau)]^s,$$

one sees by the usual ‘unfolding’ trick that

$$I(s, \phi, \varphi, \xi_1, \xi_2) = \frac{1}{2} \int_{\Gamma \backslash H} \phi(\tau) \overline{\varphi(\tau)} \left( \begin{array}{ccc} 1 & \xi_2 & 0 \\ \xi_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( y_1^2 y_2^2 \right)^s d^H \tau.$$ 

Let $\mathcal{F}$ be a fundamental domain for

$$\left\{ \left( \begin{array}{ccc} 1 & e \\ 1 & f \\ 0 & 1 \end{array} \right) \right| e, f \in \mathbb{Z} \right\} \backslash H.$$
Then we may identify \( H \setminus \Gamma_1 \) with \( \Gamma_1 \setminus \mathcal{F} \). Further, the cosets of \( \Gamma_2 \setminus \mathcal{F} \) are in exactly two-to-one correspondence with the cosets of \( \Gamma_1 \setminus \mathcal{F} \). Since \( \phi, \varphi, [\det \tau]^s \), and \( d H_\tau \) are all invariant under

\[
\begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix},
\]

one may rewrite the last line as

\[
\int_{\Gamma_2 \setminus \mathcal{F}} \phi(\tau) \overline{\varphi} \left( \begin{pmatrix}
1 \\
\xi_2 \\
1 \\
\xi_1 \\
1
\end{pmatrix} (y_1^2 y_2)^s \right) \, d H_\tau.
\]

Substituting its Fourier expansion (1.5) for \( \phi(\tau) \), and again using the unfolding trick, one sees that this in turn is equal to

\[
\int_{\Gamma_2 \setminus \mathcal{F}} \sum_{m,n=1}^{\infty} a_{m,n} (mn)^{-1} W_{\mu_1,\nu_2} \left( \begin{pmatrix}
mn & m \\
1 & 1
\end{pmatrix} \right)^{\circ \tau} (y_1^2 y_2)^s \, d H_\tau.
\]

However,

\[
\Gamma_\infty \setminus \mathcal{F} \equiv \{ \tau \in H \mid x_1, x_2, x_3 \in (0,1), 0 < y_1, y_2 \in \mathbb{R} \}.
\]

So, substituting in the Fourier expansion of \( \varphi \), and switching the summations with the \( x_i \)-integrations \((i = 1, 2, 3)\) (justified by the absolute convergence of (1.5)), one obtains

\[
I(s, \phi, \varphi, \xi_1, \xi_2) = \int_0^\infty \int_0^\infty \sum_{m,n=1}^{\infty} a_{m,n} \overline{b}_{m',n'} (mnm')^{-1}
\cdot \sum_{\gamma \in \Gamma_\infty \setminus \Gamma^2} W_{\mu_1,\nu_2} \left( \begin{pmatrix}
mn & m \\
1 & 1
\end{pmatrix} \right)^{\circ \tau}
\cdot \overline{W}_{\mu_1,\nu_2} \left( \begin{pmatrix}
mn' & m' \\
1 & 1
\end{pmatrix} \right)^{\gamma \circ \tau} (y_1^2 y_2)^s \, d H_\tau.
\]

Now from its definition (1.1) and (1.2), one sees that

\[
W \left( \begin{pmatrix}
mn' & m' \\
1 & 1
\end{pmatrix} \right)^{\circ \tau} \left( \begin{pmatrix}
1 & \xi_2 \\
1 & \xi_1 \\
1
\end{pmatrix}
\right) = e^{[m'(x_1 + \xi_1 y_1) + n'(x_2 + \xi_2 y_2)]} W \left( \begin{pmatrix}
mn' & m' \\
1 & 1
\end{pmatrix} \right)^{\circ \tau} \left( \begin{pmatrix}
y_1 y_2 \\
y_1 \\
1
\end{pmatrix}
\right).
\]
Further, the action of
\[
\gamma = \begin{pmatrix}
* & * \\
c & d \\
1 & 1
\end{pmatrix}
\]
on \tau sends \(x_1 \to cx_3 + dx_1\). But then (3.3) implies that the integrand in (3.2) corresponding to this choice of \(\gamma\) is \(e^{m'cx_3}\) times a function independent of \(x_3\). Integrating with respect to \(x_3\), we see that a nonzero contribution occurs only when \(c = 0\). Similar consideration of the \(x_1\) and \(x_2\) integrals reduces the sum to the case \(\gamma = I_3, \ m = m', \ n = n'\). So
\[
I(s, \phi, \varphi, \xi, \xi_2) = \int_0^\infty \int_0^\infty \sum_{m, n - 1} a_{m, n} b_{m, n}(mn)^{-2} W_{\nu_1, \nu_2}(\begin{pmatrix}
mny_1 y_2 \\
my_1
\end{pmatrix})
\cdot W_{\mu_1, \mu_2}(\begin{pmatrix}
mny_1 y_2 \\
my_1
\end{pmatrix}) e^{-m\xi_1 y_1 - n\xi_2 y_2}(y_1 y_2)^2 (y_1 y_2)^{-3} dy_1 dy_2.
\]

Finally, it is easy to see that \(D(s, \phi, \varphi), G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi, \xi_2)\) converge absolutely for \(\text{Re}(s)\) sufficiently large. This allows one to perform the remaining interchanges of integration and summation in (3.4). Then making the change of variables \(my_1 \to y_1, ny_2 \to y_2\) gives the result. \(\square\)

REMARKS. (i) A similar argument shows that
\[
\int_{\Gamma \setminus H} \phi(\tau) \overline{\varphi(\tau)} \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ \xi_1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} E(s, \tau) d^H \tau
\]
is independent of \(\xi_3\).

(ii) Note that by (1.1), (1.2), and (3.1), \(G\) is given for \(\text{Re}(s)\) sufficiently large as an integral over \((\mathbb{R}^+)^2 \times \mathbb{R}^6\). As we shall see below, \(G\) has a meromorphic continuation to the entire \(s\) plane, and is essentially a product of gamma functions.

(iii) Theorem 3.2 extends without change to the case where only one of \(\phi\) and \(\varphi\) is a cusp form. Slightly modified, one should be able to extend it to the case where neither \(\phi\) nor \(\varphi\) is a cusp form; the corresponding extension of the \(\text{GL}(2)\) Rankin-Selberg integral is given in Zagier [19] (the idea is to truncate the fundamental domain for \(\Gamma \setminus H\)). Since we do not give the details of this here, we assume it as hypothesis H.1 below.

For convenience in stating the next result, let
\[
G^*(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi, \xi_2) = \pi^{-3s/2} \Gamma\left(\frac{3s}{2}\right) G(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi, \xi_2)
\]
and
\[
D^*(s, \phi, \varphi, \xi, \xi_2) = G^*(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi, \xi_2) \xi(3s) D(s, \phi, \varphi).
\]
Also, let $f$ be a function on $\mathbb{R}^2$ such that

(i) $f(\xi_1, \xi_2) = f(-\xi_2, -\xi_1),$

(ii) $\int_{\mathbb{R}^2} f(\xi_1, \xi_2) e^{-\xi_1 y_1 - \xi_2 y_2} d\xi_1 d\xi_2$ is rapidly decreasing,

and set

$$G_\ast^*(s, \nu_1, \nu_2, \mu_1, \mu_2) = \int_{\mathbb{R}^2} G^*(s, \nu_1, \nu_2, \mu_1, \mu_2, \xi_1, \xi_2) f(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

$$D_\ast^*(s, \phi, \varphi) = G_\ast^*(s, \nu_1, \nu_2, \mu_1, \mu_2) \zeta(3s) D(s, \phi, \varphi).$$

Let us denote the GL(3)-Petersson inner product of two automorphic forms, at least one of which is a cusp form, by

$$\langle \phi, \varphi \rangle = \int_{\Gamma \setminus \mathbb{H}} \phi(\tau) \overline{\varphi(\tau)} dH;$$

observe that $\langle \phi, \varphi \rangle = \langle \tilde{\phi}, \tilde{\varphi} \rangle$. Then we have

**Proposition 3.3.** Let $\phi$ and $\varphi$ be $\Gamma$-automorphic forms, of types $(\nu_1, \nu_2), (\mu_1, \mu_2)$ respectively, at least one of which is a cusp form. Then

1. $D^*(s, \phi, \varphi, \xi_1, \xi_2)$ has a meromorphic continuation to the entire $s$ plane, and satisfies the functional equation

   $$D^*(s, \phi, \varphi, \xi_1, \xi_2) = D^*(1 - s, \phi, \varphi, -\xi_1, -\xi_2).$$

2. $D^*(s, \phi, \varphi, 0, 0)$ is holomorphic when $\phi$ and $\varphi$ are orthogonal, and otherwise is holomorphic except for simple poles at $s = 0, 1$, of residues $-\frac{3}{2} \langle \phi, \varphi \rangle$, $\frac{3}{2} \langle \phi, \varphi \rangle$, respectively.

3. $D_\ast^*(s, \phi, \varphi)$ has a meromorphic continuation to the entire $s$ plane, and satisfies the functional equation

   $$D_\ast^*(s, \phi, \varphi) = D_\ast^*(1 - s, \tilde{\phi}, \tilde{\varphi}).$$

4. There exists an $f$ satisfying (3.5) such that $G_\ast^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$ has an analytic continuation to the entire $s$ plane which is never zero.

**Proof.** Observe that

$$\varphi \left( \begin{array}{ccc} 1 & \xi_2 & 
 1 & \xi_1 
 \end{array} \right) = \tilde{\phi} \left( \begin{array}{ccc} 1 & -\xi_1 & 
 1 & -\xi_2 
 \end{array} \right).$$

Hence (1), (2), and (3) follow by combining Corollary 2.5, Theorem 3.2, and Remark (i) above. As for (4), note that for $\operatorname{Re}(s)$ sufficiently large,
Take a function $f_1$ whose Fourier transform is concentrated at a point in $[0, \infty] \times [0, \infty]$ where the Whittaker functions do not vanish (presumably they never do, but this is not needed). Symmetrize by setting

$$f(\xi_1, \xi_2) = f_1(\xi_1, \xi_2) + f_1(-\xi_2, -\xi_1).$$

Then it is clear that (4) holds for this $f$. □

The use of the extra variables $\xi_1, \xi_2$ was suggested to me by P. Sarnak.

Similar reasoning should give the meromorphic continuation and functional equation of $D^*(s, \phi, \varphi, \xi_1, \xi_2)$, $D^\#(s, \phi, \varphi)$ even when neither $\phi$ nor $\varphi$ is a cusp form (note, though, that the possibilities for the location and order of the poles are more diverse). Since we have not verified this here, we make it

**HYPOTHESIS H.1.** (1) of Proposition 3.3 holds for $\phi, \varphi$ Eisenstein series.

Note that (3) and (4) of Proposition 3.3 follow from Hypothesis H.1 in this case just as in the proof of Proposition 3.3. Making use of this, one can sharpen Proposition 3.3 by giving an explicit gamma factor. Namely, put

$$\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2) = \pi^{-s/2} \Gamma\left(\frac{s - 2\nu_1 - \nu_2 - 2\mu_1 - \mu_2 + 2}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 - \nu_2 - 2\mu_1 - \mu_2 + 1}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 - \nu_2 + 2\mu_1 + \mu_2 - 1}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 + 2\nu_2 + 2\mu_1 - \mu_2 - 2}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 + 2\nu_2 + 2\mu_1 + 2\mu_2 - 2}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 + 2\nu_2 - \mu_2 + \mu_1 - 1}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 + 2\nu_2 + \mu_1 + 2\mu_2 - 2}{2}\right) \cdot \Gamma\left(\frac{s + \nu_1 + 2\nu_2 + \mu_1 - \mu_2 - 2}{2}\right).

We shall show

**Theorem 3.4.** Assume Hypothesis H.1. Let $\phi$ and $\varphi$ be $\Gamma$-automorphic forms, of types $(\nu_1, \nu_2), (\mu_1, \mu_2)$ respectively, at least one of which is a cusp form. Then

$$\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2) \xi(3s) D(s, \phi, \varphi)$$

has a meromorphic continuation to the entire $s$ plane, which is holomorphic when $\phi$ and $\varphi$ are orthogonal, and otherwise is holomorphic except for simple poles at $s = 0, 1$. Further, $\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2) \xi(3s) D(s, \phi, \varphi)$ satisfies the functional equation

$$\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2) \xi(3s) D(s, \phi, \varphi) = \Gamma^*(1 - s, \nu_2, \nu_1, \mu_2, \mu_1) \xi(3(1 - s)) D(1 - s, \phi, \varphi).$$

Since the proof of Theorem 3.4 requires the Euler product for $D(s, \phi, \varphi)$, it is deferred to §4 below.

**4. The Euler product for the convolution.** Throughout this section assume that $a_{m,n}, b_{m,n}$ are the Fourier coefficients of normalized Hecke eigenforms $\phi, \varphi$ respectively (cf. §1). Also, write $(m, n)$ for the greatest common divisor of $m$ and $n$. The existence of an Euler product for $D(s, \phi, \varphi)$ is established by the following lemma.
Lemma 4.1. Let $(mn, m'n') = 1$. Then $a_{m'm',nn'} = a_{m,n}a_{m',n'}$.

Proof. Induct on $mm'n'n'$, using Proposition 1.5(2). □

Corollary 4.2. Let

$$D_p(s, \phi, \varphi) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{p^j, 1} p^{-j} p^{-k} p^{-2js-ks}. $$

Then

$$D(s, \phi, \varphi) = \prod_p D_p(s, \phi, \varphi). $$

To simplify this, we have

Lemma 4.3. $a_{p^j, 1} p^k = a_{p^{j+1}, 1} a_{1, p^k} - \sum_{t=1}^{\min(j+1, k)} a_{p^{j+1-t}, 1} a_{1, p^{k-t}}.$

Proof. We again induct, this time on $j$. By Proposition 1.5(2), we have

$$a_{p^{j+1}, 1} p^k = a_{p^{j+1}, 1} a_{1, p^k} - \sum_{t=1}^{\min(j+1, k)} a_{p^{j+1-t}, 1} a_{1, p^{k-t}}. $$

Then, using the inductive hypothesis, one sees that the sum in the resulting expression telescopes. This gives the lemma. □

Combining Lemma 4.3 and Corollary 4.2, we see that the evaluation of the Euler product reduces to the computation of the two series

$$S_p(s, \phi, \varphi)_1 = \sum_{l=0}^{\infty} a_{1, p^l} b_{1, p^l} p^{-ls},$$

$$S_p(s, \phi, \varphi)_2 = \sum_{l=0}^{\infty} a_{1, p^l} b_{1, p^{l+1}} p^{-ls},$$

since one can obtain the remaining needed expressions from these by using the involution $\sim$ and by reversing the roles of $\phi$ and $\varphi$. To give them, put

$$1 - a_{1, p} p^{-s} + a_{p, 1} p^{-2s} - p^{-3s} = \prod_{i=1}^{3} (1 - \alpha_i p^{-s}),$$

$$1 - b_{1, p} p^{-s} + b_{p, 1} p^{-2s} - p^{-3s} = \prod_{i=1}^{3} (1 - \beta_i p^{-s}),$$

and

$$L_p(s, \phi \otimes \varphi) = \prod_{i,j=1}^{3} (1 - \alpha_i \beta_j p^{-s})^{-1}.$$

Then we have

Proposition 4.4. (1)

$$S_p(s, \phi, \varphi)_1 = \left[ 1 - a_{p, 1} b_{1, p} p^{-2s} + (a_{p, 1} a_{1, p} + b_{p, 1} b_{1, p} - 2) p^{-3s} - a_{1, p} b_{1, p} p^{-4s} + p^{-6s} \right] L_p(s, \phi \otimes \varphi).$$

(2)

$$S_p(s, \phi, \varphi)_2 = \left[ b_{1, p} - a_{1, p} b_{p, 1} p^{-s} + (a_{1, p}^2 - a_{p, 1}) p^{-2s} + (b_{1, p}^2 - b_{p, 1}) p^{-3s} - a_{1, p} b_{1, p} p^{-4s} + a_{p, 1} p^{-5s} \right] L_p(s, \phi \otimes \varphi).$$
PROOF. By Proposition 1.5, we have
\[ a_{1,p'} = \sum_{i+j+k=l} \alpha_i \alpha_j \alpha_k \]
and a similar expression for \( b_{1,p'} \). Thus
\[ S_p(s, \phi, \varphi) = \sum_{i+j+k=0}^{\infty} \sum_{i+j+k=i'}^{\infty} \sum_{j'=0}^{\infty} \alpha_i \alpha_j \alpha_k \beta_i \beta_j \beta_k p^{-i} \]
\[ = \sum_{i,j,k=0}^{\infty} \sum_{i'=0}^{i+j+k-i'}^{\infty} \sum_{j'=0}^{i'+j+k-i'}^{\infty} \alpha_i \alpha_j \alpha_k \beta_i \beta_j \beta_k p^{-i-j+k-i'}. \]

Summing the geometric series here, simplifying, and making use of the relations (4.1), (1) follows. Part (2) is proved in a similar way. □

We can now prove the main result of this section.

THEOREM 4.5. \( \xi(3s)D(s, \phi, \varphi) = \prod_p L_p(s, \phi \otimes \varphi) \).

PROOF. By Lemma 4.3 one sees that
\[ (4.2) \]
\[ D_p(s, \phi, \varphi) = S_p(s, \phi, \varphi) S_p(2s, \tilde{\phi}, \tilde{\varphi}) (1 + p^{-3s}) \]
\[ - p^{-3s} [ S_p(s, \phi, \varphi) S_p(2s, \tilde{\phi}, \tilde{\varphi}) + \tilde{S}_p(s, \phi, \varphi) \tilde{S}_p(2s, \tilde{\phi}, \tilde{\varphi}) ] . \]

Each of these terms is evaluated by the formulas of Proposition 4.4. After substituting these formulas into (4.2) and combining terms, one sees that \( D_p(s, \phi, \varphi) \) is \( L_p(s, \phi \otimes \varphi) L_p(2s, \tilde{\phi} \otimes \tilde{\varphi}) \) times a polynomial of degree 21 in \( p^{-s} \). However, an explicit computation shows that this polynomial is exactly
\[ (1 - p^{-3s}) L_p(2s, \tilde{\phi} \otimes \tilde{\varphi})^{-1} . \]

Applying Corollary 4.2, the theorem is proved. □

REMARK. This computation can be further explained via a combinatorial identity involving certain Schur polynomials, since the Fourier coefficients \( a_{p^*, p'} \) may be expressed by these polynomials [1, 24]; this allows its generalization to \( \text{GL}_n \).

For example, in the case of the Eisenstein series \( E_{\nu_1, \nu_2}, E_{\mu_1, \mu_2} \) studied by Bump, we see from Proposition 1.6 that
\[ (4.3) \]
\[ \prod_p L_p(s, E_{\nu_1, \nu_2} \otimes E_{\mu_1, \mu_2}) = \xi(s - 2
\begin{align*}
\xi(s - 2\nu_1 - \nu_2 - 2\bar{\mu}_1 - \bar{\mu}_2 + 2) \\
\xi(s + \nu_1 - \nu_2 - 2\bar{\mu}_1 - \bar{\mu}_2 + 1) \xi(s + \nu_1 - \nu_2 + \bar{\mu}_1 + 1) \\
\xi(s + \nu_1 - \nu_2 + \bar{\mu}_1 + 1 \bar{\mu}_2 - 1) \xi(s + \nu_1 + 2\nu_2 - 2\bar{\mu}_1 - \bar{\mu}_2) \\
\xi(s + \nu_1 + 2\nu_2 - \bar{\mu}_2 + \bar{\mu}_1 - 1) \xi(s + \nu_1 + 2\nu_2 + \bar{\mu}_1 + 2\bar{\mu}_2 - 2).
\end{align*}
However, $G_f^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$ depends only on $f, s, \nu_1, \nu_2, \mu_1,$ and $\mu_2,$ and not on $\phi$ and $\varphi.$ Hence (4.3), together with the analytic continuation of the Riemann zeta function, its functional equation, and Theorem 4.5, implies Theorem 3.4. Note that the $\xi_1$ and $\xi_2$ variables guarantee, by virtue of Proposition 3.3(4), that no extra poles are introduced when one replaces $G_f^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$ by $\Gamma^*(s, \nu_1, \nu_2, \mu_1, \mu_2)$.

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