A RENEWAL THEOREM FOR RANDOM WALKS
IN MULTIDIMENSIONAL TIME

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ABSTRACT. Let $X, X_1, X_2, \ldots$ be a family of integer valued, independent and identically distributed random variables with positive mean and finite (positive) variance. Let $S_n = X_1 + X_2 + \cdots + X_n$. The asymptotic behavior of the weighted sum $R(k) = \sum a_n P(S_n = k)$, with summation over $n \geq 1$, is investigated as $k \to +\infty$. In the special case $a_n = d_r(n)$, the number of solutions of the equation $n = n_1 n_2 \cdots n_r$ in positive integers n_j, $1 \leq j \leq r$, $R(k)$ becomes the renewal function $Q(k)$ for a random walk in r-dimensional time whose terms are distributed as X. Under some assumptions on the magnitude of a_n and of $A(x) = \sum_{n \leq x} a_n$, (i) it is shown that $R(k)$ is asymptotically distribution free as $k \to +\infty$, (ii) the proper order of magnitude of $R(k)$ is determined, and under some further restrictions on $A(x)$, (iii) a simple asymptotic formula is given for $R(k)$. From (i), the known asymptotic formula for $Q(k)$ with $r = 2$ or 3 is deduced under the sole assumption of finite variance. The relaxation of previous moment assumptions requires a new inequality for the sum of the divisor function $d_r(n)$, $1 \leq n \leq x$, which by itself is of interest.

I. Introduction. Let $X, X_1, X_2, \ldots$ be a family of integer valued, independent and identically distributed random variables with positive mean $E(X) = \mu < +\infty$ and finite positive variance $V(X) = \sigma^2$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Let $d_r(n)$ denote the number of solutions of $n = n_1 n_2 \cdots n_r$ in positive integers n_j. For a fixed integer $r \geq 2$, we consider the sum

$$(1.1) \qquad Q(k) = \sum_{n=1}^{+\infty} d_r(n) P(S_n = k).$$

Assume that X is aperiodic in the sense that $E(\exp(itX)) \neq 1$ for any $t \neq 0$. Then, the now classical renewal theorem (see Doob (1948), Erdös, Feller and Pollard (1949) and Chung and Pollard (1952)) states that, for $r = 1$, i.e. with $d_1(n) = 1$ for all n, $Q(k) \to 1/\mu$ as $k \to +\infty$.

The case $r \geq 2$ has recently been the subject of active investigations. Notice that $Q(k)$ in (1.1), when $r \geq 2$, can be redefined as follows. Let K_r be the set of all r-vectors $\mathbf{n} = (n_1, n_2, \ldots, n_r)$, $n_i \geq 1$ integer, and let $X_\mathbf{n}$, $\mathbf{n} \in K_r$, be independent copies of X. Let

$$S_\mathbf{n} = \sum X_\mathbf{j},$$

where the summation is over all $\mathbf{j} = (j_1, j_2, \ldots, j_r)$ such that $1 \leq j_i \leq n_i$, $1 \leq i \leq r$. Then

$$Q(k) = \sum_{\mathbf{n} \in K_r} P(S_\mathbf{n} = k).$$

Now, Ney and Wainger (1972) proved that, if $r = 2$ and if $E(|X|^4) < +\infty$, then

$$Q(k) \sim (\log k)/\mu \quad \text{as } k \to +\infty.$$

M. Maejima and T. Mori (1984) replaced the moment condition by $E(|X|^3) < +\infty$ and proved that, for $r = 2$ or 3,

(1.2) $$Q(k) \sim (\log k)^{r-1}/(r-1)!\mu \quad \text{as } k \to +\infty.$$

Their proof is based on the following form of the central limit theorem (see Petrov (1975), p. 207):

$$P(S_n = k) = \phi_n(k; \mu, \sigma^2)[1 + (\beta/6\sigma^2 n^{1/2})(\xi_{n,k}^3 - 3\xi_{n,k})] + c_{n,k}/n(1 + |\xi_{n,k}|^3),$$

where $\beta = E[(X - \mu)^3]$,

(1.3) $$\phi_n(k) = \phi_n(k; \mu, \sigma^2) = \exp(-\tfrac{1}{2}\xi_{n,k}^2)/(2\pi n)^{1/2}\sigma,$$

with

(1.4) $$\xi_{n,k} = (n\mu - k)/\sigma n^{1/2},$$

and $c_{n,k}$ are constants such that $c_n = \sup_k |c_{n,k}| \to 0$ as $n \to +\infty$. The limitation $r = 2$ and 3 is due to the lack of good asymptotic results for the sum of $d_r(n)$ over n.

In a recent paper, Galambos and Kátai (1986) gave an error term in (1.2) for $r = 2$ or 3, and proved that, for arbitrary $r \geq 2$, as $N \to +\infty$,

$$\limsup \frac{1}{N} \sum_{k=N+1}^{2N} |Q(k) - s(\log k\mu^{-1})| < +\infty$$

with an explicitly given polynomial $s(\cdot)$ of degree $r - 1$, under the assumption that the third absolute moment of X is finite. One of the main reasons for this last assumption on the third moment is the appeal to the quoted central limit theorem for first establishing that the asymptotic behavior of $Q(k)$, as $k \to +\infty$, can always be reduced to that of

(1.5) $$Q_1(k) = \sum_{n=1}^{+\infty} d_r(n)\phi_n(k).$$

The relation of $Q(k)$ and $Q_1(k)$ is made more precise in Galambos and Kátai (1986a) under a weaker moment assumption. As a matter of fact, it is established that, whatever r,

(1.6) $$Q(k) = Q_1(k) + o(1) \quad \text{as } k \to +\infty$$

if

$$\int_{|x| \geq z} x^2 \, dF(x) = O(z^{-a})$$

with a suitable $0 < a < 1$, where F denotes the distribution function of X. Furthermore, Galambos and Kátai (1986a) also prove that

$$(1.7) \qquad Q_1(k) > c_1(\log k)^{r-1}$$

with a suitable constnt $c_1 > 0$ which may depend on r.

The aim of the present paper is to further relax the moment assumption and to prove the following results.

THEOREM 1. *Let X be integer valued, aperiodic, with finite positive mean μ and finite (positive) variance σ^2. Then, for any fixed $r \geq 2$,*

$$(1.8) \qquad Q(k) = Q_1(k) + o((\log k)^{r-1}) \quad as \ k \to +\infty.$$

Consequently,

$$(1.9) \qquad Q(k)/Q_1(k) \to 1 \quad as \ k \to +\infty.$$

COROLLARY. *Let $r = 2$ or 3. Then, for integer valued, aperiodic X with finite positive mean μ and variance, (1.2) holds.*

Note that, on account of (1.7), (1.8) indeed implies (1.9). In addition, it should be noted that (1.8) and Lemma 2 of Maejima and Mori (1984) immediately yield the corollary.

The proof of Theorem 1 is based upon the following assertions.

LEMMA 1. *Under the hypothesis stated in Theorem 1*

$$(1.10) \qquad \sup_k (1 + \xi_{n,k}^2)|P(S_n = k) - \phi_n(k)| < \varepsilon(n)n^{-1/2},$$

where $\varepsilon(n) \to 0$ as $n \to +\infty$.

For the proof see Bhattacharya and Ranga Rao (1976), Theorem 22.1, pp. 231–237.

LEMMA 2. *Let $\varepsilon > 0$ be an arbitrary constant. Then for $x^\varepsilon \leq y \leq x$,*

$$(1.11) \qquad \sum_{x \leq n \leq x+y} d_r(n) < c(r, \varepsilon)y(\log x)^{r-1},$$

where $c(r, \varepsilon)$ is some positive constant depending on r and ε.

We shall use Lemma 2 only with $\varepsilon = 1/2$.

The proof of Theorem 1 is given in the next section within the framework of a more general formulation. Lemma 2, which appears to be of interest on its own, is proved in §3.

2. A renewal theorem for a class of weighted renewal functions. Let $a_n \geq 0$, $n \geq 1$, be a sequence of real numbers, and set

$$(2.1) \qquad A(x) = \sum_{n \leq x} a_n.$$

Assume that

$$(2.2) \qquad a_n = O(n^\varepsilon)$$

for every $\varepsilon > 0$, where the constant implied by the order term may depend on ε. Furthermore, let us assume that, with some positive constants c_1, c_2, c_3 and c_4, the inequalities

$$(2.3) \qquad c_1 h L(x) \leq A(x + h) - A(x) \leq c_2 h L(x)$$

and

$$(2.4) \qquad c_3 \leq L(h)/L(x) \leq c_4$$

hold with a positive function $L(x)$ for all $x \geq 1$ and $\sqrt{x} \leq h \leq x$.

Let

$$(2.5) \qquad R(k) = \sum_{n=1}^{+\infty} a_n P(S_n = k)$$

and

$$(2.6) \qquad R_1(k) = \sum_{n=1}^{+\infty} a_n \phi_n(k).$$

We now prove the following result.

THEOREM 2. *Assume that the conditions stated for* X *in Theorem 1 and the order assumptions* (2.2), (2.3) *and* (2.4) *hold. Then*

$$(2.7) \qquad R(k) = R_1(k) + o(R_1(k)) \quad as \ k \to +\infty.$$

Furthermore, with suitable positive constants c_5 *and* c_6,

$$(2.8) \qquad c_5 \leq R_1(k)/L(k) \leq c_6.$$

Before proceeding to the proof, let us remark that all assumptions of Theorem 2 which relate to a_n are satisfied when $a_n = d_r(n)$ with arbitrary r. Indeed, (2.2) is well known (see Hardy and Wright (1960) for $r = 2$, which trivially extends to $r \geq 2$ by induction), and the lower inequality of (2.3) with $L(x) = (\log x)^{r-1}$ is very simple to prove (see Galambos and Kátai (1986a) for details). The upper inequality of (2.3), on the other hand, is our Lemma 2. Finally, (2.4) is evident when $L(x)$ is a power of $\log x$. Consequently, Theorem 2 implies Theorem 1. The inequalities in (2.8) are new even for the special case $a_n = d_r(n)$ for $r \geq 4$. The fact that (2.8) does not extend to an asymptotic formula without some further assumptions on the sequence a_n is pointed out by Kawata (1961) who investigated $R(k)$ in the special case $A(x) \sim ax$ with some real number a.

PROOF OF THEOREM 2. We first observe that the inequality $c_7 (\log x)^{-c} < L(x) < c_8 (\log x)^c$, where c, c_7 and c_8 are suitable positive constants, immediately follows frim (2.4).

Let

$$H(n, k) = |P(S_n = k) - \phi_n(k)|$$

and

$$(2.9) \qquad T(k) = \sum_{n=1}^{+\infty} a_n H(n, k).$$

In view of the elementary inequality

$$(2.10) \qquad |R(k) - R_1(k)| \leq T(k),$$

we have to show that $T(k) = o(R_1(k))$, which in turn is equivalent to

$$(2.11) \qquad T(k) = o(L(k)),$$

assuming that (2.8) has been proved. We thus have to establish (2.8) and (2.11).
   We first prove (2.11). From Lemma 1 we have

$$(2.12) \qquad H(n,k) \leq \varepsilon(n)/\sqrt{n}(1 + \xi_{n,k}^2),$$

and we may assume that $\varepsilon(n)$ tends to zero monotonically. Let $M_1 = k/2\mu$ and $M_2 = 2k/\mu$. If $n < M_1$, then $\xi_{n,k}^2 \geq k^2/4\sigma^2 n$, while for $n > M_2$, $\xi_{n,k}^2 \gg n$. (Here, and in what follows, $a \gg b$ signifies the existence of a positive constant c such that $a > cb$.) Thus, the contribution of the integers $n < M_1$ and $n > M_2$ to $T(k)$ in (2.9) can be majorized by the sums $\sum_{n<M_1} a_n \sqrt{n}/k^2$ and $\sum_{n>M_2} a_n/n^{3/2}$. On account of (2.2), both sums above are $O(k^{-1/4})$, and thus

$$T(k) = \sum_{n=M_1}^{M_2} a_n H(n,k) + O(k^{-1/4}).$$

Because, as was pointed out earlier, $L(k)$ can go to zero at most as fast as a power of $\log k$, the error term above is $o(L(k))$. Hence, it remains to estimate the sum on the right-hand side above, and to show that

$$(2.13) \qquad T^*(k; M_1, M_2) = \sum_{n=M_1}^{M_2} a_n H(n,k) = o(L(k)).$$

Let now $m_j = j(2k/\mu)^{1/2} + M_1$, $j = 0, 1, 2, \ldots, s$, where s is determined by the inequality $m_s \leq M_2 < m_{s+1}$. Then, by (2.12),

$$T^*(k; M_1, M_2) \leq \varepsilon(M_1) \sum_{j=0}^{s} \frac{1}{(1 + \xi_{n_j,k}^2)\sqrt{m_j}} \sum_{m_j \leq n < m_{j+1}} a_n,$$

where n_j is defined as an integer in $[m_j, m_{j+1})$ for which $\xi_{n,k}^2$ takes its minimum in the interval $[m_j, m_{j+1})$. To estimate the inner sum, we observe that $m_{j+1} - m_j \geq \sqrt{m_j}$, and thus (2.3) is applicable. This, together with (2.4), yields

$$T^*(k; M_1, M_2) \leq c_9 \varepsilon(M_1) L(k) \sum_{j=0}^{s} \frac{1}{(1 + \xi_{n_j,k}^2)}.$$

The sum appearing on the right-hand side can be majorized by the convergent series $\sum 1/j^2$, and thus (2.13) is proved.
   For establishing (2.8), first note that $\phi_n(k) \gg n^{-1/2}$ in the range

$$n \in [k/\mu - \sigma\sqrt{k}, k/\mu + \sigma\sqrt{k}].$$

Hence, keeping only these terms in $R_1(k)$, the lower inequality of (2.8) follows by an appeal to (2.3) and (2.4). For the upper inequality of (2.8) we start with showing that the contributions of those terms to $R_1(k)$ whose index n satisfies either $n < k/\mu - \sqrt{k}\,\sigma(\log k)^2$ or $n > k/\mu + \sqrt{k}\,\sigma(\log k)^2$ are negligible. Indeed,

for these n with the additional limitation $n < 2k/\mu$, $|\xi_{n,k}| \gg (\log k)^2$ as well as $|\xi_{n,k}| \gg (\log n)^2$, while for $n \geq 2k/\mu$, $|\xi_{n,k}| \gg \sqrt{n}$. Hence, in view of (2.2), the total contribution of these terms to $R_1(k)$ is smaller than an arbitrary negative power of k, which is negligible in terms of $L(k)$ (recall that it can go to zero at most as fast as a negative power of $\log k$). Next, we split up the remaining values of n into intervals of the type $[k/\mu - \sigma(j+1)\sqrt{k}, k/\mu - \sigma j\sqrt{k})$ and $[k/\mu + \sigma j\sqrt{k}, k/\mu + \sigma(j+1)\sqrt{k})$ where $0 \leq j \leq (\log k)^2$. Over the jth interval of either type above, $cj^2 \leq \xi_{n,k}^2 \leq c^*(j+1)^2$ with some positive constants c and c^*. Thus, for these n,

$$\Phi_n(k) \ll k^{-1/2} \exp(-c^*(j+1)^2)$$

and their contribution to $R_1(k)$ is of the magnitude

$$k^{-1/2} \exp(-c^*(j+1)^2)[A(k/\mu + (j+1)\sigma\sqrt{k}) - A(k/\mu + j\sigma\sqrt{k})],$$

where $A(x)$ is the function defined in (2.1). Now, upon utilizing (2.3) and (2.4), the summation over j of these contributions to $R_1(k)$ yields the upper inequality of (2.8). This completes the proof of Theorem 2.

For the validity of (2.7), condition (2.3) can be weakened at the expense of assuming more than just the finiteness of the second moment of X. In fact, with no change in the proof of the corresponding result in Galambos and Kátai (1986a), the following result can be obtained.

THEOREM 3. *Let $a_n \geq 0$ satisfy (2.2). Let $L(x)$ be a positive function for which (2.4) holds. Assume that $A(x) < cxL(x)$ with some positive constant c and that the lower inequality of (2.3) is valid. Let X satisfy the conditions of Theorem 2 as well as the condition $\int_{|x| \geq z} x^2 \, dF(x) = O(z^{-a})$ with a suitable constant $0 < a < 1$, where $F(x)$ is the distribution function of X. Then (2.7) holds.*

In order to give an asymptotic formula for $R_1(k)$ we have to assume more than (2.3).

THEOREM 4. *Assume that conditions (2.2), (2.3) and (2.4) hold for $a_n \geq 0$. Furthermore assume that there exists a positive function $\rho(x)$, tending to zero monotonically, such that*

(2.14) $(A(x+h) - A(x))/hL^*(x) \to 1$ *(as $x \to +\infty$)*

uniformly in $h \in (\rho(x)\sqrt{x}, \sqrt{x})$, where $L^*(x)$ is a very slowly varying function in the sense that, as $x \to +\infty$,

$$L^*(Y(x))/L^*(x) \to 1 \quad \text{whenever } (\log Y(x))/(\log x) \to 1.$$

Then, as $k \to +\infty$,

$$R_1(k) \sim (1/\mu)L^*(k/\mu).$$

PROOF. From (2.3) and (2.14) it follows that $L(x)$ and $L^*(x)$ have the same order of magnitude, i.e. $L(x) \ll L^*(x)$ and $L^*(x) \ll L(x)$. Now, let $u(k)$ be a function tending to infinity with k and satisfying $u(k) = O(\log k)$. Let

$$J = [N_1, N_2], \quad N_1 = k/\mu - \sigma\sqrt{k}\,u(k), \quad N_2 = k/\mu + \sigma\sqrt{k}\,u(k).$$

By elementary estimates, upon utilizing (2.3), we get

$$\sum_{n \notin J} a_n \phi_n(k) = o(L^*(k)) \quad \text{as } k \to +\infty.$$

Thus, as $k \to +\infty$,

$$R_1(k) = R_1^*(k) + o(L^*(k))$$

where

$$R_1^*(k) = \sum_{n \in J} a_n \phi_n(k).$$

Choose an integer M such that $\rho(n)\sqrt{n} \leq M \leq \sqrt{n}$ for each $n \in J$. It is easy to see that

$$\phi_n(k) = \frac{1}{M} \sum_{j=0}^{M-1} \phi_{n+j}(k) + o(\phi_n(k)) \qquad (k \to +\infty)$$

uniformly in $n \in J$. Hence,

$$R_1^*(k) = \frac{1}{M} \sum_{n \in J} a_n \sum_{j=0}^{M-1} \phi_{n+j}(k) + o(R_1^*(k)),$$

which, by changing the order of summation, becomes

$$R_1^*(k) = \sum_{m=N_1}^{N_2+M} \phi_m(k) \frac{1}{M} \left[\sum_{t=0}^{M-1} a_{m-t} \right]^* + o(R_1^*(k)),$$

where the star at the end of the inner sum means that the summation extends only to those m for which $m - t \in J$. This implies restriction only if either $m < N_1 + M$ or $m > N_2$. The total contribution of these terms, however, does not exceed

$$L^*(k) \sum_{N_1 \leq m \leq N_1+M} \phi_m(k) + L^*(k) \sum_{N_2 \leq m \leq N_2+M} \phi_m(k),$$

which is $o(L^*(k))$. Therefore, since, in view of (2.14),

$$\frac{1}{M} \sum_{t=0}^{M-1} a_{m-t} = (1 + o(1))L^*\left(\frac{k}{\mu}\right)$$

uniformly in $m \in J$,

$$R_1^*(k) = \left\{ \sum_{m=N_1+M}^{N_2} \phi_n(k) \right\} L^*\left(\frac{k}{\mu}\right) + o(R_1^*(k)).$$

This now completes the proof upon observing that the sum appearing on the right side converges to $1/\mu$ as $k \to +\infty$ (see formula (34) in Galambos and Kátai (1986) and the hint for its proof).

Asymptotic formulas have been developed only in a few special cases (see W. L. Smith (1958), Kawata (1961) and Greenwood et al. (1982)) for $R(k)$. Several other papers deal with general renewal functions $R(k)$ whose assumptions on a_n, however, are unrelated to the current investigation (our assumption (2.2) is essential in relation to renewal problems in multidimensional time). In fact, the example given in Remark (b) on p. 568 by Embrechts et al. (1984) clearly shows this claim: it is demonstrated there that, under our assumption (2.2), their conclusion, and earlier related results, cannot be applied.

3. Proof of Lemma 2. By the nature of Lemma 2, its proof is of a quite different character when compared with the rest of the paper. The basic ideas we shall use here can be found in the paper of Linnik and Vinogradov (1957).

Clearly, it is sufficient to prove the assertion for $y = x^\varepsilon$. Let the integers $b_n, n = 1, 2, \ldots, [y]$, be defined by the formula $b_n = [x] + n$, where $[z]$ signifies the integer part of z. For an integer m, let $p(m)$ denote the smallest and $P(m)$ the largest prime factor of m. For every b_n, let $b_n = c_n r_n$, where $P(c_n) \leq y^{1/2}$ and $p(r_n) > y^{1/2}$. Since $d_r(m)$ is multiplicative, i.e., $d_r(m_1 m_2) = d_r(m_1) d_r(m_2)$ for coprime m_1 and m_2, and, for primes p and integers $a \geq 1$,

$$d_r(p^a) = \binom{a + r - 1}{r - 1},$$

we have that $d_r(b_n) \ll d_r(c_n)$. Hence, for Lemma 2, we have to develop an estimate of the sum

$$(3.1) \qquad\qquad B = \sum_{n \leq y} d_r(c_n).$$

Now, upon noting that

$$d_r(c_n) = \sum_{m_1 m_2 = c_n} d_{r-1}(m_1),$$

we can write

$$(3.2) \qquad\qquad B = B_1 + B_2,$$

where

$$(3.3) \qquad\qquad B_i = \sum_{n \leq y} \sum_i d_{r-1}(m_1), \qquad i = 1, 2,$$

with $\sum_i$ signifying summation over m_1 satisfying $m_1 m_2 = c_n$ and $m_1 \leq y$ in $\sum_1$ while $m_1 > y$ in $\sum_2$. The sum B_1 can easily be estimated. By interchanging summations, we have

$$(3.4) \qquad \begin{aligned} B_1 &\leq \sum_{m_1 \leq y} d_{r-1}(m_1) \sum_{\substack{n \leq y \\ c_m \equiv 0 \ (\mathrm{mod}\ m_1)}} 1 \\ &\ll y \sum_{m_1 \leq y} \frac{d_{r-1}(m_1)}{m_1} \ll y (\log y)^{r-1}. \end{aligned}$$

The estimation of B_2 is much more involved, and we do it in several steps. Let $m_1 = p_1 p_2 \cdots p_s$ be the prime decomposition of m_1 with $p_1 \leq p_2 \leq \cdots \leq p_s$. Let t be defined so that $t = p_1 p_2 \cdots p_k \leq y$ and $t p_{k+1} > y$. Hence, t is well determined by m_1. Let us now split B_2 into two parts, $B_2 = B_{2,1} + B_{2,2}$, say, according as $P(t) \leq \log x$ or $P(t) > \log x$.

Since, for fixed n and t,

$$\sum_{\substack{t \mid m_1 \\ m_1 \mid c_n}} d_{r-1}(m_1) \ll x^\eta$$

with arbitrary $\eta > 0$, $B_{2,1} \ll x^\eta y \sum 1/t$, where the summation is extended over those t for which $y^{1/2} \leq t \leq y$ and $P(t) \leq \log x$. For estimating this sum we appeal to the following well known result of Rankin

$$(3.5) \qquad \sum_{\substack{t \leq y \\ P(t) \leq \log x}} 1 \ll 2^{(2 \log x)/\log \log x}.$$

We thus have

$$(3.6) \qquad B_{2,1} = O(y).$$

Let now $P(t) > \log x$. It is obvious that $d_{r-1}(uv) \leq d_{r-1}(u)d_{r-1}(v)$ for all integers u and v. Thus, for fixed n and t,

$$\sum_{\substack{m_1 | c_n \\ t | m_1}} d_{r-1}(m_1) \leq d_{r-1}(t) \prod_{q > p_k} (1 + d_{r-1}(q)),$$

where q runs over the prime divisors of c_n/t. Since $c_n \leq x + y < 2x$, we get

$$\sum_{m_1 | c_n} d_{r-1}(m_1) \leq d_{r-1}(t) r^{(\log x)/(\log p_k)}.$$

Hence,

$$B_{2,2} \leq \sum_1 d_{r-1}(t) r^{(\log x)/(\log P(t))} \sum_{\substack{c_n \equiv 0 \pmod t \\ n \leq y}} 1$$

$$\ll y \sum_1 \frac{d_{r-1}(t)}{t} r^{(\log x)/(\log P(t))} = y B_3,$$

where $\sum_1$ denotes summation over t with $y^{1/2} \leq t \leq y$ and $P(t) > \log x$.

We split $B_3 = B_{3,1} + B_{3,2}$, where in $B_{3,1}$, $P(t) < x^\gamma$ with some $\gamma > 0$, which may depend on r and ε, and in $B_{3,2}$, $P(t) > x^\gamma$.

Note that, in $B_{3,2}$, $(\log x)/(\log P(t))$ is bounded. Thus, if we write $P(t) = p$, $t = pt^*$, then

$$B_{3,2} \ll \sum_{x^\gamma < p < y} \frac{1}{p} \sum^* \frac{d_{r-1}(t^*)}{t^*},$$

where $\sum^*$ is summation over t^* such that $P(t^*) \leq p$ and $y^{1/2}p^{-1} < t^* < yp^{-1}$. The sum involving t^* is easily seen to be $O((\log x)^{r-1})$. The well-known asymptotic formula

$$(3.7) \qquad \sum_{p \leq z} \frac{1}{p} = \log \log z + C + o(1) \qquad (z \to +\infty)$$

thus yields

$$(3.8) \qquad B_{3,2} = O((\log x)^{r-1}).$$

We finally estimate $B_{3,1}$. Writing again $t = pt^*$, where $p = P(t)$,

$$B_{3,1} \ll \sum_{\log x < p \leq x^\gamma} \frac{1}{p} r^{(\log x)/\log p} \sum^{**} \frac{d_{r-1}(t^*)}{t^*},$$

where $\sum^{**}$ denotes summation over t^* such that $y^{1/2} \leq t \leq y$ and $p \leq x^\gamma$. We further increase above if we sum over all t^* with $y^{1/4} \leq t^*$, namely, the mentioned limitations in t and p imply this bound for t^* if γ is chosen sufficiently small. We, of course, keep the additional restriction $P(t^*) \leq p$. Since $t^*/y^{1/4} \geq 1$, we further increase if we plug in the additional factor $(t^*/y^{1/4})^\delta$ with a suitably chosen $\delta > 0$. This last step again is an idea of Rankin [10]. Now, if we choose

$$\delta = \frac{1}{3}(\log p)^{-1} \log \frac{\log y^{1/4}}{\log p}$$

it follows (see Indlekofer (1980))

$$\sum^{**} \frac{d_{r-1}(t^*)}{t^*} \ll \exp\left\{ \sum_{\substack{q \leq p \\ q \text{ prime}}} \frac{r-1}{q} - c\frac{\log y^{1/4}}{\log p} \log \frac{\log y^{1/4}}{\log p} \right\}$$

$$\leq \exp\left[(r-1)\log\log p - c\frac{\log x^{\varepsilon/4}}{\log p} \log \frac{\log x^{\varepsilon/4}}{\log x^\gamma} \right]$$

with a positive constant c. Thus

$$(3.9) \qquad B_{3,1} \ll \sum_{\log x < p \leq x^\gamma} \frac{1}{p} \exp\left[(r-1)\log\log p + \frac{\log x}{\log p} u_r(\varepsilon, \gamma) \right],$$

where $u_r(\varepsilon, \gamma) = \log r - (c\varepsilon/4)\log(\varepsilon/4\gamma)$. Upon choosing γ small enough, such that $u_r(\varepsilon, \gamma) \leq -1$, summation over p in the estimate (3.9) of $B_{3,1}$ yields

$$(3.10) \qquad B_{3,1} \ll \exp[(-1/\gamma)(\log x^\gamma)^{r-1}],$$

where the trivial estimate $-(\log x)/(\log p) \leq -1/\gamma$ as well as the well-known estimate

$$\sum_{p \leq x^\gamma} \frac{(\log p)^{r-1}}{p} \ll (\log x^\gamma)^{r-1} \qquad (r \geq 2)$$

were utilized. Hence, since

$$B = B_1 + B_2 = B_1 + B_{2,1} + B_{2,2} \leq B_1 + B_{2,1} + yB_3$$
$$\leq B_1 + B_{2,1} + y(B_{3,1} + B_{3,2}),$$

the estimates (3,4), (3.6), (3.10) and (3.8) establish Lemma 2.

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